University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

**SOLUTION TO FINAL EXAMINATION**

**Directions.** Do all six problems (weights are indicated). This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Calculators are allowed but not essential – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

**1.** (35 points)
One circularly polarized photon is trapped between two parallel perfectly conducting plates, separated by a distance $L$, which are parallel to the photon's electric and magnetic fields.

**a.** (10 points) If the photon has the smallest definite energy possible under these circumstances, at a point halfway between the plates describe a possible motion of its electric field vector.

**Solution.** The electric field of the light $\vec{E}$ is parallel to the conducting plates, so $\vec{E}$ must vanish at the plates. Therefore we have

$$\vec{E}(0,t) = \vec{E}(L,t) = 0 \ .$$

Of course, $\vec{E}(z,t)$ is a solution to the wave equation:

$$\Big(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\vec{E}(z,t) = 0 \ .$$

In order to satisfy both the wave equation and the boundary conditions, we choose

$$\vec{E}(z,t) = E_0 \sin{(kz)} \exp{(-i\omega t)}\big(\hat{x} \pm i\hat{y}\big) \ ,$$

where

$$k = \frac{n\pi}{L} \quad \text{where } n = 1, 2, ...$$

The smallest definite energy is achieved when $k = \pi/L$ (consequently $\omega = \pi c/L$), in which case we have for the electric field:

$$\vec{E}(z,t) = E_0 \sin{\Big(\frac{\pi z}{L}\Big)} \exp{\Big(-i\frac{\pi ct}{L}\Big)}\big(\hat{x} \pm i\hat{y}\big) \ .$$

Halfway between the plates ($z = \frac{L}{2}$), we find that

$$\vec{E}\Big(\frac{L}{2}, t\Big) = E_0 \exp{\Big(-i\frac{\pi ct}{L}\Big)}\big(\hat{x} \pm i\hat{y}\big) \ .$$

So $\vec{E}$ moves in a circle with angular frequency $\omega = \pi c/L$.

**b.** (10 points) Same for its magnetic field vector.

**Solution.** From Maxwell's equations,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ .$$

Therefore we have

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$
$$\frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} \ .$$

From the above relations, it follows that the spatial dependence of $\vec{B}(z,t)$ is given by $\cos{(\pi z/L)}$. Halfway between the plates $\vec{B} = 0$.

**c.** (15 points) For this plate configuration, making no restriction on the photon energy, evaluate the density of photon states

$$\frac{d^2N}{dL\, d\lambda}$$

where $N$ is the number of states and $\lambda$ is the photon wavelength $\ll L$. Take into account the possible states of circular polarization.

**Solution.** We have from part (**a.**) that $k = n\pi/L$, so in terms of wavelength $\lambda$ (recalling $k = 2\pi/\lambda$):

$$n = \frac{2L}{\lambda} \; .$$

Since we have 2 possible polarization states, the total number of states $N$ as a function of wavelength is

$$N = \frac{4L}{\lambda} \; .$$

The derivative with respect to wavelength is

$$\frac{dN}{d|\lambda|} = \frac{4L}{\lambda^2}$$

and so we obtain for the density of photon states

$$\frac{d^2N}{d|\lambda|dL} = \frac{4}{\lambda^2} \; .$$

**2.** (35 points)
A linearly ($\hat{\mathbf{x}}$) polarized plane EM wave travelling along $\hat{\mathbf{z}}$ is incident on an opaque baffle located in the plane $z = 0$. The baffle has two slits cut in it, which are of infinite extent in the $\hat{\mathbf{y}}$ direction. In the $\hat{\mathbf{x}}$ direction, the slit widths are each $a$ and their center-to-center distance is $d$. (Obviously $d > a$, but you may *not* assume that $d \gg a$.) The top and bottom slits are each an equal distance from $x = 0$.

The diffracted image is viewed on a screen located in the plane $z = L$, where $L \gg d$; also $\lambda L \gg d^2$, where $\lambda$ is the EM wavelength.

Quarter-wave plates are placed in each slit. They are identical, except that the top plate's "slow" (high-index) axis is along $(\hat{\mathbf{x}}+\hat{\mathbf{y}})/\sqrt{2}$ ($+45°$ with respect to the $\hat{\mathbf{x}}$ axis), while the bottom plate's slow axis is along $(\hat{\mathbf{x}}-\hat{\mathbf{y}})/\sqrt{2}$ ($-45°$ with respect to the $\hat{\mathbf{x}}$ axis).

**a.** (15 points) What is the state of polarization of the diffracted light that hits the center of the screen, at $x = y = 0$? Explain.

**Solution.** Light that exits the top slit (slow axis at $+45°$) is in a state

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \; ,$$

or left-hand circular polarization; light that exits the bottom slit (slow axis at $-45°$) is in a state of polarization

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \; ,$$

or right-hand circular polarization. At the center of the screen, light from each slit contributes equally; the state of polarization is proportional to

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ;$$

it is $\hat{\mathbf{x}}$ polarized like the incident beam. (See Fowles page 34 and Table 2.1 for the Jones vectors and matrices.)

**b.** (20 points) At what diffracted angle $\theta_x$ does the first minimum of the irradiance occur?

**Solution.** Right- and left-hand polarized states are orthogonal; they do not interfere. To see this formally (though this is not required as part of the solution), consult Fowles Eq. 3.11; the interference term there is proportional to

$$\mathbf{E}_2^* \cdot \mathbf{E}_1 \propto \begin{pmatrix} 1 & -i^* \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 1 + i^2 = 0 \; .$$

Since there is no interference between the light from the top and bottom slit, the resulting irradiance is just twice that expected from a single slit of width $a$. According to Fowles Eq. 5.18, this pattern is proportional to

$$\left(\frac{\sin\beta}{\beta}\right)^2 \; ,$$

where in this problem's notation

$$\beta = \frac{1}{2}ka\sin\theta_x$$

and $k = 2\pi/\lambda$. The first minimum occurs at $\beta = \pi$, or

$$\sin\theta_x = \frac{\lambda}{a} \; .$$

**3.** (35 points)
A lens has an $f$-number (ratio of focal length to diameter) equal to $F$. The lens is used to concentrate sunlight on a ball whose diameter is equal to the diameter of the sun's image. The ball is convectively and conductively insulated, but it freely radiates energy outward so that its temperature can approach an equilibrium value $T_b$.

**a.** (10 points) The sun subtends a half-angle of $\approx 0.005$ radians. Is the size of its image "diffraction-limited", *i.e.* determined largely by the effects of diffraction? Make an order-of-magnitude argument assuming that the lens is a typical camera lens, with a radius of order $10^{-2}$ m.

**Solution.** The image of a distant point source formed at the focal plane of a lens is actually a Fraunhofer diffraction pattern where the aperture is the lens opening. The image becomes diffraction limited when the size of the image is near the size of an Airy disk. From this condition, we have the *Rayleigh criterion*:

$$2\theta > \frac{1.22\lambda}{D} ,$$

where $\theta$ is the half-angle subtended by the sun ($\approx 0.005$ rad), $\lambda$ is the wavelength of the light, and $D$ is the diameter of the lens. We can assume $\lambda \approx 600$ nm for the sun, $D = 2 \times 10^{-2}$ m, so the image is not diffraction limited if

$$\theta > 2.5 \times 10^{-5} \text{ rad} ,$$

which is clearly satisfied in this problem. Thus the image is not diffraction limited.

(Note that this question requires only an order-of-magnitude analysis. So you don't need to know anything about the details of Rayleigh's criterion, or Airy disks, to get full credit; all that you need to say is that the diffraction angle is of order $\lambda/D$, which here is much smaller than the sun's angular width.)

**b.** (25 points) Assuming the sun to be a blackbody of temperature $T$, calculate the ball's temperature $T_b$. Neglect reflection by the lens. (*Hint:* your answer should depend only on $T$ and $F$.)

**Solution.** The sun radiates total power

$$P_S = \sigma T_S^4 \cdot 4\pi R_S^2 ,$$

where $T_S$ is the sun's surface temperature and $R_S$ is the sun's radius. The lens collects a fraction of this light power given by

$$\frac{\Delta\Omega}{\Omega} = \frac{\pi(D/2)^2}{4\pi R_{ES}^2} ,$$

where $R_{ES}$ is the distance from the earth to the sun. The entirety of this light is focused on the ball. The ball re-radiates power

$$P_b = \sigma T_b^4 \cdot 4\pi R_b^2 .$$

In equilibrium we have the light power absorbed by the ball equal to the light power radiated by the ball. Setting the two equal, we obtain

$$T_S^4 \frac{D^2}{4} \frac{R_S^2}{R_{ES}^2} = T_b^2 r_b^2 .$$

To relate this result to $F$, we note that $2r_b/f = 1/F$ where $f$ is the focal length. Also we have $r_b/f \approx R_S/R_{ES} \approx \theta$, where $\theta$ is the half-angle subtended by the sun. Employing the above relations in the equation relating the sun's temperature to the ball's temperature:

$$T_b^4 = T_S^4 \frac{d^2\theta^2}{4r_b^2} = \frac{1}{16F^2} T_S^4 .$$

So we find that

$$T_b = \frac{T_S}{2}\sqrt{\frac{1}{F}} .$$

**4.** (30 points)
You are given a Hamiltonian

$$\mathcal{H} = \frac{1}{2}(LR + RL) ,$$

where $R$ and $L$ are two operators such that

$$[L, R] = E_0$$

with $E_0$ a constant. You are also given an eigenfunction $u_E(x)$ of $\mathcal{H}$, such that

$$\mathcal{H}u_E = Eu_E \ ,$$

where $E$, another constant, is the energy eigenvalue.

Prove that

$$\mathcal{H}(Ru_E) = (E + E_0)(Ru_E) \ ,$$

*i.e.* $R$ is a *raising operator*.

**Solution.**

$$\begin{aligned}
H(Ru) &= \frac{1}{2}\big(LRR + RLR\big)u \\
&= \frac{1}{2}\big(LRR - RLR + 2RLR\big)u \\
&= \frac{1}{2}\big([L, R]R + 2RLR\big)u \\
&= \frac{1}{2}\big(E_0R + 2RLR\big)u \\
&= \frac{1}{2}\big(E_0R + RLR - RRL + RRL + RLR\big)u \\
&= \frac{1}{2}\big(E_0R + R[L, R] + RRL + RLR\big)u \\
&= \frac{1}{2}\big(2E_0R + R(RL + LR)\big)u \\
&= \frac{1}{2}\big(2E_0R + R(2H)\big)u \\
&= (E + E_0)Ru \ .
\end{aligned}$$

**5.** (35 points).
Consider a harmonic oscillator potential

$$V(x) = \frac{1}{2}m\omega_0^2x^2$$

in one dimension. An *even* number $N$ of particles of mass $m$ are placed in this potential. There are no special interactions between the particles – no significant mutual electrostatic repulsion, gravitational attraction, *etc.*, compared to the strength of their interaction with the harmonic potential itself.

You may use what you already know about the levels of a harmonic oscillator.

The system is in its ground state, *i.e.* $T = 0$ Kelvin.

Calculate the total energy $E$ of the $N$-particle system, relative to the bottom of the well, for the cases

**a.** (10 points) The $N$ particles are *distinguishable*.

**Solution.** The energy levels of a harmonic oscillator are

$$E_n = \Big(n + \frac{1}{2}\Big)\hbar\omega_0 \ ,$$

where $n = 0, 1, 2 \dots$. Nothing prevents mutually noninteracting distinguishable particles from occupying the same spatial wavefunction. At $T = 0$ this will be the ground state $n = 0$. Then

$$E = \frac{N}{2}\hbar\omega_0 \ .$$

**b.** (10 points) The $N$ particles are identical *bosons*.

**Solution.** Same as (**a.**). All the identical bosons are in the ground state at $T = 0$.

**c.** (15 points) The $N$ particles are identical spin $\frac{1}{2}$ *fermions*.

**Solution.** Each spatial wavefunction can accommodate two identical spin $\frac{1}{2}$ fermions, one with spin up, one with spin down. At $T = 0$ the lowest occupied state is the ground state, with energy

$$E_0 = \frac{1}{2}\hbar\omega_0 \ ,$$

while the highest-energy occupied state has (Fermi) energy equal to

$$E_F = E_0 + \Big(\frac{N}{2} - 1\Big)\hbar\omega_0 \ .$$

The total energy is $N$ times the average energy, which is the mean of $E_0$ and $E_F$:

$$\begin{aligned}
E &= N\frac{E_0 + E_F}{2} \\
&= \frac{N}{2}\Big(2E_0 + \Big(\frac{N}{2} - 1\Big)\hbar\omega_0\Big) \\
&= \frac{N^2}{4}\hbar\omega_0 \ .
\end{aligned}$$

**6.** (30 points)
In the rest frame $\mathcal{S}'$ of a star, ignoring the gravitational redshift, some of the photons emitted by the star arise from a particular atomic transition with an unshifted wavelength $\lambda'$. When these photons are observed on earth, they are shifted to longer wavelength $\lambda = \lambda' + \Delta\lambda$ because the star is receding from the earth with velocity $\beta_0 c$ due to the Hubble expansion of the universe. Astronomers measure this redshift by means of the parameter $z$, defined by

$$z \equiv \frac{\Delta\lambda}{\lambda'} .$$

For light, the relativistic Doppler shift is

$$\omega = \frac{\omega'}{\gamma_0(1-\beta_0\cos\theta)} .$$

**a.** (10 points) In the "neighboring star" limit $\beta_0 \ll 1$, show that $\beta_0$ is approximately equal to the measured $z$.

**Solution.**

$$\begin{aligned}
\omega &= \frac{\omega'}{\gamma_0(1-\beta_0\cos\theta)} \\
&= \frac{2\pi c}{\lambda} \\
1/\lambda &= \frac{1/\lambda'}{\gamma_0(1-\beta_0\cos\theta)} \\
\frac{\lambda}{\lambda'} &= \gamma_0(1-\beta_0\cos\theta) \\
\frac{\Delta\lambda}{\lambda'} &= \gamma_0(1-\beta_0\cos\theta) - 1 \\
z &= \gamma_0(1-\beta_0\cos\theta) - 1 .
\end{aligned}$$

For a receding star, $\cos\theta = -1$, and so

$$z = \gamma_0(1+\beta_0) - 1 .$$

When $\beta_0 \ll 1$, $\gamma_0 \approx 1$ to *second* order in $\beta_0$. Then

$$z \approx 1 + \beta_0 - 1 = \beta_0 .$$

**b.** (10 points) In the "distant star" limit $\gamma_0 \gg 1$, derive an expression for $\gamma_0$ in terms of the measured $z$.

**Solution.** When $\gamma_0 \gg 1$, $\beta_0 \approx 1$. Then, from the solution to (**a.**),

$$\begin{aligned}
z &\approx 2\gamma_0 - 1 \\
\gamma_0 &\approx \frac{1+z}{2} .
\end{aligned}$$

Full credit is given with or without the "1" term.

**c.** (10 points) The observation of Supernova 1987A marked the dawn of a new astronomy, in which humans are able to detect fermions (neutrinos) as well as bosons (photons) from (spatially or temporally) resolved sources outside the solar system. About a dozen such neutrinos were detected in each of two huge underground water tanks. The photons from Supernova 1987A were redshifted by

$$z \approx 10^{-5} .$$

Taking the Hubble constant to be

$$H_0 \approx 0.7 \times 10^{-10}\text{yr}^{-1} ,$$

for how many years did the neutrinos from SN1987A travel before humans observed them?

**Solution.** From Rohlf Eq. (19.17) (necessary for solving assigned problem 19.18), the velocity $v$ with which SN1987A is receding from Earth is

$$v = H_0 d ,$$

where $d$ is its present distance from Earth. Neutrinos are nearly massless and travel essentially at the speed of light $c$. Using the result of part (**a.**), the travel time $T$ of the neutrinos from SN1987A was

$$\begin{aligned}
T &= \frac{d}{c} \\
&= \frac{v}{H_0 c} \\
&= \frac{\beta_0}{H_0} \\
&\approx \frac{z}{H_0} \\
&\approx \frac{1\times 10^{-5}}{0.7\times 10^{-10}\ \text{yr}^{-1}} \\
&\approx 1.4\times 10^{5}\ \text{yr} .
\end{aligned}$$