University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

SOLUTION TO EXAMINATION 1

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (58 points)
In a free-electron laser, a beam of relativistic electrons is subjected to a transverse magnetic field that varies sinusoidally with lab coordinate z, the (average) beam direction:

$$\mathbf{B} = \hat{\mathbf{x}} B_0 \cos\frac{2\pi z}{\lambda_0}$$

where B_0 and λ_0 are constants. In the lab, the z component of the electrons' velocity is

$$v_z = \beta_0 c$$

where β_0 is a constant.

a. (8 points) Consider a Lorentz frame $\mathcal{S}'$ moving with velocity

$$\boldsymbol{\beta}_0 c = \hat{\mathbf{z}}\beta_0 c$$

with respect to the lab. The Lorentz transformation for electromagnetic fields is

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma_0(\mathbf{E}_{\perp} + \boldsymbol{\beta}_0 \times c\mathbf{B}_{\perp}) \\ c\mathbf{B}'_{\perp} &= \gamma_0(c\mathbf{B}_{\perp} - \boldsymbol{\beta}_0 \times \mathbf{E}_{\perp})\ , \end{aligned}$$

where $\gamma_0 \equiv (1-\beta_0^2)^{-1/2}$. Calculate the electric field $\mathbf{E}'$ seen in $\mathcal{S}'$; continue to express it in terms of $2\pi z/\lambda_0$.

Solution. There are no electric or parallel magnetic fields in the lab frame so the third equation gives us the total electric field seen in $\mathcal{S}'$:

$$\begin{aligned} \mathbf{E}' &= \boldsymbol{\beta}_0 \times c\mathbf{B}_{\perp} \\ &= \gamma_0\beta_0 c B_0 \cos\Big(\frac{2\pi z}{\lambda_0}\Big)(\hat{z} \times \hat{x}) \\ &= \gamma_0\beta_0 c B_0 \cos\Big(\frac{2\pi z}{\lambda_0}\Big)\hat{y}\ . \end{aligned}$$

b. (8 points) Defining

$$2\pi z/\lambda_0 \equiv \omega'_0 t'\ ,$$

where t' is the time as observed at the origin of $\mathcal{S}'$, compute ω'_0 in terms of the constants previously given.

Solution. We need a Lorentz transformation to relate (ct, z) to (ct', z'). Taking advantage of the fact that $z' = 0$, we minimize algebra by choosing the inverse transformation:

$$\begin{aligned} z &= \gamma_0 z' + \gamma_0\beta_0 ct' \\ &= 0 + \gamma_0\beta_0 ct' \\ &= \gamma_0\beta_0 ct' \\ \frac{2\pi}{\lambda_0} z &= \frac{2\pi}{\lambda_0}\gamma_0\beta_0 ct' \\ &\equiv \omega'_0 t' \\ \omega'_0 &= \frac{2\pi\gamma_0\beta_0 c}{\lambda_0}\ . \end{aligned}$$

c. (8 points) Consider an electron of charge $-e$ and mass m whose *average* position is

$$\langle x', y', z' \rangle = (0, 0, 0)$$

as observed in $\mathcal{S}'$. In this frame, its velocity is so small that you may ignore $\mathbf{v}' \times \mathbf{B}'$

with respect to $\mathbf{E}'$. In frame $\mathcal{S}'$, making this approximation, compute the electron's motion $y'(t')$. (In case you didn't get part **b.** exactly right, leave your answer in terms of ω_0'.)

Solution. The Lorentz force on the electron in $\mathcal{S}'$ is given by:

$$\mathbf{F}' = -e\mathbf{E}' - e\mathbf{v}' \times \mathbf{B}' \ ,$$

but we can ignore the $\mathbf{v}' \times \mathbf{B}'$ term since $\mathbf{v}'$ is always small. So then we have an equation for the acceleration:

$$m_e \mathbf{a}' = -e\mathbf{E}'.$$

Plugging in $\mathbf{E}'$ from (**a.**) and integrating twice with respect to t',

$$y'(t') = \frac{e}{m}\gamma_0\beta_0 c B_0 \frac{\cos\omega_0' t'}{(\omega_0')^2} \ .$$

d. (8 points) The electric dipole moment $\mathbf{p}$ of a distribution of N point charges q_i at positions $\mathbf{r_i}$ is defined as

$$\mathbf{p} = \sum_{i=1}^{N} \mathbf{r}_i q_i \ .$$

The power $P(t)$ radiated by a charge distribution with time-varying dipole moment $\mathbf{p}(t)$ is

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{(d^2\mathbf{p}/dt^2)^2}{c^3} \ .$$

As seen in $\mathcal{S}'$, calculate $\langle P' \rangle$, the *time-averaged* power radiated by a single electron in the free-electron laser.

Solution. The second derivative of the time-varying electric dipole moment for a single electron in the free electron laser is given by:

$$\frac{d^2\mathbf{p}'}{dt^2} = -e\mathbf{a}' \ ,$$

where $\mathbf{a}'$ is the acceleration found in (**c.**):

$$\mathbf{a}' = -\frac{e}{m}\gamma_0\beta_0 c B_0 \cos\omega_0' t'.$$

Plugging this result into the formula given above for radiated power P',

$$P' = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^4\gamma_0^2\beta_0^2 c^2 B_0^2}{m^2 c^3}\cos^2\omega_0' t' \ .$$

Since $\langle \cos^2\omega_0' t' \rangle = 1/2$, the time average of P' is given by:

$$\langle P' \rangle = \frac{1}{4\pi\epsilon_0}\frac{1}{3}\frac{e^4\gamma_0^2\beta_0^2 B_0^2}{m^2 c} \ .$$

e. (8 points) Energy and time both transform as the 0^{th} component of a four-vector. Calculate $\langle P \rangle$, the time-averaged power radiated by a single electron as observed in the lab. You may leave your answer in terms of $\langle P' \rangle$.

Solution. Since both energy and time transform as the 0^{th} component of a four-vector, if we have measured a change in energy of the electron $\Delta E'$ and a change in time $\Delta t'$ in the electron's rest frame $\mathcal{S}'$, then the same quantities in the lab frame are given by $\Delta E = \gamma_0 \Delta E'$ and $\Delta t = \gamma_0 \Delta t'$. So

$$\begin{aligned}\langle P' \rangle &= \Delta E'/\Delta t' \\ &= \Delta E/\Delta t \\ &= \langle P \rangle \ .\end{aligned}$$

This was also solved in problem set 3, problem 3!

f. (10 points) For light, the relativistic Doppler shift is

$$\omega = \frac{\omega'}{\gamma_0(1 - \beta_0\cos\theta)} \ .$$

Calculate the ratio λ_0/λ, where λ_0, as before, is the characteristic length describing the spatial variation in the lab of the free-electron laser's magnetic field, and λ is the wavelength of the light that its electrons radiate in the forward direction, as observed in the lab. Express this ratio in terms of γ_0, in the limit $\beta_0 \to 1$.

Solution. In $\mathcal{S}'$, we know from (**c.**) that the electron is oscillating with angular frequency ω_0'.

Then, in $\mathcal{S}'$, the EM radiation produced by that electron has the same angular frequency. A forward observer in $\mathcal{S}$, upon whom the beam impinges with $\theta = 0$, sees this radiation with a Doppler shifted angular frequency ω_0 that, by the above Doppler formula, is equal to

$$\omega_0 = \frac{\omega_0'}{\gamma_0(1-\beta_0)} \ .$$

Substituting $\omega_0 = 2\pi c/\lambda$, and plugging in the value of ω_0' from (**b.**),

$$\begin{aligned}
\frac{2\pi c}{\lambda} &= \frac{2\pi\gamma_0\beta_0 c}{\lambda_0}\frac{1}{\gamma_0(1-\beta_0)} \\
\frac{\lambda_0}{\lambda} &= \frac{\beta_0}{1-\beta_0} \\
&= \frac{1+\beta_0}{1+\beta_0}\frac{\beta_0}{1-\beta_0} \\
&= \gamma_0^2\beta_0(1+\beta_0) \ .
\end{aligned}$$

This reduces to $2\gamma_0^2$ in the relativistic limit, a famous (and simple) result. If, say, λ_0 is 0.1 m and the electron energy is 500 MeV ($\gamma^2 \approx 10^6$), the FEL or wiggler can be made to radiate in the far UV ($\lambda \approx 0.05\ \mu$m), where no conventional laser is available.

g. (8 points) What is the state of polarization of the free-electron laser's light? Explain.

Solution. From (**c.**) we know that the electron oscillates in the $\hat{y}$ direction. The on-axis radiation from the dipole is propagating in the $\hat{z}$ direction, so light is polarized orthogonal to $\hat{z}$. From the formula derived in class describing dipole radiation, we know the radiation is polarized in the $\hat{\theta}$ direction where the vertical axis is defined by the direction the dipole oscillates in. Hence we can conclude that the light is linearly polarized in the $\hat{y}$ direction.

2. (42 points)
Semi-infinite regions $y > L$ and $y < -L$ are filled by perfect conductor, while the intervening slab $-L < y < L$ is filled by dielectric with constant dielectric constant ϵ and permeability μ.

a. (8 points) In SI units, write Maxwell's equations for $\mathbf{E}$ and $\mathbf{H}$ inside the dielectric. Do not write any terms involving free charges or free currents, which both vanish there.

Solution. Within the dielectric, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, where ϵ and μ are constants. The source-free equations are

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= 0 \\
\mathbf{B} &= \mu\mathbf{H} \\
\Rightarrow \nabla\cdot\mathbf{H} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\
\mathbf{B} &= \mu\mathbf{H} \\
\Rightarrow \nabla\times\mathbf{E} &= -\mu\frac{\partial\mathbf{H}}{\partial t}
\end{aligned}$$

The useful source-dependent equations are the variety that depend on free rather than total charges and currents, because free charges and currents are zero in the dielectric. These are

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho_{\text{free}} = 0 \\
\mathbf{D} &= \epsilon\mathbf{E} \\
\Rightarrow \nabla\cdot\mathbf{E} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\nabla\times\mathbf{H} &= \mathbf{J}_{\text{free}} + \frac{\partial\mathbf{D}}{\partial t} = \frac{\partial\mathbf{D}}{\partial t} \\
\mathbf{D} &= \epsilon\mathbf{E} \\
\Rightarrow \nabla\times\mathbf{H} &= \epsilon\frac{\partial\mathbf{E}}{\partial t}
\end{aligned}$$

b. (8 points) Prove that E_x and E_z both must vanish at $y = \pm L$.

Solution. Electric fields vanish in perfect conductors, because the infinitely mobile free charges instantaneously rearrange themselves to shield out any externally applied electric field. Consider a rectangular loop with long side S and short side s. One long side lies in the conductor, parallel to the plane $y = L$; the other long side lies in the dielectric. In the limit $s \to 0$, the right-hand side of Faraday's law,

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\oint \mathbf{B}\cdot d\mathbf{a} \ ,$$

vanishes because the area vanishes, and the contributions of the short parts to the rectangular loop on the left-hand side also vanish. The only nonvanishing contribution to the left-hand side is $\mathbf{E}_{\|} \cdot \mathbf{S}$ in the dielectric. This proves that $\mathbf{E}_{\|}$ in the dielectric must vanish at $|y| = L$. This includes $\mathbf{E}_{\|}$ in either the $\hat{x}$ or $\hat{z}$ directions, which both are parallel to the interface.

For parts **c.** and **d.** only, assume, for $-L < y < L$, that the fields are given by

$$\mathbf{E}_{\text{physical}} = \Re\big(\hat{\mathbf{y}} E_2 \exp\big(i(kz - \omega t)\big)\big)$$
$$\mathbf{H}_{\text{physical}} = \Re\big((\hat{\mathbf{x}} H_1 + \hat{\mathbf{z}} H_3) \exp\big(i(kz - \omega t)\big)\big) \ ,$$

where E_2, H_1, and H_3 are unknown complex constants, and k and ω are unknown real constants.

c. (8 points) Prove that $H_3 = 0$.

Solution. As usual we require the complex electromagnetic fields to satisfy Maxwell's equations (not just their (physical) real part). When their dependence on $\mathbf{r}$ and t is of the form $\exp\big(i(\mathbf{k} \cdot \mathbf{r} - \omega t)\big)$, the operators $\nabla\cdot$ and $\nabla\times$ reduce to $i\mathbf{k}\cdot$ and $i\mathbf{k}\times$, respectively, while the operator $\partial/\partial t$ reduces to $-i\omega$. Using the first Maxwell equation in (**a.**),

$$\begin{aligned} \nabla \cdot \mathbf{H} &= 0 \\ i\mathbf{k} \cdot (\hat{x} H_1 + \hat{z} H_3) &= 0 \\ \mathbf{k} &= \hat{z} k \\ \Rightarrow ikH_3 &= 0 \ . \end{aligned}$$

d. (10 points) Calculate the ratio H_1/E_2 in terms of known quantities.

Solution. Using the methods of (**c.**), the second Maxwell equation in (**a.**) requires

$$\begin{aligned} ik(\hat{z} \times \hat{y}) E_2 &= +i\omega\mu \hat{x} H_1 \\ -ik\hat{x} E_2 &= +i\omega\mu \hat{x} H_1 \\ -\frac{k}{\mu\omega} &= \frac{H_1}{E_2} \end{aligned}$$

while the fourth Maxwell equation in (**a.**) requires

$$\begin{aligned} ik(\hat{z} \times \hat{x}) H_1 &= -i\omega\epsilon \hat{y} E_2 \\ ik\hat{y} H_1 &= -i\omega\epsilon \hat{y} E_2 \\ \frac{H_1}{E_2} &= -\frac{\epsilon\omega}{k} \ . \end{aligned}$$

Setting the two values of H_1/E_2 equal,

$$\begin{aligned} -\frac{k}{\mu\omega} &= -\frac{\epsilon\omega}{k} \\ \frac{k}{\omega} &= \sqrt{\epsilon\mu} \ . \end{aligned}$$

Plugging this value for k/ω into either of the equations for H_1/E_2,

$$\frac{H_1}{E_2} = -\sqrt{\frac{\epsilon}{\mu}} \ .$$

e. (8 points) Can a linear combination of a right-hand and a left-hand circularly polarized plane wave in the region $-L < y < L$ propagate in the z direction? If not, why not? If so, what combination(s) would be possible? Explain fully.

Solution. From (**b.**) we know that $E_x = 0$ at $|y| = L$. In this part (only!) we are asked to assume that a plane wave is propagating in the $\hat{z}$ direction within the dielectric. This means that $\mathbf{E}$ must be $\perp$ to $\hat{z}$ and it cannot depend on x and y. So, if E_x vanishes at the boundaries $|y| = L$, it must vanish throughout the dielectric. Thus the only nonzero component of $\mathbf{E}$ lies in the $\hat{y}$ direction. We then ask, what combination of RH polarization ($\propto \hat{x} - i\hat{y}$) and LH polarization ($\propto \hat{x} + i\hat{y}$) add to pure $\hat{y}$ polarization? Evidently, the RH and LH waves must have equal amplitude and opposite sign.