"The purpose of physics is to understand the universe... the purpose of mathematics is, well, obscure to me..."

- Prof. Seamus Davis, UC Berkeley

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

If you're interested in the possibility of magnetic monopoles, you might want to look up a paper by Blas Cabrera (Physical Review Letters, vol. **48**, no. 20, 1982 pp. 1378-81), where the possible detection of a single magnetic monopole is discussed. There have been no further monopoles detected since that time, so this report remains unconfirmed. There is also an excellent discussion of magnetic monopoles in J.D. Jackson's *Classical Electrodynamics*.

A discussion of the additional problem presented in discussion section this week can be found in a paper by Robert Romer (American Journal of Physics vol. **50**, no. 12, 1982 pp. 1089-93).

Problem 1

(a)

We use Gauss's law and choose a cylindrical surface of radius r centered on the axis (we'll call it $\hat{z}$) of the parallel plate capacitor, far from the edges of the capacitor ($r \ll b$). Then:

$$\int \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} = \kappa \pi r^2/\epsilon_0, \tag{1}$$

where κ is the surface charge density of the capacitor. We find directly from Eq. (1) that:

$$\vec{E} = (\kappa/\epsilon_0)\hat{z}. \tag{2}$$

Since there is a current I, the surface charge density changes with time by an amount:

$$\frac{d\kappa}{dt} = \frac{I}{\pi b^2}, \tag{3}$$

where we assume the current is flowing in the $\hat{z}$ direction. So from Eqs. (2) and (3), we find that:

$$\frac{d\vec{E}}{dt} = \frac{I}{\epsilon_0 \pi b^2}\hat{z}. \tag{4}$$

(b)

The Ampere-Maxwell equation, since there is no real (conduction) current J between the plates of the capacitor, reduces to:

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{5}$$

Then using our result from part (a) and integrating (we choose an Amperian loop centered on the z-axis of radius r), we find:

$$\oint \vec{B} \cdot d\ell = 2\pi r B_\phi = \frac{\mu_0 I r^2}{b^2} \tag{6}$$

Thus we find the magnetic field in the $\hat{\phi}$ direction to be:

$$B_\phi = \frac{\mu_0 I r}{2\pi b^2}. \tag{7}$$

(c)

Far from the capacitor, there is no changing electric field and therefore only conduction current, so this is the familiar Ampere's law:

$$\oint \vec{B} \cdot d\ell = \mu_0 I_{encl}, \tag{8}$$

from which we find a magnetic field in the $\hat{\phi}$ direction:

$$B_\phi = \frac{\mu_0 I}{2\pi r}. \tag{9}$$

which you will note is equivalent to Eq. (7) when $r \to b$. Also note that inside the capacitor, the magnetic field grows with r while far from the capacitor the field falls as $1/r$.

(d)

Let's consider the electric field in two different regions. First, we'll consider $\vec{E}$ far from the capacitor in the vicinity of one of the long axial leads (as in part (c)). The changing current produces a changing magnetic field, and from Maxwell's equations we know this creates an electric field:

$$\oint \vec{E} \cdot d\ell = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}. \tag{10}$$

From Eq. (9), we see that $\frac{\partial \vec{B}}{\partial t}$ is given by:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\mu_0}{2\pi r}\frac{\partial I}{\partial t}\hat{\phi}. \tag{11}$$

We choose an Amperian loop as indicated in Fig. 1. There is no electric field perpendicular to the wire (along $\hat{r}$). This can be deduced from symmetry considerations. Suppose there was an electric field in the $\hat{r}$ direction. How does it know whether to point in the $+\hat{r}$ or $-\hat{r}$ direction? That has to be decided by either the direction of the current or the change in current. If we reverse these quantities, the electric field in the $\hat{r}$ direction should reverse. But on the opposite sides of the wire, these quantities have opposite signs! The only way this can be true is if the electric field in the $\hat{r}$ direction is zero.

Furthermore, we know that the electric field must go to zero as $r \to \infty$. But since $\oint \vec{E} \cdot d\ell \neq 0$, it must be the case that we have an electric field in the $\hat{z}$ direction which varies with r. In other words, it is apparent that the electric field is larger closer to the wire (z-axis).

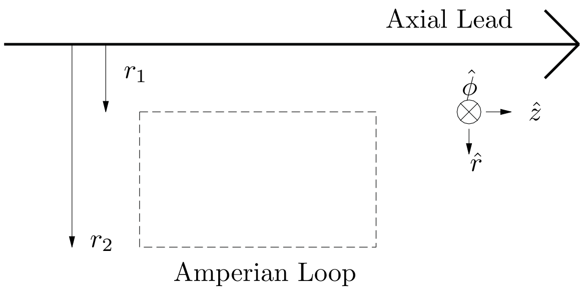

Figure 1

This can be done explicitly, of course, from Eqs. (10) and (11):

$$E(r_2) - E(r_1) = \frac{\mu_0}{2\pi}\frac{\partial I}{\partial t}\ln r_1/r_2. \tag{12}$$

Let's now consider the electric field inside the capacitor, far from the fringe (as in part (b)). Once again we apply Eq. (10), but in this case:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\mu_0 r}{2\pi b^2}\frac{\partial I}{\partial t}\hat{\phi}, \tag{13}$$

We see that there is also a component of the electric field in the $\hat{z}$ direction which varies with r by utilizing similar arguments as those presented above:

$$E(r) = -\frac{\mu_0 r^2}{4\pi b^2}\frac{\partial I}{\partial t}. \tag{14}$$

Problem 2

We can simplify the problem by thinking of C_2 and C_3 as two capacitors in series or in parallel, respectively (Fig. 1). The capacitance C of a parallel plate capacitor is given by:

$$C = \frac{\epsilon A}{d} \tag{15}$$

where A is the area of the plates and d is the plate separation. So for C_1:

$$C_1 = \frac{\epsilon_0 A}{d} \tag{16}$$

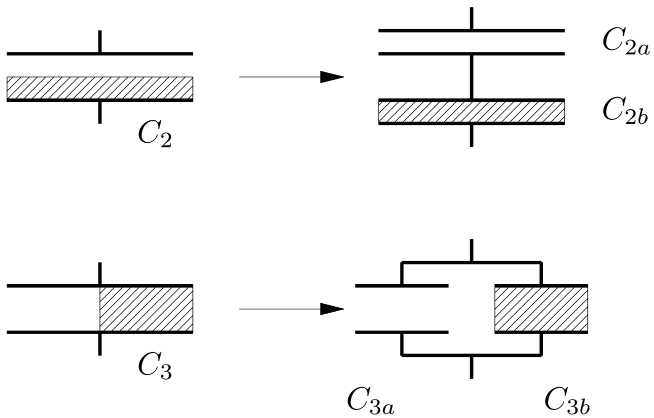

Figure 2

For C_2 we break up the problem into two parts, solving for C_{2a} and C_{2b} (shown in Fig. 1), then determining C_2 using:

$$C_2 = \left(\frac{1}{C_{2a}} + \frac{1}{C_{2b}}\right)^{-1}. \tag{17}$$

From Eq. (15) we can find C_{2a} and C_{2b}, where:

$$C_{2a} = \frac{\epsilon_0 A}{d/2} = 2C_1 \tag{18}$$

and

$$C_{2b} = \frac{\epsilon A}{d/2} = 2\frac{\epsilon}{\epsilon_0}C_1. \tag{19}$$

So with a wee bit of algebra, we find that:

$$C_2 = \frac{2C_1}{\epsilon_0/\epsilon + 1}. \tag{20}$$

Similarly for C_3, we break up the capacitor into two parts C_{3a} and C_{3b}, and then solve for C_3 using:

$$C_3 = C_{3a} + C_{3b}. \tag{21}$$

We use Eq. (15) to solve for C_{3a} and C_{3b}, finding:

$$C_{3a} = \frac{\epsilon_0 A/2}{d} = \frac{1}{2}C_1 \tag{22}$$

and

$$C_{3b} = \frac{\epsilon A/2}{d} = \frac{\epsilon}{2\epsilon_0}C_1. \tag{23}$$

So here the overall capacitance is given by:

$$C_3 = \frac{C_1}{2}(\epsilon/\epsilon_0 + 1). \tag{24}$$

Problem 3

The energy per unit volume U stored in an electromagnetic wave is given by:

$$U = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) = \epsilon E^2. \tag{25}$$

If we then time average the energy, we find that the average energy stored is:

$$\langle U \rangle = \epsilon E_0^2 \int \cos^2(\omega t) dt = \frac{\epsilon E_0^2}{2}. \tag{26}$$

The average power P dissipated per unit volume is given by the relation:

$$P = \frac{J^2}{\sigma}, \tag{27}$$

where the current density is given by Ohm's law:

$$J = \sigma E. \tag{28}$$

Once again taking the time average, we find:

$$\langle P \rangle = \frac{\sigma E_0^2}{2}. \tag{29}$$

The Q-factor is the ratio of these two quantities, $\langle U \rangle$ and $\langle P \rangle$, multiplied by the frequency:

$$Q = \frac{\epsilon\omega}{\sigma}. \tag{30}$$

If we plug in the numbers for seawater, we find that $Q \approx 1.1$. This suggests that decimeter waves cannot propagate very far in seawater, since the energy in the wave falls to $1/e$ its initial value in about one decimeter!

Problem 4

First, we can write down the the electric and magnetic fields of the incident, transmitted and reflected waves:

$$\begin{gathered}
\hat{z}E_i \sin(ky - \omega t) \\
\hat{x}B_i \sin(ky - \omega t) \\
\\
\hat{z}E_0 \sin(k_0 y - \omega t) \\
\hat{x}B_0 \sin(k_0 y - \omega t) \\
\\
\hat{z}E_r \sin(ky + \omega t) \\
\hat{x}B_r \sin(ky + \omega t)
\end{gathered} \tag{31}$$

We note that $k_0 = nk$ since the transmitted wave is in glass. Then we can impose the condition

$$|B| = |\sqrt{\epsilon\mu}E| \tag{32}$$

on each of the waves, and demand that the Poynting vector, $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$, is along the direction of propagation of the waves. This fixes the amplitudes and signs of the magnetic fields with respect to the electric fields:

$$\hat{z}E_i \sin(ky - \omega t)$$
$$\hat{x}\sqrt{\epsilon_0\mu_0}E_i \sin(ky - \omega t)$$

$$\hat{z}E_0 \sin(k_0 y - \omega t)$$
$$\hat{x}\sqrt{\epsilon\mu_0}E_0 \sin(k_0 y - \omega t)$$

$$\hat{z}E_r \sin(ky + \omega t)$$
$$-\hat{x}\sqrt{\epsilon_0\mu_0}E_r \sin(ky + \omega t) \tag{33}$$

Now we consider the fields at $y = 0$, the interface between the block of glass and vacuum. We require that the electric and magnetic fields parallel to the surface of the glass satisfy:

$$E_{\shortparallel} = E'_{\shortparallel}$$
$$\frac{B_{\shortparallel}}{\mu} = \frac{B'_{\shortparallel}}{\mu'}. \tag{34}$$

After substitution, this leaves us with two equations:

$$-E_i + E_r = -E_0$$
$$E_i + E_r = \sqrt{\frac{\epsilon}{\epsilon_0}}E_0. \tag{35}$$

We can then eliminate E_0 from these equations yielding the ratio of E_r to E_i:

$$\frac{E_r}{E_i} = \frac{\sqrt{\epsilon/\epsilon_0} - 1}{\sqrt{\epsilon/\epsilon_0} + 1}. \tag{36}$$

The energy is proportional to E^2 (as can be readily seen by considering the Poynting vector $\vec{S}$), and in this case the index of refraction $n = \sqrt{\epsilon/\epsilon_0}$. Thus the ratio of reflected to incident energy U_r/U_i is given by:

$$\frac{U_r}{U_i} = \left(\frac{E_r}{E_i}\right)^2 = \left(\frac{n-1}{n+1}\right)^2. \tag{37}$$

For $n = 1.6$, 5% of the energy is reflected.

Problem 5

If there were magnetic charges, a magnetic charge density ρ_m and a magnetic current density $\vec{J}_m$ would appear in Maxwell's equations. To avoid confusion, let's denote the traditional electric charge density ρ_e and electric current density $\vec{J}_e$. We can place both of these, with some constants c_1 and c_2 which will be defined later, in Maxwell's equations to make them nice and symmetric:

$$\vec{\nabla} \cdot \vec{E} = \rho_e/\epsilon_0$$
$$\vec{\nabla} \cdot \vec{B} = c_1\rho_m$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + c_2\vec{J}_m$$
$$\vec{\nabla} \times \vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} + \mu_0\vec{J}_e \tag{38}$$

We can go further and work out a relationship between magnetic charge density and current density. We begin by demanding that magnetic charges and currents satisfy the continuity equation, namely:

$$\vec{\nabla} \cdot \vec{J}_m + \frac{\partial \rho_m}{\partial t} = 0. \tag{39}$$

Then if we take the divergence of the new third Maxwell's equation, we get:

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = -\vec{\nabla} \cdot \frac{\partial \vec{B}}{\partial t} + c_2\vec{\nabla} \cdot \vec{J}_m. \tag{40}$$

There is a vector derivative rule that states for any vector field $\vec{A}$, $\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$. So the left-hand side of (40) is 0. The derivatives on the right hand side, $\vec{\nabla}$ and $\frac{\partial}{\partial t}$, can be swapped and we get:

$$-\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{B} + c_2\vec{\nabla} \cdot \vec{J}_m = 0. \tag{41}$$

From the second Maxwell equation we know that $\vec{\nabla} \cdot \vec{B} = c_1\rho_m$, so we find:

$$-c_1\frac{\partial \rho_m}{\partial t} + c_2\vec{\nabla} \cdot \vec{J}_m = 0. \tag{42}$$

If we then apply the continuity equation, Eq. (39), we find that $c_1 = -c_2 \equiv c$.

Thus the final form of Maxwell's equations is:

$$\vec{\nabla}\cdot\vec{E} = \rho_e/\epsilon_0$$
$$\vec{\nabla}\cdot\vec{B} = c\rho_m$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} - c\vec{J}_m$$
$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} + \mu_0\vec{J}_e, \quad (43)$$

where c is a constant of proportionality between the magnetic charge unit and the magnetic field it produces (the equivalent of $1/\epsilon_0$ for electric fields).

Problem 6

The first part of this problem is to calculate the magnetic field $\vec{B}$ inside the magnetized iron. We can use the auxiliary field $\vec{H}$ to make our job a little easier. We know that:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}. \quad (44)$$

Also, we have the relation:

$$\oint \vec{H}\cdot d\vec{\ell} = I_{free}, \quad (45)$$

where in our problem $I_{free} = 0$ everywhere. We choose an Amperian loop as pictured in Fig. 3 ($\vec{M}$ is in the $\hat{z}$ direction), taking advantage of the planar symmetry of the problem (we can assume the iron plate is infinite). Since the component of $\vec{H}$ perpendicular to the surface of the iron plate must be zero based on symmetry, and outside the iron plate $\vec{H}\to 0$ as $y\to\pm\infty$, we conclude that in fact $\vec{H} = 0$ everywhere.

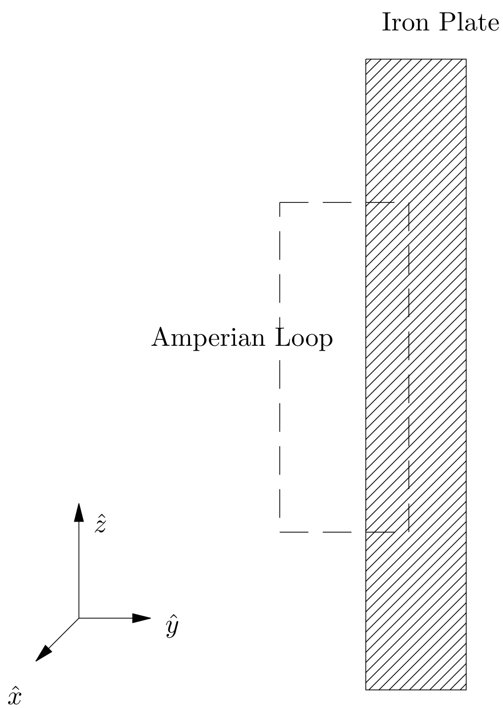

Figure 3

I would like to pause here and point out that this conclusion is not entirely trivial. If there is no free current I_{free}, that does not necessarily mean that $\vec{H} = 0$ everywhere. The fundamental reason for this is that in order to completely determine a vector field you must know both its curl and divergence. Only in cases where we have planar, cylindrical, toroidal or solenoidal symmetry can we conclude that $\vec{\nabla}\cdot\vec{H} = 0$, and get $\vec{H}$ quickly. This is different from Ampere's law with $\vec{B}$ where we always know $\vec{\nabla}\cdot\vec{B} = 0$. So, be careful when using $\vec{H}$!

Anyhow, in this case it's no problem, we find that:

$$\vec{B} = \mu_0\vec{M}. \quad (46)$$

inside the iron plate and zero outside the plate.

This problem now reduces to the traditional problem of solving for the cyclotron orbit of a moving charged particle in a magnetic field. An important difference, as pointed out by Paul Wright in section (thanks!), is that in this case we need to be careful about relativistic corrections to the radius of the cyclotron orbit.

To find the radius of the cyclotron orbit R, we balance the Lorentz force qvB with the relativistic centrifugal force $\gamma mv^2/R$. This tells us:

$$R = \frac{\gamma mv}{qB}. \quad (47)$$

where R is the radius of the circular orbit. If you take a look at Fig. 4, hopefully the simple geometric arguments suggested convince you that in fact:

$$\sin\theta = d/R = \frac{dqBc}{\gamma mvc} = \frac{dqBc}{pc}, \quad (48)$$

where θ is the deflection angle and d is the thickness of the plate. The rest of the problem is working out the correct units...

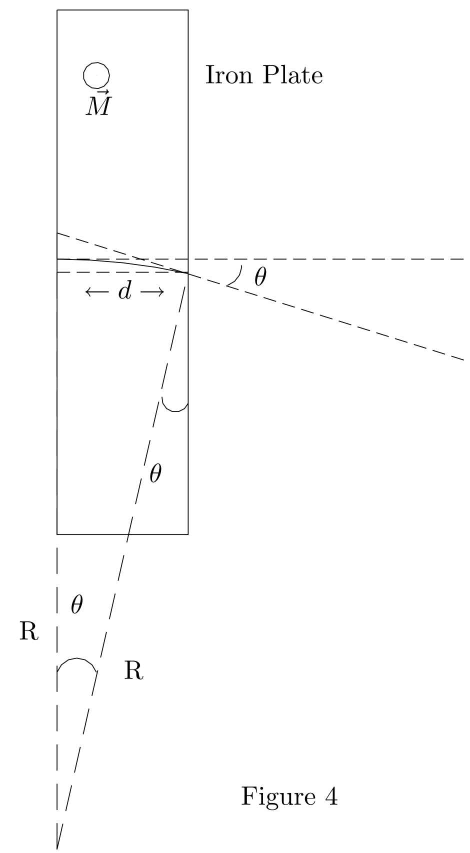

Figure 4

First let's get B in SI units. $B = \mu_0 M = 4\pi\times10^{-7}$ N/A^2 $\cdot 1.5\times10^{29}$ electron magnetic moments per m^3 $\cdot 9\times10^{-24}$ J/T, or about 1.7 T. Then $dqcB = 10^8$ eV, so $\sin\theta \approx \theta = \frac{dqBc}{pc} = 10^{-2}$ rad. That's only about half a degree, so not too big of a deflection...

Problem 7

Fowles 1.4

The 3D wave equation is:

$$\nabla^2 f = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} \tag{49}$$

We employ spherical coordinates, and since our wavefunction is a function only of r, ∇^2 is also a function only of r:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right). \tag{50}$$

Plugging in the spherical harmonic wavefunction $f = \frac{1}{r}e^{\imath(kr-\omega t)}$, we get:

$$\nabla^2 f = -\frac{k^2}{r}e^{\imath(kr-\omega t)} = -k^2 f. \tag{51}$$

If we evaluate the right-hand side of Eq. (49), and use the fact that $k = \omega/u$, we find that:

$$\frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = -k^2 f. \tag{52}$$

which verifies that f is a solution to the 3D wave equation.

Problem 8

Fowles 1.6

(a)

Let's begin by deriving

$$u_g = u - \lambda\frac{\partial u}{\partial \lambda}. \tag{53}$$

We can begin by using Fowles (1.33), the definition of the group velocity:

$$u_g = \frac{d\omega}{dk} = \frac{d\omega}{d\lambda}\cdot\frac{d\lambda}{dk}, \tag{54}$$

where $k = 2\pi/\lambda$ is the wave vector. We can also express ω in terms of u and λ:

$$\omega = ku = \frac{2\pi u}{\lambda}. \tag{55}$$

If you take the derivative of ω with respect to λ:

$$\frac{d\omega}{d\lambda} = -\frac{2\pi u}{\lambda^2} + \frac{2\pi}{\lambda}\frac{du}{d\lambda}. \tag{56}$$

Now we calculate $\frac{d\lambda}{dk}$:

$$\frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi}. \tag{57}$$

If we then substitute the expressions in Eqs. (56) and (57) into Eq. (54), we arrive at our result:

$$u_g = u - \lambda\frac{\partial u}{\partial \lambda}. \tag{58}$$

(b)

We use similar tricks to derive the result:

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}. \tag{59}$$

We begin by noting that

$$\frac{1}{u_g} = \frac{dk}{d\omega} = \frac{dk}{d\lambda_0}\cdot\frac{d\lambda_0}{d\omega} \tag{60}$$

Let's write the wave vector in terms of λ_0 and n:

$$k = \frac{2\pi n}{\lambda_0} \tag{61}$$

We can then take some derivatives, and find that:

$$\frac{dk}{d\lambda_0} = -\frac{2\pi n}{\lambda_0^2} + \frac{2\pi}{\lambda_0}\frac{dn}{d\lambda_0} \tag{62}$$

and

$$\frac{d\lambda_0}{d\omega} = -\frac{\lambda_0^2}{2\pi c}. \tag{63}$$

Substituting these results back into Eq. (60) gives us the answer we were looking for:

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}. \tag{64}$$

That's all folks!