Above the front door of Niels Bohr's cottage was nailed a horseshoe. A visitor who saw it exclaimed: "Being as great a scientist as you are, do you really believe that a horseshoe above the entrance to a home brings good luck?"
"No," answered Bohr, "I certainly do not believe in this superstition. But you know," he added with a smile, "they say that it does bring luck even if you don't believe in it!"

- George Gamow, excerpted from *Thirty Years that Shook Physics.*

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

This problem makes sense only if you make some rather poorly motivated approximations. In particular, we must assume that we are far away from resonance (namely that $\omega_0^2 - \omega^2 \gg \gamma\omega$). We also can assume that there are very few electrons (namely that $\frac{Ne^2}{m\epsilon_0} \ll 1$, which is well-motivated by the fact that κ is much less than n, i.e. few absorbers). If we make these approximations, the results follow almost immediately. If you don't make these assumptions, then the results are clearly incorrect (see Figure 6.1 in Fowles, which is nothing like the equations Fowles asks us to derive). Thus, we'll make these assumptions!

Then we can apply these approximations to equations 6.34 and 6.35 in Fowles. We find that:

$$n^2 - \kappa^2 \approx n^2 \approx 1 + \frac{Ne^2}{m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2} \tag{1}$$

Using a first order Taylor expansion, we then find that:

$$n \approx 1 + \frac{Ne^2}{2m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2} \tag{2}$$

Since n is approximately 1, κ is given by:

$$\kappa \approx \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2}. \tag{3}$$

Problem 2

Once again Fowles attempts to confuse us by implying the above results can be applied in the solution to this problem when they can't. This is because the above equations break down in the vicinity of the resonance, which is where we must work to solve this problem. So here we bring back the $\gamma^2\omega^2$ terms, but continue to assume there are few electrons. In this case our formulas for κ and n are given by:

$$n \approx 1 + \frac{Ne^2}{2m\epsilon_0}\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{4}$$

and

$$\kappa \approx \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{5}$$

If we take the derivative of n with respect to ω and set it equal to zero, we find two positive roots yielding the values for the max and min of the the function n, namely

$$\omega = \omega_0\sqrt{1 \pm \gamma/\omega_0}.$$

It is safe to assume, since damping is small, that this value can be approximated by the first order Taylor expansion:

$$\omega = \omega_0 \pm \gamma/2.$$

If we plug these values into our expression for κ (Eq. 5), we see that these values are those where κ attains half its maximum value.

Problem 3

We are given that $\sigma = 6.8 \times 10^7$ mho/m and that $N_e = 1.5 \times 10^{28}$ electrons/m^3. Using these values in the appropriate Fowles formulas gives us the desired answers...

(a)

Plasma frequency

$$\omega_p = \sqrt{\frac{Ne^2}{m\epsilon_0}} = 6.9 \times 10^{15}\mathrm{s}^{-1}$$

(b)

Relaxation time

$$\tau = \frac{\mu_0\sigma c^2}{\omega_p^2} = 1.6 \times 10^{-13}\mathrm{s}$$

(c)

Our frequency with a wavelength of 10^{-6} m is given by

$$\omega = \frac{2\pi c}{\lambda} = 1.9 \times 10^{15}\mathrm{s}^{-1}.$$

Real and imaginary parts of the index of refraction can be derived from from Fowles Eqs. 6.55 and 6.56:

$$n^2 - \kappa^2 = 1 - \frac{\omega_p^2}{\omega^2 + \tau^{-2}}$$

$$2n\kappa = \frac{1}{\omega\tau}\frac{\omega_p^2}{\omega^2 + \tau^{-2}}$$

Clearly, $\omega_p, \omega \gg \tau^{-1}$, so we get

$$n^2 - \kappa^2 \approx 1 - \left(\frac{\omega_p}{\omega}\right)^2 = -12.2$$

$$2n\kappa \approx \frac{1}{\omega\tau}\frac{\omega_p^2}{\omega^2} = 0.044.$$

We can then solve these equations for n and κ, and with a little algebra we get:

$$n = 0.006$$

$$\kappa = 3.5.$$

(d)

The reflectance is given by the Hagen-Rubens formula,

$$R = 1 - \sqrt{\frac{8\omega\epsilon_0}{\sigma}} \approx 1.$$

Problem 4

This problem, as Prof. Strovink pointed out, is a little bit tricky. Our experimenter finds 1 event in 10^6 interactions. Now we want to be 90% sure we find a second event. How many interactions do we need? The basic problem is that we don't really know the average number of events we should see per 10^6 interactions, which is needed to calculate how many more interactions are necessary to be 90% sure we'll see a second event. So we'll have to try to figure out some function describing our confidence in the value of a we have measured and then convolve it with the probability for seeing a second event.

The probability density function, in this case that for Poisson statistics, is given by:

$$f_p(x) = \frac{e^{-a}a^x}{x!} \tag{6}$$

where x is a non-negative integer and a is the average value of x. The probability density function (with known parameter a) allows us to predict the frequency with which random data x will take on some particular value.

We first want to calculate a value for $a = n{\cdot}p$ (where n is the number of interactions and p is the probability for an event), a distribution function based on the most likely values for a and our confidence in those values. The most likely value of p from the data is 10^{-6}.
For a conservative upper limit (without assuming very much about the prior probability distribution), we can estimate that the lower limit of p (at a 95% confidence level) must be that for which the probability of seeing one event in 10^6 interactions is at least 5%:

$$f_p(1) = e^{-np}np \geq 0.05, \tag{7}$$

which tells us that $np \geq 0.05$, or $p \geq 5 \times 10^{-8}$. Furthermore, we know that the upper limit at 95% confidence level on p can be found from:

$$f_p(1) = e^{-np}np \leq 0.95, \tag{8}$$

which tells us that $p \leq 5.14 \times 10^{-6}$.

We want to be 90% sure we'll see a second event. We could guess that if we're 95% sure p is bigger than $p_{\min} = 5 \times 10^{-8}$ and 95% sure that we'll see at least one more event after n_2 interactions using this value for p, we'll be 90% sure to see a second event. We're 95% sure we'll see at least one more event if the probability to see zero events is less than 0.05:

$$f_p(0) = e^{-n_2 p_{\min}} \leq 0.05,$$

which gives us

$$n_2 \geq 6 \times 10^7 \text{ interactions.}$$

This is a conservative upper limit on the number of interactions we need before we'll see another event.

The above analysis gives us some idea of what our "likelihood distribution" $\tilde{L}$ for p looks like... it is peaked at 1×10^{-6} and nears zero at both 5×10^{-8} and 5.14×10^{-6}. It is well described by the function

$$\tilde{L}(p) = \frac{p}{p_0} e^{-p/p_0}$$

where $p_0 = 10^{-6}$ is the most likely value for p. It is not an accident that this looks exactly like $f_p(1)$. This, as Prof. Strovink explained to me, is just Bayes' assumption of a uniform prior probability distribution – meaning that we assume we found the most likely value of p_0 in our experiment and the distribution of probabilities is that from, in our case, Poisson statistics.

A little more mathematical rigor can be applied if we take our "likelihood distribution" $\tilde{L}$ for p and convolve it with the restriction that the probability for zero events must be less than 10%. This approach yields more or less the same result as Rohlf's answer, which is reasonable since in the Bayesian approach we assume a prior probability distribution with $p_0 = 10^{-6}$ as the central value. Rohlf just assumed that *a priori* we knew the probability for an event to occur would be $p_0 = 10^{-6}$. If we try this approach, we find that:

$$\frac{\int_0^\infty \tilde{L}(p) e^{n_2 p} dp}{\int_0^\infty \tilde{L}(p) dp} \leq 0.1$$

Plugging in our assumed prior probability distribution or "likelihood distribution" and using the substitution $u = (1/p_0 + n_2)p$ in the numerator's integral, we find:

$$\frac{1}{p_0} \int_0^\infty \frac{p}{p_0} e^{-p/p_0} e^{-n_2 p} dp = \frac{1}{p_0^2 \cdot (1/p_0 + n_2)^2} \leq 0.1.$$

From which we can calculate $n_2 = 2.16 \times 10^6$ for 90% CL that we will see a second event. This is far smaller than our original rough estimate, but assumes a prior probability distribution. This is probably the more correct approach.

Well, as you can probably tell, this problem was quite difficult for me, so don't feel too bad if you had some trouble as well...

Problem 5

As evidenced by the sampling of problems from Rohlf, we can guess his two main interests are particle physics and beer.

The expression for the number of particles from the ideal gas law is:

$$N = \frac{PV}{kT}. \tag{9}$$

We can take the derivative of N with respect to time to obtain:

$$\frac{dN}{dt} = \frac{P}{kT}\frac{dV}{dt}. \tag{10}$$

We know from the statement of the problem that the number of CO_2 molecules is proportional to the surface area of the beer bubbles,

$$\frac{dN}{dt} = Cr^2.$$

Also, assuming a spherical bubble, we know that

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Plugging these into Eq. (10), we find that:

$$\frac{dr}{dt} = \frac{CkT}{4\pi P}, \tag{11}$$

which indicates the radius of the bubble increases linearly with time.

Problem 6

The power given off by the sun is S (power radiated per unit area) times the surface area of the sun, which is:

$$P_S = (\sigma T_S^4) \cdot (4\pi R_S^2) \tag{12}$$

The portion of this power received by the earth is scaled down by the emissivity factor ϵ (earth is treated as a gray body) and the cross-sectional area of the earth over the surface area of a sphere with a radius equal to the distance between the earth and the sun:

$$P_E^{(in)} = \epsilon(\sigma T_S^4) \cdot (4\pi R_S^2) \frac{\pi R_E^2}{4\pi R_{ES}^2} \tag{13}$$

The power re-radiated by the earth is given by:

$$P_E^{(out)} = (\epsilon \sigma T_S^4)(4\pi R_E^2). \tag{14}$$

In equilibrium, $P_E^{(in)} = P_E^{(out)}$. If we equate these expressions, we find that:

$$\frac{T_E^4}{T_S^4} = \frac{1}{4}\frac{R_S^2}{R_{ES}^2}, \tag{15}$$

from which we deduce that

$$T_E = 290 \text{ K}.$$

Problem 7

(a)

We apply Wien's law to get the peak of the earth's blackbody spectrum. First, the constant can be derived from plugging in the known parameters for the sun:

$$\lambda_{\max} = \frac{C}{T}$$

$$C = \lambda_{\max,\text{sun}} T_S = (0.58\ \mu\text{m})(5800\text{ K}) = 3364\mu\text{m K}.$$

Applying Wien's law to the earth, we get:

$$\lambda_{\max,\text{earth}} = \frac{C}{T_E} = 11.2\ \mu\text{m}.$$

(b)

If half the power re-radiated from the dirt is radiated back into the greenhouse at every interface with the walls, in equilibrium the power radiated by the dirt should be twice that incident from the sun. The power from the sun hitting the dirt of the greenhouse is:

$$P_S^{(dirt)} = (\sigma T_S^4)(4\pi R_S^2)\frac{\text{Area of dirt}}{4\pi R_{ES}^2}, \tag{16}$$

The power re-radiated by the dirt is:

$$P_{out}^{(dirt)} = \sigma T_{\text{dirt}}^4 \cdot (\text{Area of dirt}) \tag{17}$$

Setting $2P_{out}^{(dirt)} = P_S^{(dirt)}$, we find that:

$$T_{\text{dirt}}^4 = \frac{T_S^4}{2}\frac{R_S^2}{R_{ES}^2}$$

$$T_{\text{dirt}} = 333 \text{ K}.$$

Problem 8

Since we only get half the light power from the sun that we used to down on the earth, the new temperature on earth T'_E is given simply by:

$$T'_E = \left(\frac{1}{2}\right)^{1/4} T_E = 252\text{ K} = -5.3\ ^o\text{F}.$$

Brrrr......