Welcome to the archival Web page for U.C. Berkeley's Physics H7ABC, Honors Physics for Scientists and Engineers, Fall 1998, Spring 1999, and Fall 1999.

Instructor:
(Prof.) Mark Strovink. I have a research web page, a standardized U.C. Berkeley web page, and a statement of research interests.

Physics H7A (*Mechanics and Vibrations*)
Problem set solutions initially composed by E.A. ("Ted") Baltz
Graduate Student Instructors: David Bacon and Elizabeth Wu

Physics H7B (*Electromagnetism and Thermal Physics*):
Most problem set solutions composed by Peter Pebler
Graduate Student Instructor: Robin Blume-Kohout

Physics H7C (*Physical Optics and Modern Physics*):
Problem set solutions composed by Graduate Student Instructor: Derek Kimball

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Most documents linked on this page are in PDF format. They are typeset except where indicated. The documents are intended to be displayed by Adobe Acrobat [Reader], version 4 or later (version 3 may also work). (You may optimize Acrobat's rendering of equations by unchecking "Use Greek Text Below:" on File-Preferences-General.)

You may right-click to download a single .pdf file (19.5 MB) that includes every displayable image on this Web page.

Physics H7A (*Mechanics and Vibrations*)
Texts: Kleppner/Kolenkow, *An Introduction to Mechanics*; French, *Vibrations and Waves*.

General Information including schedules and rooms.
Course Outline including all reading assignments.
Questionnaire that was filled out by the students.
Typos in Kleppner & Kolenkow

Practice Exam 1	Solution to Practice Exam 1
Exam 1	Solution to Exam 1
Exam 2	Solution to Exam 2
Final Exam	Solution to Final Exam

You may right-click to download a single .tar.gz archive (1.0 MB) that includes the source files required to build every H7A file linked above.

Physics H7B (*Electromagnetism and Thermal Physics*)
Text: Purcell, *Electricity and Magnetism*.

General Information including schedules and rooms.
Course Outline including all reading assignments.
Special Relativity Notes* also used in H7C.

Problem Set 1	Solution Set 1
Problem Set 2	Solution Set 2
Problem Set 3	Solution Set 3
Problem Set 4	Solution Set 4
Problem Set 5	Solution Set 5
Problem Set 6	Solution Set 6
Problem Set 7	Solution Set 7
Problem Set 8	Solution Set 8
Problem Set 9	Solution Set 9
Problem Set 10	Solution Set 10
Problem Set 11	Solution Set 11
Problem Set 12	Solution Set 12
Problem Set 13	Solution Set 13

Midterm	Solution to Midterm
Final Exam	Solution to Final Exam*

* handwritten
You may right-click to download a single .tar.gz archive (20 MB) containing the source files required to build every typeset H7B file linked above.

Physics H7C (*Physical Optics and Modern Physics*)
Texts: Fowles, *Introduction to Modern Optics*; Rohlf, *Modern Physics from alpha to Z0*.

General Information including schedules and rooms.
Course Outline including all reading assignments.
Note on H7C texts, required and supplementary.
Errata for Fowles, *Introduction to Modern Optics*, second edition.
Special Relativity Notes* also used in H7B.

Problem Set 1	Solution Set 1
Problem Set 2	Solution Set 2
Problem Set 3	Solution Set 3
Problem Set 4	Solution Set 4
Problem Set 5	Solution Set 5
Problem Set 6	Solution Set 6
Problem Set 7	Solution Set 7
Problem Set 8	Solution Set 8
Problem Set 9	Solution Set 9
Problem Set 10	Solution Set 10
Problem Set 11	Solution Set 11
Problem Set 12	Solution Set 12

Midterm 1	Solution to Midterm 1
Midterm 2	Solution to Midterm 2
Final Exam	Solution to Final Exam

* handwritten

You may right-click to download a single .tar.gz archive (0.5 MB) containing the source files required to build every typeset H7C file linked above.

University of California, Berkeley
Physics H7A, Fall 1998 (Strovink)

General Information (1 Sep 98)

Instructors: Prof. Mark Strovink, 437 LeConte; (LBL) 486-7087; (home, before 10) 486-8079; (UC) 642-9685. Email: `strovink@lbl.gov`. Web: `http://d0lbln.lbl.gov/`. Office hours: M 3:15-4:15, Tu 10-11, Th 10-11. Mr. David Bacon, 214 LeConte; (UC) 642-5430; (home, before 1 AM) 666-9867. Email: `dabacon@wco.com`. Office hours: W 12-2. Ms. Elizabeth Wu, 214 LeConte; (UC) 642-5430. Email: `ecmwu@euler.berkeley.edu`. Office hours in 262 LeConte: M 10-11. You may also get help in the 7A Course Center, 262 LeConte.

Lectures: Tu-Th 11:10-12:30, 2 LeConte. Lecture attendance is essential, since not all of the course content can be found in either of the course texts.

Labs: In the second week, in **2**70 LeConte, please enroll in one of *only 3* special H7A lab sections [(A) #134, M 4-6; (B) #241, Th 4-6; or (C) #312, F 8-10]. Section 134 is taught by Ms. Wu and Sections 241 and 312 are taught by Mr. Bacon. If you can make more than one of these lab (and section, see below) slots, please attempt to enroll in the earliest of these lab slots. Depending on crowding, you may be asked to move to a later lab. During "off" weeks not requiring lab apparatus, your lab section will still meet, in (#134) 336 LeConte, (#241) 395 LeConte, or (#312) 335 LeConte.

Discussion Sections: Beginning in the second week, please enroll in *the only one* of the 1 hr H7A discussion sections corresponding to your H7A lab section: (A) #134, W 1-2, 343 LeConte; (B) #241, Tu 4-5, 395 LeConte; (C) #312, W 8-9, 385 LeConte. You are especially encouraged to attend discussion section regularly. There you will learn techniques of problem solving, with particular application to the assigned exercises.

Texts (both required): Kleppner/Kolenkow, *An Introduction to Mechanics*, (McGraw-Hill, 1973). A.P. French, *Vibrations and Waves*, Paper Edition (Norton, 1971).

Problem Sets: Twelve problem sets are assigned and graded, with solutions provided as part of the Syllabus. They are due on Wednesday at 5 PM on weeks (including Thanksgiving) in which there is no exam, beginning in week 2. Deposit problem sets in the box labeled "H7A" outside 201 LeConte. You are encouraged to attempt all the problems. Students who do not do them find it almost impossible to learn the material and to succeed on the examinations. Work independently. Credit for collective effort, which is easy to identify, will be divided among the collectivists. Late papers will not be graded. Your lowest problem set score will be dropped, in lieu of due date extensions for any reason.

Syllabus: H7A has two syllabus cards. The first card is mandatory; it will be collected at the time of the first in-class examination. This card pays for the experiment descriptions and instructions that you will receive from your GSI at the beginning of each laboratory. The second syllabus card is optional. It entitles you to pick up printed solutions to the problem set assignments from Copy Central. These solutions will also be made available on the Web. Both cards will be available for purchase at Copy Central beginning in the second week of class.

Exams: There will be two 80-minute in-class examinations and one 3-hour final examination. Before confirming your enrollment in this class, please check that its final Exam Group 9 does not conflict with the Exam Group for any other class in which you are enrolled. Please verify that you will be available for both in-class examinations (Th 24 Sep and Th 5 Nov, 11:10-12:30), and for the final examination, F 11 Dec, 5-8 PM. Except for unforeseeable emergencies, it will not be possible for these in-class or final exams to be rescheduled. Passing H7A requires passing the final exam.

Grading: 35% in-class examinations; 20% problem sets; 40% final exam; 5% lab. Grading is not "curved" -- it does not depend on your performance relative to that of your H7A classmates. Rather it is based on comparing your work to that of a generation of earlier lower division Berkeley physics students, with due allowance for educational trends.

COURSE OUTLINE

Week No.	Week of...	Lecture chapter	Topic (**K&K** = Kleppner/Kolenkow, *Intro. to Mechanics*) (**French** = *Vibrations & Waves)* (**Feynman** = *Lectures on Physics* Vol. II)	Problem Set No.	Due 5 PM on...	Lab
		K&K				
1	24-Aug	1.1-8	Introduction; vectors, kinematics.			none
						(do experiment in lab=expt) (have discussion in lab=disc)
2	31-Aug	1.9-2.3	Motion in polar coordinates; Newton's laws; units.	1	Wed 2 Sep	expt
3	7-Sep	2.4-2.5	LABOR DAY HOLIDAY Application of Newton's laws; forces.	2	9-Sep	disc
4	14-Sep	3.1-Note3.1	Momentum; center of mass.	3	16-Sep	expt
5	21-Sep *24-Sep*	4.1-6	Work; kinetic energy. **EXAM 1** (covers PS 1-3)			disc
6	28-Sep	4.7-14	Potential energy; nonconservative forces; energy conservation; power; collisions.	4	30-Sep	disc
7	5-Oct	6.1-7	Angular momentum; fixed axis rotation; rotation with translation.	5	7-Oct	disc
8	12-Oct	7.1-5 8.1-8.4	Vector angular momentum; conservation thereof. Noninertial systems; fictitious forces.	6	14-Oct	expt
9	19-Oct	8.5-Note8.2 9.1-9.5	Rotating coordinate systems; equivalence principle. Central forces.	7	21-Oct	expt
10	26-Oct	9.6-9.7 **French**	Planetary motion.	8	28-Oct	disc
		10-15,43-45,77-89	Damped forced harmonic oscillator.			
11	2-Nov *5-Nov*	62-70,92-96	Transient response. **EXAM 2** (covers PS 1-8)			disc
12	9-Nov	19-27,119-129 161-170,189-196	Coupled oscillator; beats. Fourier expansion in normal modes.	9	11-Nov	disc
13	16-Nov	201-209,213-215, 228-234	Waves: travelling, sinusoidal, modulated; phase and group velocity.	10	18-Nov	expt
14	23-Nov	45-62,209-212 170-178,274-279	Longitudinal waves; sound. Boundary reflections of waves; Doppler effect.	11	25-Nov	disc
	26,27-Nov		THANKSGIVING HOLIDAY			
15	30-Nov	**Feynman** II.2-1,2,3,4,5;II.40-1,2,3	Fluid statics and nonviscous dynamics. LAST LECTURE (review)	12	Fri 4 Dec	disc
16	7-Dec *11-Dec*	5-8 PM	**FINAL EXAM** (Group 9) (covers PS 1-12)			

University of California, Berkeley

Physics H7A Fall 1998 (Strovink)

STUDENT QUESTIONNAIRE

Please complete this questionnaire if you are considering enrolling in Physics H7A. Among other purposes, it will be used to make up the initial class roll.

Are you qualified for this course? Your interest in Honors Physics is already a good indication that you are. It is unlikely that the issue of your eligibility will come into question. If it does, you will be interviewed promptly by the instructor.

Last name__________________________First_______________________Initial_________

Registration No.___________________ Year (Freshman, etc.)_____________

SAT scores:

SAT I Quantitative______ SAT II Math 1______

SAT I Verbal______ SAT II Math 1C______

SAT II Math 2______

SAT II Physics______ SAT II Math 2C______

AP Calculus Exam: Grade______ AB BC (circle one)

AP Physics Exam: Grade______ B C (circle one)

Year of last math course_______ Where taken?________________________________

Course title and no.____

Course text_______________________________________ Grade received_______

Year of last physics course____ Where taken?________________________________

Course title and no.____

Course text_______________________________________ Grade received_______

List majors that you are considering at Berkeley_______________________________

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Why are you (might you be) interested in Honors in place of standard lower-division physics at Berkeley?____

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University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

TYPOS IN KLEPPNER & KOLENKOW

This is a list of typos in Kleppner & Kolenkow that have been brought to the attention of H7A F'98 staff by students in the course. Thanks to them for pointing these out.

Page 205, Example 5.2:

In the equation below "the differential of f is", replace the second dy with dx.

Page 276, Note 6.2:

In the equation following $E = K + U$, the first term $\frac{1}{2}l^2\dot{\phi}^2$ should be multiplied by m.

Page 392, Equation 9.22:

The first instance of x should be x^2 instead.

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PROBLEM SET 1

1. Specify the properties of two vectors $\mathbf{a}$ and $\mathbf{b}$ such that

(a.) $\mathbf{a}+\mathbf{b}=\mathbf{c}$ and $|\mathbf{a}|+|\mathbf{b}|=|\mathbf{c}|$.

(b.) $\mathbf{a}+\mathbf{b}=\mathbf{a}-\mathbf{b}$.

(c.) $\mathbf{a}+\mathbf{b}=\mathbf{c}$ and $|\mathbf{a}|^2+|\mathbf{b}|^2=|\mathbf{c}|^2$.

(d.) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$.

(e.) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}|=|\mathbf{b}|$.

2. K&K problem 1.2 "Find the cosine of the angle between $\mathbf{A}=(3\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ and $\mathbf{B}=(-2\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}})$."

3. The relation between Cartesian (x,y,z) and spherical polar (r,θ,ϕ) coordinates is:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

Consider two points on a sphere of radius R: (R,θ_1,ϕ_1) and (R,θ_2,ϕ_2). Use the dot product to find the cosine of the angle θ_{12} between the two vectors which point to the origin from these two points. You should obtain:

$$\cos\theta_{12} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2).$$

4. New York has North Latitude ($=90^\circ-\theta$) $=41^\circ$ and West Longitude ($=360^\circ-\phi$) $=74^\circ$. Sydney has South Latitude ($=\theta-90^\circ$) $=34^\circ$ and East Longitude ($=\phi$) $=151^\circ$. Take the earth to be a sphere of radius 6370 km; use the result of Problem 3.

(a.) Find the length in km of an imaginary straight tunnel bored between New York and Sydney.

(b.) Find the distance of the shortest possible low-altitude flight between the two cities. (*Hint:* The "great circle" distance along the surface of a sphere is just $R\theta_{12}$, where θ_{12} is the angle between the two points, measured in radians.)

5. K&K problem 1.6 "Prove the law of sines using the cross product. It should only take a couple of lines. (*Hint*: Consider the area of a triangle formed by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, where $\mathbf{A}+\mathbf{B}+\mathbf{C}=0$.)"

6. K&K problem 1.11 "Let $\mathbf{A}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\mathbf{A}=(\mathbf{A}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}+(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$."

7. If the air velocity (velocity with respect to the air) of an airplane is $\mathbf{u}$, and the wind velocity with respect to the ground is $\mathbf{w}$, then the ground velocity $\mathbf{v}$ of the airplane is

$$\mathbf{v}=\mathbf{u}+\mathbf{w}.$$

An airplane files a straight course (with respect to the ground) from P to Q and then back to P, with air speed $|\mathbf{u}|$ which is always equal to a constant U_0, regardless of the wind. Find the time required for one round trip, under the following conditions:

(a.) No wind.

(b.) Wind of speed W_0 blowing from P to Q.

(c.) Wind of speed W_0 blowing perpendicular to a line connecting P and Q.

(d.) Wind of speed W_0 blowing at an angle θ from a line connecting P and Q.

(e.) Show that the round trip flying time is always least for part (a.).

(f.) What happens to the answers to (b.)-(d.) when $W_0 > U_0$? Interpret this limiting condition physically.

8. A particle moves along the curve $y=Ax^2$ such that its x position is given by $x=Bt$ ($t=$ time).

(a.) Express the vector position $\mathbf{r}(t)$ of the particle in the form

$$\mathbf{r}(t)=\mathbf{i}f(t)+\mathbf{j}g(t)\ \ [\text{or } \hat{\mathbf{x}}f(t)+\hat{\mathbf{y}}g(t)]$$

where $\mathbf{i}$ and $\mathbf{j}$ [or $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$] are unit vectors, and $f(t)$ and $g(t)$ are functions of t.

(b.) Find the (vector) velocity $\mathbf{v}(t)$ as a function of t.

(c.) Find the (vector) acceleration $\mathbf{a}(t)$ as a function of t.

(d.) Find the (scalar) speed $|\mathbf{v}(t)|$ as a function of t.

(e.) Find the (vector) average velocity $\langle \mathbf{v}(t_0) \rangle$ between $t = 0$ and $t = t_0$ where t_0 is any positive time.

9. Below are some measurements taken on a stroboscopic photograph of a particle undergoing accelerated motion. The distance s is measured from a fixed point, but the zero of time is set to coincide with the first strobe flash:

time (sec)	distance (m)
0	0.56
1	0.84
2	1.17
3	1.57
4	2.00
5	2.53
6	3.08
7	3.71
8	4.39

Plot a *straight-line* graph, based on these data, to show that they are fitted by the equation

$$s = a(t - t_0)^2/2,$$

where a and t_0 are constants, and extrapolate the line to evaluate t_0.

University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

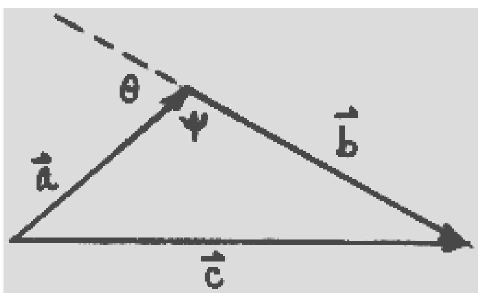
SOLUTION TO PROBLEM SET 1

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. You may remember the law of cosines from trigonometry. It will be useful for several parts of this problem, so we will state it here. If the lengths of the sides of a triangle are a, b, and c, and the angle opposite the side c is ψ, then

$$c^2 = a^2 + b^2 - 2ab\cos\psi$$

(a.) When two vectors add up to a third vector, the three vectors form a triangle. If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then the angle opposite the side formed by $\mathbf{c}$ is $180° - \theta$.



The law of cosines then tells us that

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(180° - \theta)$$

From trigonometry, remember that

$$\cos(180° - \theta) = -\cos\theta$$

which gives

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

We know that $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{c}|$. Squaring this equation, we get

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|$$

If we compare this with the equation above, we can see that $\cos\theta$ has to be equal to one. This only happens when $\theta = 0°$. What this means is that the two vectors are parallel to each other, and they point in the same direction. If the angle between them were $180°$, then they would be parallel but point in opposite directions.

(b.) This part is simple. Just subtract the vector $\mathbf{a}$ from both sides to see that $\mathbf{b} = -\mathbf{b}$. The only way that this can happen is if $\mathbf{b} = \mathbf{0}$, the zero vector.

(c.) This part can also be done by the law of cosines. Like part (a.), we have the following two equations

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

This is just the law of cosines again, where θ is the angle between $|\mathbf{a}|$ and $\mathbf{b}|$. The problem states that

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$$

Comparing this with the equation above, we find that $\cos\theta = 0$. This happens at $\theta = \pm 90°$. This means that the vectors must be perpendicular to each other.

(d.) Yet again, we can use the law of cosines. If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then the angle between $\mathbf{a}$ and $-\mathbf{b}$ is $180° - \theta$. The lengths of the sum and difference are

$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta$$
$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

For these to be equal, we need $\cos\theta = 0$, which happens when $\theta = \pm 90°$. Again, this means that the vectors are perpendicular.

(e.) Guess what? Yup, law of cosines. We know that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$. Adding $\mathbf{a}$ to $\mathbf{b}$ is going to look like two vectors stuck together to form two sides of a triangle. If the angle between the vectors is θ, the law of cosines gives

$$|\mathbf{a}+\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{a}|^2\cos\theta$$

where we have used the fact that $\mathbf{a}$ and $\mathbf{b}$ have the same length. We also know that $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}|$. Using this we get

$$|\mathbf{a}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{a}|^2\cos\theta$$

Dividing by $|\mathbf{a}|^2$, we find a condition on the angle θ

$$\cos\theta = -\frac{1}{2} \quad\Rightarrow\quad \theta = 120°$$

2. K&K problem 1.2

We can use the dot product, also known as the inner product, of two vectors here. Remember that

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$$

where θ is the angle between the vectors. We can use the formula for computing the dot product from the vector components

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

The vectors are given as follows: $\mathbf{A} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{B} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$. Multiplying, we find that $\mathbf{A}\cdot\mathbf{B} = -10$. We need the lengths of $\mathbf{A}$ and $\mathbf{B}$. Remember that $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$. This tells us that $|\mathbf{A}|^2 = 11$ and $|\mathbf{B}|^2 = 14$. Dividing, we find that

$$\cos\theta = \frac{-10}{\sqrt{11\cdot 14}} = -0.805$$

3. Using the formulas on the problem set, we can convert the points on the surface of the sphere to Cartesian coordinates.

$$\begin{aligned}
x_1 &= R\sin\theta_1\cos\phi_1\\
y_1 &= R\sin\theta_1\sin\phi_1\\
z_1 &= R\cos\theta_1\\
x_2 &= R\sin\theta_2\cos\phi_2\\
y_2 &= R\sin\theta_2\sin\phi_2\\
z_2 &= R\cos\theta_2
\end{aligned}$$

As in problem 2, we need to know the length of these vectors in order to calculate the angle between them from the dot product. It is fairly obvious that the lengths of the vectors are just R, because that is the radius of the sphere; we will show this explicitly.

$$\begin{aligned}
|(R,\theta,\phi)|^2 = R^2(&\sin^2\theta\cos^2\phi\\
&+\sin^2\theta\sin^2\phi + \cos^2\theta)
\end{aligned}$$

Using the fact that $\sin^2\phi + \cos^2\phi = 1$, we get

$$|(R,\theta,\phi)|^2 = R^2\left(\sin^2\theta + \cos^2\theta\right)$$

We can just repeat the previous step for θ now and get

$$|(R,\theta,\phi)| = R$$

as we expected in the first place. Now we calculate the dot product. Let $\mathbf{x}_1$ be the vector to the first point and $\mathbf{x}_2$ be the vector to the second point. We find that

$$\begin{aligned}
\mathbf{x}_1\cdot\mathbf{x}_2 = R^2(&\sin\theta_1\sin\theta_2\cos\phi_1\cos\phi_2\\
&+\sin\theta_1\sin\theta_2\sin\phi_1\sin\phi_2 + \cos\theta_1\cos\theta_2)
\end{aligned}$$

This can be simplified if we remember the formula for the cosine of a sum of two angles.

$$\cos(\theta\pm\phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

Using this formula, we get the result

$$\begin{aligned}
\mathbf{x}_1\cdot\mathbf{x}_2 = R^2(&\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\\
&+\cos\theta_1\cos\theta_2)
\end{aligned}$$

To get the angle we just divide by the lengths of each vector, which are both R. This gives the final result.

$$\begin{aligned}
\cos\theta_{12} = &\cos\theta_1\cos\theta_2\\
&+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)
\end{aligned}$$

4. This problem is an application of the results of problem 3.

(a.) A straight tunnel between Sydney and New York can be represented by the *difference* of the vectors pointing to their locations. To say it another way, the distance between the ends of two vectors is the length of the difference of the vectors. Adjusting for the fact that latitude and longitude are not quite the same as the coordinates θ and ϕ, we find the polar coordinates of the cities.

$$\begin{aligned}
\mathbf{X}_{\mathrm{NY}} &= (6370\,\mathrm{km}, 49^\circ, 286^\circ)\\
\mathbf{X}_{\mathrm{Sydney}} &= (6370\,\mathrm{km}, 124^\circ, 151^\circ)
\end{aligned}$$

Converting to Cartesian coordinates (x, y, z) using the formulas from problem 3 we get

$$\begin{aligned}
\mathbf{X}_{\mathrm{NY}} &= (1325\,\mathrm{km}, -4621\,\mathrm{km}, 4179\,\mathrm{km})\\
\mathbf{X}_{\mathrm{Sydney}} &= (-4619\,\mathrm{km}, 2560\,\mathrm{km}, -3562\,\mathrm{km})
\end{aligned}$$

The distance between New York and Sydney through the earth is just $|\mathbf{X}_{\mathrm{NY}} - \mathbf{X}_{\mathrm{Sydney}}|$. The result of the calculation is

$$\text{Distance} = 12,117\,\text{km}$$

(b.) Using the result from problem 3 to calculate the angle between Sydney and New York, we find that $\cos\theta_{12} = -0.809$, thus $\theta_{12} = 144.0°$. To calculate the distance along the earth's surface we need to express this angle in radians. The conversion formula is

$$\theta(\text{radians}) = \frac{\pi}{180°}\theta(\text{degrees})$$

Thus $\theta_{12} = 2.513$ radians. Multiplying this by the radius of the earth, we get the "great circle" distance between New York and Sydney:

$$\text{Distance} = 16,010\,\text{km}$$

5. K&K problem 1.6

This question asks you to prove the law of sines using the cross product. Let A, B, and C be the lengths of the vectors making the three sides of the triangle. Let a, b and c be the angle opposite each of those sides. The law of sines states that

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

Remember that the *length* of the cross product of two vectors is equal to the *area* of the parallelogram defined by them. Remember also that the the length of the cross product is equal to the product of the lengths times the sine of the angle between them: $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta$. We have three vectors to play with in this problem, and using the cross product we can compute the area of the triangle from any two of them. We find that

$$\text{Area} = AB \sin c = BC \sin a = AC \sin b$$

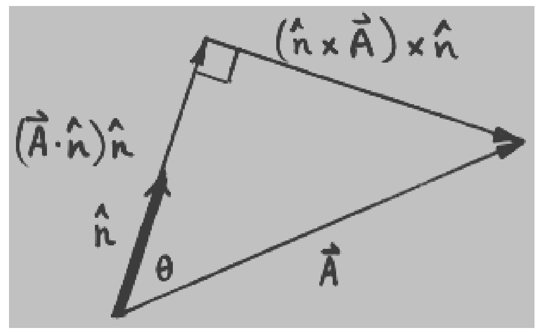
We just divide the whole thing by ABC and we recover the law of sines.

6. K&K problem 1.11

Let $\mathbf{A}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \mathbf{A}) \times \hat{\mathbf{n}}$$

Form a triangle from the three vectors in this equation. Let $\mathbf{B} = (\mathbf{A} \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}}$ and let $\mathbf{C} = (\hat{\mathbf{n}} \times \mathbf{A}) \times \hat{\mathbf{n}}$. Let the angle between $\mathbf{A}$ and $\hat{\mathbf{n}}$ be θ. What this formula does is to break up the vector $\mathbf{A}$ into a piece parallel to $\hat{\mathbf{n}}$ and a piece perpendicular to $\hat{\mathbf{n}}$. $\mathbf{B}$ gives the parallel piece. Its length is just $|\mathbf{B}| = |\mathbf{A}| \cos\theta$. The length of the perpendicular piece must then be $|\mathbf{A}| \sin\theta$.



Examining the vector $\mathbf{C}$, we see that inside the parentheses is a vector whose length is $|\mathbf{A}| \sin\theta$ and is perpendicular to $\hat{\mathbf{n}}$. This vector is then crossed into $\hat{\mathbf{n}}$. Since it is perpendicular to $\hat{\mathbf{n}}$, the length of the final vector is $|\mathbf{A}| \sin\theta$, which is what we want. Now we are just concerned with the direction. The first cross product is perpendicular to the plane containing $\hat{\mathbf{n}}$ and $\mathbf{A}$. The second cross product is perpendicular to the first, thus it is coplanar with $\hat{\mathbf{n}}$ and $\mathbf{A}$. It is also perpendicular to $\hat{\mathbf{n}}$. Thus it represents the component of $\mathbf{A}$ that is perpendicular to $\hat{\mathbf{n}}$. Be careful about the sign here.

A useful vector identity that you will be seeing again is the so-called "BAC-CAB" rule. It is an identity for the triple cross product.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Its fairly obvious why this is called the BAC-CAB rule. Using this rule, we see that

$$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{A}) = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{A}) - \mathbf{A}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})$$

This immediately gives

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \mathbf{A}) \times \hat{\mathbf{n}}$$

Of course we haven't derived the BAC-CAB rule here. It's a mess.

7. The idea in all of the parts of this problem is that the plane must oppose any perpendicular wind speed to maintain its straight path. If the wind is blowing with a speed v perpendicular to the path, the plane's airspeed must be $-v$ perpendicular to the path. The airspeed is $\mathbf{u}$, the wind speed relative to the ground is $\mathbf{w}$, and the ground speed is $\mathbf{v} = \mathbf{u} + \mathbf{w}$. $|\mathbf{u}| = U_0$. Let the total distance traveled be D.

(a.) No wind, $\mathbf{w} = \mathbf{0}$ so $\mathbf{u} = \mathbf{v}$. $T = D/U_0$.

(b.) Wind of speed W_0 blowing parallel to the path. When the wind is going with the plane, $v = W_0 + U_0$, when it opposes the plane, the ground speed is $v = U_0 - W_0$. The time for the first leg is $T_1 = D/2(U_0 + W_0)$. The time for the second leg is $T_2 = D/2(U_0 - W_0)$. The total time is the sum

$$T = \frac{D}{2}\left(\frac{1}{U_0 + W_0} + \frac{1}{U_0 - W_0}\right)$$

This can be simplified, and we get the final answer, which agrees with part (a.) when $W_0 = 0$.

$$T = \frac{DU_0}{U_0^2 - W_0^2}$$

(c.) Wind of speed W_0 blowing perpendicular to the path. This part is a little harder. The plane will not be pointed straight along the path because it has to oppose the wind trying to blow it off course. The airspeed of the plane in the perpendicular direction will be W_0, and we know what the total airspeed is, so we can calculate the airspeed along the path.

$$U_0^2 = U_\perp^2 + U_\parallel^2 \quad \Rightarrow \quad U_\parallel = \sqrt{U_0^2 - W_0^2}$$

The wind has no component along the path of motion, so the airspeed in the parallel direction is the same as the ground speed in the parallel direction. The ground speed is furthermore the same on both legs of the trip. The final answer again agrees with part (a.) when $W_0 = 0$

$$T = \frac{D}{\sqrt{U_0^2 - W_0^2}}$$

(d.) Wind of speed W_0 blowing at an angle θ to the direction of travel. The plane again needs to cancel the component of the wind blowing in the perpendicular direction. The perpendicular component of the wind speed is $W_{0\perp} = W_0 \sin\theta$. As in part (c.) the airspeed in the parallel direction can be computed

$$U_0^2 = U_\perp^2 + U_\parallel^2 \quad \Rightarrow \quad U_\parallel = \sqrt{U_0^2 - W_0^2 \sin^2\theta}$$

In this case, the wind has a component along the direction of travel. This parallel component is $W_{0\parallel} = W_0 \cos\theta$. On one leg of the trip, this adds to the ground velocity. On the other leg, it subtracts. This gives us the following formula:

$$T = \frac{D}{2}\left(\frac{1}{\sqrt{U_0^2 - W_0^2 \sin^2\theta} + W_0\cos\theta} + \frac{1}{\sqrt{U_0^2 - W_0^2 \sin^2\theta} - W_0\cos\theta}\right)$$

This can be simplified considerably:

$$T = \frac{D\sqrt{U_0^2 - W_0^2 \sin^2\theta}}{U_0^2 - W_0^2}$$

If you look at this carefully, you will realize that it reduces to the correct answer for parts (a.), (b.), and (c.) with the proper values for W_0 and θ. If $\theta = 90°$, the wind blows perpendicular to the path and we get the result from part (c.). If $\theta = 0°$, the wind blows parallel to the direction of travel and we recover the result from part (b.).

(e.) (f.) This part requires some calculus. We need to do a minimization of a function. What this part asks is to study the travel time as a function of wind speed for an arbitrary angle θ. We need to consider the result from part (d.) as a function of the wind speed:

$$T(W_0) = \frac{D\sqrt{U_0^2 - W_0^2 \sin^2\theta}}{U_0^2 - W_0^2}$$

For now we are going to ignore the fact that it also depends on U_0 and θ. Remember that functions have maxima and minima at places where the derivative vanishes, so we need to take the derivative of T with respect to W_0:

$$\frac{d}{dW_0}T(W_0) = D\Bigg(\frac{2W_0\sqrt{U_0^2 - W_0^2\sin^2\theta}}{(U_0^2 - W_0^2)^2} - \frac{W_0\sin^2\theta}{\sqrt{U_0^2 - W_0^2\sin^2\theta}(U_0^2 - W_0^2)}\Bigg)$$

The derivative is clearly zero when $W_0 = 0$. In this case the travel time $T = D/U_0$ as in part (a.). There is another case we have to worry about though. We divide out what we can to get an equation for another value where the derivative vanishes

$$U_0^2\sin^2\theta - W_0^2\sin^2\theta = 2U_0^2 - 2W_0^2\sin^2\theta$$

This gives us the other point where the derivative is zero

$$W_0^2 = U_0^2\frac{2 - \sin^2\theta}{\sin^2\theta}$$

Notice that this point always occurs when the wind speed is greater than the air speed. No progress can be made against the wind if this is the case, so the trip cannot occur. The final possibility to consider is the case where the wind speed is the same as the air speed. Looking at the formula, the time taken is infinite. The only possibility is that the minimum is at $W_0 = 0$. The final piece of this problem is to observe what happens to the "time taken" when $W_0 > U_0$. For one thing, it becomes negative. In some circumstances it can even become imaginary. There is really no interpretation of this other than "ask a stupid question, get a stupid answer". The answer doesn't make sense because the question didn't make sense. The trip cannot occur when $W_0 > U_0$, so it is meaningless to ask how long it would take.

8. A particle moves along the curve $y = Ax^2$ and its x position is given by $x = Bt$.

(a.) We can just plug the x equation into the y equation to get the y position as a function of time, $y = AB^2t^2$. In vector form, the position is then

$$\mathbf{r}(t) = \hat{\mathbf{x}}Bt + \hat{\mathbf{y}}AB^2t^2$$

(b.) The vector velocity is obtained from the vector position by differentiating with respect to t

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \hat{\mathbf{x}}B + \hat{\mathbf{y}}2AB^2t$$

(c.) The vector acceleration is obtained from the vector velocity by again differentiating with respect to t

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \hat{\mathbf{y}}2AB^2$$

(d.) The scalar speed is just the length of the velocity vector. Remember that $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$.

$$|\mathbf{v}(t)| = \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)} = \sqrt{B^2 + 4A^2B^4t^2}$$

(e.) The vector average velocity is the integral of the velocity vector over a time interval, divided by the time interval. In general, the (time) average of a quantity $\mathbf{A}$ is given by

$$\langle\mathbf{A}\rangle = \frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}\mathbf{A}(t)dt$$

Applying this formula, we see the integral that needs to be evaluated:

$$\begin{aligned}\langle\mathbf{v}(t_0)\rangle &= \frac{1}{t_0}\int_0^{t_0}\mathbf{v}(t)dt \\ &= \frac{1}{t_0}\int_0^{t_0}(\hat{\mathbf{x}}B + \hat{\mathbf{y}}2AB^2t)dt\end{aligned}$$

We could also use the fact that the integral of the velocity is the position to get a simpler looking formula for the average velocity

$$\langle\mathbf{v}(t_0)\rangle = \frac{1}{t_0}\left(\mathbf{r}(t_0) - \mathbf{r}(0)\right)$$

Evaluating this integral, we get an answer that is not surprising

$$\langle\mathbf{v}(t_0)\rangle = \hat{\mathbf{x}}B + \hat{\mathbf{y}}AB^2t_0$$

This is just $(\mathbf{r}(t_0) - \mathbf{r}(0))/t_0$! The average velocity is just the distance traveled divided by the time it took.

9. The idea behind this problem is to make a graph of position vs. time data and show that they fit the equation $s = a(t - t_0)^2/2$. In addition you are supposed to find t_0. The way to do this is to plot the square root of the distance vs. time, which will give a straight line graph: $\sqrt{s} = \sqrt{(a/2)}(t - t_0)$. The slope of this graph is approximately 0.168, so we can use that to extrapolate back to zero. We find that the graph reaches zero at about $t = -4.45$, so this means that $t_0 = -4.45$ to make the distance traveled equal zero at $t = -4.45$.

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PROBLEM SET 2

1. Calculate the following centripetal accelerations as fractions or multiples of g ($= 9.8$ m/sec^2):

(a.) The acceleration toward the earth's axis of a person standing on the earth at 45° latitude.

(b.) The acceleration of the moon toward the earth.

(c.) The acceleration of an electron moving around a proton at a speed of 2×10^6 m/sec in a circular orbit of radius 0.5 Angstroms (1 Angstrom $= 10^{-10}$ m).

(d.) The acceleration of a point on the rim of a bicycle wheel of 26 in diameter, traveling at a constant speed of 25 mph.

2. K&K problem 1.17 "A particle moves in a plane...".

3. K&K problem 1.20 "A particle moves outward along...".

4. At t=0 an object is released from rest at the top of a tall building. At the time t_0 a second object is dropped from the same point.

(a.) Ignoring air resistance, show that the time at which the objects have a vertical separation l is given by

$$t = \frac{l}{gt_0} + \frac{t_0}{2}.$$

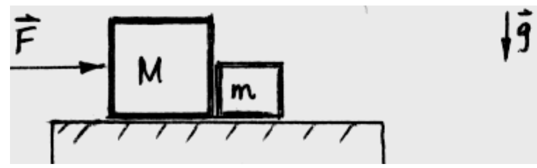
How do you interpret this result for $l < gt_0^2/2$?

(b.) The above formula implies that there is an optimum value of t_0 such that the separation l reaches some specified value l_0 at the earliest possible value of t. Calculate this optimum value of t_0 and interpret the result.

5. K&K problem 1.21 "A boy stands at the peak...".

6. K&K problem 2.1 "A 5-kg mass moves under the...".

7. In the figure, two blocks are in contact on a table. The coefficient of sliding friction between the blocks and the table is μ. A force $\mathbf{F}$ is applied to M as shown and the blocks begin to slide. Find the contact force between the two blocks. Is it the same if the force is applied to m instead of M? Does the contact force depend on μ?



8. K&K problem 2.5 "The Atwood's machine shown in...".

9. In the figure, the pulley axle has no friction and the pulley and cords have no mass. As the system is studied for various values of the external applied upward force $\mathbf{F}$, it is found that there are regimes (ranges of $|\mathbf{F}|$) for which

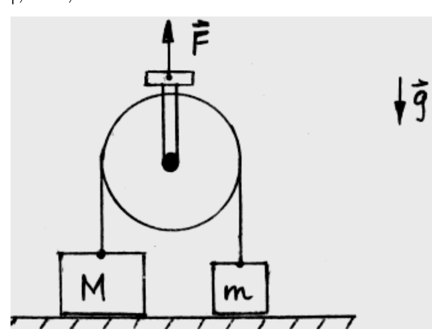
(i) Neither block moves.

(ii) Only the small block with mass m ($m < M$) moves.

(iii) Both blocks move.

(a.) Find the values of $|\mathbf{F}|$ which define the transitions between regimes (i) and (ii), and between regimes (ii) and (iii).

(b.) Find the accelerations of the masses within the regimes (b) and (c), expressed as functions of $|\mathbf{F}|$, M, and m.



10. K&K problem 2.6 "In a concrete mixer...".

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SOLUTION TO PROBLEM SET 2

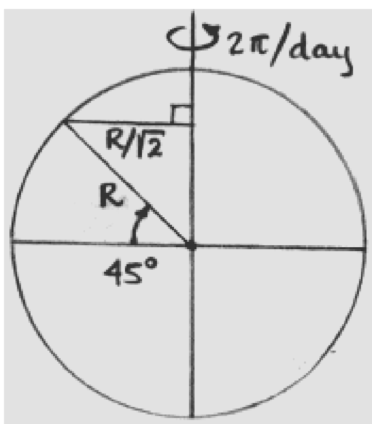
Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. This problem is a simple application of the formula for centripetal acceleration. An object moving at speed v in a circular path of radius r has a centripetal acceleration directed inward:

$$\mathbf{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{\mathbf{r}}$$

If we know instead the angular velocity ω of the object, we can use it instead. Remember that an object with angular velocity ω going on a circular path of radius r has speed $v = \omega r$. Plugging into the above formula, we get

$$\mathbf{a}_{\text{centripetal}} = -\omega^2 r\hat{\mathbf{r}}$$



(a.) Standing at 45° latitude, the distance to the axis of the earth's rotation is just $r_{\text{earth}}/\sqrt{2}$. This is the radius to be used. The speed is just the angular velocity times this radius. The angular velocity is easy to guess, it's 2π/day. Using $r_{\text{earth}} =$ 6370 km, the distance to the axis is 4504 km.

Plugging in these numbers, the acceleration is 0.0238 m/sec^2. This is $2.4 \times 10^{-3}g$, a very small acceleration compared to the acceleration of gravity.

(b.) This is the same sort of calculation, except that the angular velocity is now $2\pi/28$ days. The mean radius of the moon's orbit is 3.84×10^8m. The centripetal acceleration is 0.0026 m/sec^2, which is $2.6 \times 10^{-4}g$.

(c.) This time you are given the speed, not the angular velocity, so we use the first formula. The acceleration is 8.0×10^{22} m/sec^2. This is $8.2 \times 10^{21}g$!

(d.) A point on the rim of a wheel is moving at the same speed that the wheel is rolling. The radius is 13 inches, and the velocity is 25 mph, so we need to convert these units. The official definition of the inch is 1 inch = 2.54 cm. This gives 1610 meters per mile. The acceleration is then 379 m/sec^2, which is $38.6g$.

2. K&K problem 1.17

In plane polar coordinates the velocity and acceleration are given by

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

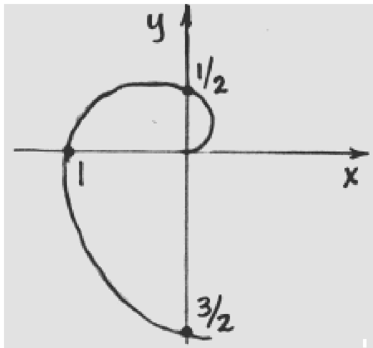
We know all of these things, so we can get a formula for the magnitudes of $\mathbf{v}$ and $\mathbf{a}$, given that both the radial and angular velocities are constant ($\ddot{r} = \ddot{\theta} = 0$).

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$
$$a = \sqrt{r^2\dot{\theta}^4 + 4\dot{r}^2\dot{\theta}^2}$$

Plugging into these, we find that $v = \sqrt{52}$ m/sec and $a = 20$ m/sec^2.

3. K&K problem 1.20

The motion of a particle is given by $r = A\theta$, $\theta = \alpha t^2/2$ and $A = (1/\pi)$ m/rad. The sketch of this motion should look something like this:



(b.) We plug the expression for θ into r to get $r = A\alpha t^2/2$. From these we can get all of the necessary derivatives. $\dot{r} = A\alpha t$, $\ddot{r} = A\alpha$, $\dot{\theta} = \alpha t$, and $\ddot{\theta} = \alpha$.

Using the equation for radial acceleration, we find an expression for points where the radial acceleration is zero.

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 \;\Rightarrow\; \ddot{r} = r\dot{\theta}^2$$

Plugging in the expressions for the r and θ variables, we find that $A\alpha = A\alpha(\alpha t^2)^2/2$. Using the fact that $\theta = \alpha t^2/2$, we arrive at the result $\theta = 1/\sqrt{2}$.

(c.) For this part, we set the magnitudes of the radial and tangential accelerations equal to each other and solve for the angle. Plugging into the formulas for the two accelerations, we get

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{aligned}$$

Plugging into the various terms, we get

$$\left| A\alpha - \frac{1}{2}A\alpha^3 t^4 \right| = \left| \frac{1}{2}A\alpha^2 t^2 + 2A\alpha^2 t^2 \right|$$

Plugging in the expression for θ where we can, we get

$$\left|1 - 2\theta^2\right| = |5\theta|$$

When the dust settles, there will be separate results depending on the value of θ. The final results are

$$\begin{aligned} \theta < \frac{1}{\sqrt{2}} &: \quad \theta = \frac{\sqrt{33} - 5}{4} \\ \theta > \frac{1}{\sqrt{2}} &: \quad \theta = \frac{\sqrt{33} + 5}{4} \end{aligned}$$

4. Two objects are dropped from a building at times $t = 0$ and $t = t_0$. The distance that the first has fallen as a function of time is just $d_1 = gt^2/2$. The distance that the second has fallen is $d_2 = g(t - t_0)^2/2$. When $t > t_0$, the separation between them is $l = d_1 - d_2$. Thus we get

$$l = \frac{1}{2}gt^2 - \frac{1}{2}g(t^2 - 2tt_0 + t_0^2) = gtt_0 - \frac{1}{2}gt_0^2$$

Solving for t as a function of l, we get

$$t = \frac{l}{gt_0} + \frac{t_0}{2}$$

When $l < gt_0^2/2$ this time is negative. This does have a sensible interpretation, unlike the negative time in the airplane problem of the previous problem set. Think of the problem as if we wanted to throw both objects at the same time, but still have the initial conditions given. At $t = 0$, when we drop the first object, from where do we have to throw the other object? The answer is we want to throw upwards from below in such a way that at time $t = t_0$, the ball has reached the peak of its path and is momentarily at rest at the point where the first ball was dropped. During the time between $t = 0$ and $t = t_0$, the separation between the objects can be negative, meaning that the second one is below the first.

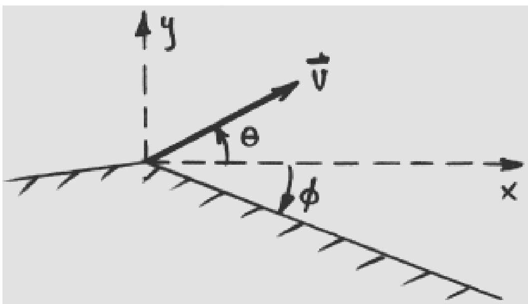
(b.) In this part we want to calculate the optimal value of t_0 so that the separation reaches some value l_0 at the earliest time possible. In other words, we want to minimize the function $t(t_0, l_0)$. First we take the derivative and set it to zero to find local extrema.

$$\frac{dt}{dt_0} = \frac{1}{2} - \frac{l_0}{gt_0^2} = 0 \;\Rightarrow\; t_0 = \sqrt{\frac{2l_0}{g}}$$

Notice that this is just the time that it takes the first object to fall a distance l_0. The endpoints here are $t_0 = 0$, where the separation remains at $l = 0$ forever and $t_0 = \infty$, where the time to reach a separation of l_0 is also infinity. Thus this best time to drop it is at $t_0 = \sqrt{2l_0/g}$. This means that the best thing to do is to drop the second object when the first object has already fallen a distance l_0.

5. K&K problem 1.21

This is another maximization problem. We want to know the optimal angle to throw a ball down a hill with slope angle ϕ. Splitting this into the x and y directions is the easiest way to do the problem. First put the origin at the top of the hill. If the ball is thrown up at an angle θ with speed v, the initial velocities are $v_x = v\cos\theta$ and $v_y = v\sin\theta$.



Taking into account the acceleration of gravity, the positions are given by

$$x = vt\cos\theta$$
$$y = vt\sin\theta - \frac{1}{2}gt^2$$

We need to know where the ground is in these coordinates. At a position x on the ground, the y coordinate is given by $y_{\text{ground}} = -x\tan\phi$. We can now find the time at which the ball hits the ground. Plugging into the equation for distance traveled in y, we get

$$y_{\text{ground}} = -x\tan\phi = vt\sin\theta - \frac{1}{2}gt^2$$
$$\Rightarrow \quad v\sin\theta - \frac{1}{2}gt = -v\cos\theta\tan\phi$$

This gives the time at which the ball hits the ground:

$$t = \frac{2v}{g}\left(\sin\theta + \cos\theta\tan\phi\right)$$

We now plug this time into the equation for distance traveled in the x direction, giving the distance that the ball traveled:

$$\begin{aligned} x &= \frac{2v^2}{g}(\cos\theta\sin\theta + \cos^2\theta\tan\phi) \\ &= \frac{2v^2}{g}(\frac{1}{2}\sin 2\theta + \cos^2\theta\tan\phi) \end{aligned}$$

Treating this as a function of θ, we can maximize the range by differentiating with respect to θ. Note that the endpoints in this problem are not interesting. Throwing the ball straight up ($\theta = 90°$) and throwing it at an angle $-\phi$ both result in the ball traveling no distance in the x direction.

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{2v^2}{g}(\cos(2\theta) - 2\cos\theta\sin\theta\tan\phi) \\ &= \frac{2v^2}{g}(\cos(2\theta) - \sin(2\theta)\tan\phi) = 0 \end{aligned}$$

Solving for θ, we get

$$\cos(2\theta) = \sin(2\theta)\tan\phi \;\Rightarrow\; \cot(2\theta) = \tan\phi$$

Remembering that $\cot\alpha = \tan(\pi/2 - \alpha)$, we see the final result:

$$\tan\left(\frac{\pi}{2} - 2\theta\right) = \tan\phi \;\Rightarrow\; \theta = \frac{\pi}{4} - \frac{\phi}{2}$$

Note that on a level surface, when $\phi = 0$, the optimal angle is 45°, as you might already know.

6. K&K problem 2.1

This is the first problem where you are asked to consider the forces causing acceleration. The force on a 5 kg mass is given by $\mathbf{F} = 4t^2\hat{\mathbf{x}} - 3t\hat{\mathbf{y}}$ Newtons. Apply Newton's second law of motion, namely $\mathbf{F} = m\mathbf{a}$, to get the acceleration, $\mathbf{a} = (4t^2/5)\hat{\mathbf{x}} - (3t/5)\hat{\mathbf{y}}$m/sec.

(a.) We can get velocity from acceleration by integrating

$$\mathbf{v}(t) - \mathbf{v}(t_0) = \int_{t_0}^{t} \mathbf{a}(t')dt'$$

Plugging the acceleration we just determined into this integral, and knowing that the velocity at $t = 0$ is zero, we get the velocity as a function of time:

$$\mathbf{v}(t) = \frac{4}{15}t^3\hat{\mathbf{x}} - \frac{3}{10}t^2\hat{\mathbf{y}} \text{ m/sec}$$

(b.) We get the position by integrating again:

$$\mathbf{r}(t) - \mathbf{r}(t_0) = \int_{t_0}^{t} \mathbf{v}(t')dt'$$

Applying this to the result of part (a.), and remembering that at $t = 0$ the mass is at the origin so $\mathbf{r}(0) = 0$, we get the position as a function of time:

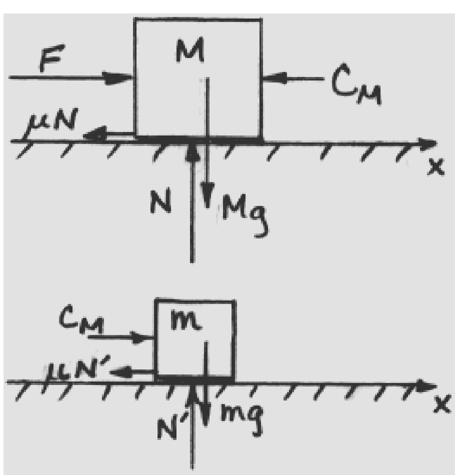
$$\mathbf{r}(t) = \frac{1}{15}t^4\hat{\mathbf{x}} - \frac{1}{10}t^3\hat{\mathbf{y}} \text{ m}$$

(c.) Now all that is left is to take the cross product of the position with the velocity. We find that

$$\mathbf{r} \times \mathbf{v} = \left(-\frac{3}{150}t^6 + \frac{4}{150}t^6\right)\hat{\mathbf{z}} = \frac{t^6}{150}\hat{\mathbf{z}}$$

7. This problem asks you to consider two blocks sliding on a table together. The larger block, with mass M, has five forces acting on it. They are $F\hat{\mathbf{x}}$, the applied force, a contact force that I will call $-C_M\hat{\mathbf{x}}$, the force of gravity $-g\hat{\mathbf{y}}$, the normal force $N\hat{\mathbf{y}}$, and the force of friction. Because there is no acceleration in the $\hat{\mathbf{y}}$ direction, we can easily find that $N = Mg$, so that there is no net force in the $\hat{\mathbf{y}}$ direction. From the normal force we can determine the force of friction. An object that is sliding with friction along a surface is acted upon by a force opposing the direction of motion with magnitude $\mu N (= \mu Mg)$, where μ is the coefficient of sliding friction. We now can write an expression for the acceleration of the large block in the $\hat{\mathbf{x}}$ direction

$$a_x = \frac{F_x}{M} = \frac{F}{M} - \frac{C_M}{M} - \mu g$$



There are two unknowns here, a_x and C_M. We need another equation. Luckily, there is another block that we can write equations about. The small block, having mass m, is affected by four forces. They are the contact force, the force of gravity, the normal force, and the force of friction. The contact force is exactly opposite to the contact force on the first block. This is due to Newton's third law of motion, which states that every force has an equal and opposite force. Thus $C_m = C_M$ and the contact force is $C_M\hat{\mathbf{x}}$. The force of gravity is just $-mg\hat{\mathbf{y}}$, the normal force exactly opposes gravity as before, $+mg\hat{\mathbf{y}}$, and the force of friction is again $-\mu mg$. We get the equation of motion for the second block, noticing that acceleration of the second block is the same as for the first block because they are moving together:

$$a_x = \frac{C_M}{m} - \mu g$$

We are only concerned with C_M here, so we can simplify the solution of these equations. Subtract the second equation from the first to get

$$\frac{F}{M} - C_M\left(\frac{1}{M} + \frac{1}{m}\right) = 0$$

We can now solve for C_M in terms of F:

$$C_M = F\left(\frac{m}{M+m}\right)$$

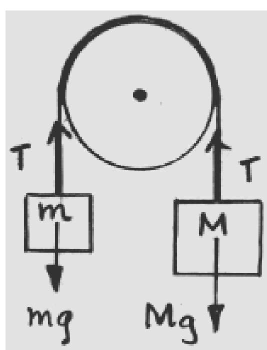
If we follow the same procedure when the force is acting on the second block, we get a very similar answer, but the mass in the numerator is the larger mass

$$C_M = F\left(\frac{M}{M+m}\right)$$

This is a factor of M/m larger, which is what we expect. The only force pushing on the second block in each case is the contact force, and the acceleration doesn't depend on which side we push the combined system from. The force of friction acts in proportion to the mass in this case, so it does not affect this argument. It can be thought of as a force that acts on the combined system, not on the individual blocks, because it is proportional to the mass. Notice that neither of these expressions depend on the value of μ, which would indicate that the friction was affecting the contact force.

8. K&K problem 2.5

This is the first pulley problem, and it won't be the last. The pulley is massless and frictionless, and supports two masses M and m by a massless rope connecting them. The first thing to notice about this problem is that the tension in the rope must be the same on both sides of the pulley. If the different sides had different tensions, there would be a tendency to cause an angular acceleration in the pulley. Since it is massless, this acceleration would be infinite, which is unphysical, so the tensions must be equal. You will see this in more detail when you do rigid body motion later this term.



The second thing to notice is that the more massive block will fall and the less massive block will rise, and their accelerations will be the same, but in opposite directions. This just means that the rope isn't stretching. For this problem I will set "down" to be the positive direction. The equation for the larger mass is

$$Ma = Mg - T \;\Rightarrow\; T = M(g-a)$$

Using the expression for tension derived in the above equation, the equation for the smaller mass is the following:

$$ma = -mg + T \;\Rightarrow\; m(a+g) = M(g-a)$$

Solving for the acceleration,

$$a = g\left(\frac{M-m}{M+m}\right)$$

The larger mass accelerates with magnitude a downward. The tension is found by plugging the acceleration into either of the starting equations

$$T = Mg - Mg\left(\frac{M-m}{M+m}\right) = g\left(\frac{2Mm}{M+m}\right)$$

9. Nope, this isn't the last pulley problem either. Again, the pulley and cords are massless and the pulley is frictionless. A force F is applied upward, and various things will happen depending on what F is. The first thing to notice is that the pulley is massless. This means that the tensions on the two ropes must be equal, otherwise a finite angular force would be applied to a massless object, which again is unphysical. The second thing to notice is that the upward force must exactly balance the sum of the tensions. If this weren't the case, there would be a net force applied to a massless object. This can't happen, so to balance the forces we just need $F = 2T$. With these two points in mind, we can do the rest of the problem.

(a.) The boundary between regimes (i) and (ii) is where the lighter block lifts off the ground. Consider the forces on this block. They are gravity $-mg$, the tension T, and the normal force N. The equation of motion is $ma = T + N - mg$. At the boundary between regimes (i) and (ii) when the block just barely can be lifted upward, the normal force N is zero, but so is the acceleration. This gives us $T = mg$. We know also that $T = F/2$, so the minimum force to lift the lighter block is $F = 2mg$. This is just twice the weight of the lighter block, which we expect because the force applied gets divided in half by the pulley. The boundary between regimes (ii) and (iii) is found in a similar way. The equation of motion is $T + N - Mg = Ma$. Again both a and N are zero at the transition point, so $T = Mg$, which gives the force $F = 2Mg$. The final results are

$$\begin{aligned} \text{regime(i)} &\;\; \{F < 2mg\} \\ \text{regime(ii)} &\;\; \{2mg < F < 2Mg\} \\ \text{regime(iii)} &\;\; \{2Mg < F\} \end{aligned}$$

(b.) Now we want to find the accelerations for regimes (ii) and (iii). This is easy because we have already determined the equations of motion. For regime (ii), only the lighter block accelerates. The equation of motion is $F/2 - mg = ma$. This gives the result

$$\text{regime (ii)} \quad a_M = 0$$
$$a_m = \frac{F}{2m} - g$$

In regime (iii), the equation of motion of the first block is the same, so we get the same result for the acceleration. The equation of motion of the larger block is $F/2 - Mg = Ma$. These give the results

$$\text{regime (iii)} \quad a_M = \frac{F}{2M} - g$$
$$a_m = \frac{F}{2m} - g$$

10. K&K problem 2.6

The cement mixer's drum has a radius R. We want to know how fast it can rotate so that the material will not stick to the walls all of the time. We just need to figure out at what speed the drum can oppose gravity all of the time. For a glob of material of mass m, the worst case is at the top. To remain in contact with the drum, at that point the glob must feel a downward force from the drum that is positive. In addition it feels the downward force mg due to gravity. So the total downward force on it must be at least mg. Now, what acceleration accompanies this force? We don't want the glob to leave the drum, so its radial velocity must remain equal to zero. The only remaining possible downward acceleration is the centripetal acceleration due to the circular motion. This is $\omega^2 R$. Equating the mass m times this acceleration to the total downward force, we conclude that $m\omega^2 R \geq mg$, or $\omega \geq \sqrt{g/R}$. For the material to not always stick, we need the final result

$$\omega < \sqrt{\frac{g}{R}}$$

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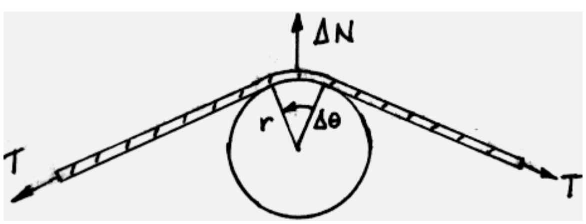
PROBLEM SET 3

1.

(a.) A clothesline is tied between two poles, 10 m apart, in a way such that the sag is negligible. When a wet shirt with a mass of 0.5 kg is hung at the middle of the line, the midpoint is pulled down by 8 cm. What is the tension in the clothesline?

(b.) A car is stranded in a ditch, but the driver has a length of rope. The driver knows that he is not strong enough to pull the car out directly. Instead, he ties the rope tightly between the car and a tree that happens to be 50 ft away; he then pushes transversely on the rope at its midpoint. If the midpoint of the rope is displaced transversely by 3 ft when he pushes with a force of 500 N ($\approx$50 kg), what force does this exert on the car? If this were sufficient to begin to move the car, and the man pushed the rope another 2 ft, how far would the car be shifted, assuming that the rope does not stretch any further? Does this seem like a practical method of dealing with the situation?

2. A string in tension T is in contact with a circular rod (radius r) over an arc subtending a small angle $\Delta\theta$ (see the figure).



(a.) Show that the force with which the string presses radially inward on the pulley (and hence the normal force ΔN with which the pulley pushes on the string) is equal to $T\Delta\theta$.

(b.) Hence show that the normal force per unit length is equal to T/r. This is a sort of *pressure* which, for a given value of T, gets bigger as r decreases. (This helps to explain why, when a string is tied tightly around a package, it cuts into the package most deeply as it passes around corners, where r is least.)

(c.) If the contact is not perfectly smooth, the values of the tension at the two ends of the arc can differ by a certain amount ΔT before slipping occurs. The value of ΔT is equal to $\mu\Delta N$, where μ is the coefficient of friction between string and rod. Deduce from this the exponential relation

$$T(\theta) = T_0 \exp(\mu\theta)$$

where T_0 is the tension applied at one end of an arbitrary arc θ of string and $T(\theta)$ is the tension at the other end.

(d.) The above result expresses the possibility of withstanding a large tension T in a rope by wrapping the rope around a cylinder, a phenomenon that has been exploited by time immemorial by sailors. Suppose, for example, that the value of μ in the contact between a rope and a (cylindrical) bollard on a dock is 0.2. For $T_0 = 100$ lb applied by the sailor, calulate the values of T corresponding to 1, 2, 3, and 4 complete turns of the rope around the bollard. (It is interesting to note that T is proportional to T_0. This allows sailors to produce a big pull or not, at will, by having a rope passing around a continuously rotating motor-driven drum. This arrangement can be described as a *force amplifier*).

3. A popular demonstration of inertia involves pulling the tablecloth out from beneath dishes with which the table is set. Suppose a tablecloth just covers the area s^2 of a square table. A dish is in the exact center of the table. The coefficient of sliding friction between the dish and the cloth is μ_1, and that between the dish and the table is μ_2. A dinner guest withdraws the cloth swiftly, but at a steady rate. Let the distance the dish moves while in contact with the moving cloth be x_1 and the distance it moves while in contact with the table be x_2.

(a.) Solve for the maximum velocity v of the dish in terms of x_1, μ_1, and g.

(b.) Do the same in terms of x_2, μ_2, and g.

(c.) Show that in order for the dish just to remain on the table,

$$x_1 = (s/2)\frac{\mu_2}{\mu_1 + \mu_2}.$$

(d.) Find the length of time during which the dish and tablecloth are in contact under conditions (c.).

(e.) A pitfall for the dinner guest is that the dish may not slide at all, but instead merely move with the cloth. How does she avoid that?

4. A piece of string of length L, which can support a maximum tension T, is used to whirl a particle of mass m in a circular path. What is the maximum speed with which the particle may be whirled if the circle is

(a.) horizontal;

(b.) vertical?

5. K&K problem 2.31 "Find the frequency of oscillation...".

6. K&K problem 2.35 "This problem involves... A block of mass m slides...".

7. Two skaters (A and B), both of mass 70 kg, are approaching one another, each with a speed of 1 m/sec. Skater A carries a bowling ball with a mass of 10 kg. Both skaters can toss the ball at 5 m/sec relative to themselves. To avoid collision they start tossing the ball back and forth when they are 10 m apart. Is one toss enough? How about two tosses, *i.e.* A gets the ball back? Plot the entire incident on a time *vs.* displacement graph, in which the positions of the skaters are marked along the x axis, and the advance of time is represented by the increasing value of the y axis. (Mark the initial positions of the skaters at $x = +$ or $-$ 5 m, and include the space-time record of the ball's motion in the diagram.)

This situation serves as a simple model of the standard view of interactions (repulsive, in the present example) between elementary particles.

8. Find the center of mass of a semicircular hoop of radius R.

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SOLUTION TO PROBLEM SET 3

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

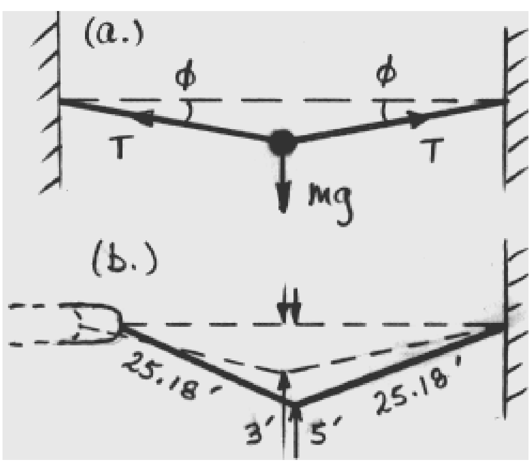
1. (a.) This problem is a simple force balance. The component of the tension in the rope pointing up must balance the force of gravity pulling down on the shirt. We are going to ignore the effect of gravity on the rope. The angle that the rope makes with the horizontal is just $\tan\phi = 8$ cm/5 m $= 0.016$. The sine of this angle is very close to this value. In fact $\sin\phi = 0.015998$, so we will use $\sin\phi = 0.016$.

The equation for the force balance in the vertical direction is just

$$2T\sin\phi = mg$$

This ensures that the force of tension balances gravity. There is a factor of two in front of the tension because the tension in each half of the rope acts on the mass. Solving this equation with the values given, the answer is

$$T = 153 \text{ N}$$



(b.) This part is similar to the first. In this case the force being applied to the rope is a man pushing on it, no gravity, but the method is the same. The force of 500 N must be balanced by tension in the rope. The angle $\tan\phi = 3/25 = 0.12$ is much larger, but the approximation $\sin\phi \approx \tan\phi \approx \phi$ is still very accurate, so we will use $\sin\phi = 0.12$. We also need $\cos\phi = 0.99$. The force of tension is found by a force balance

$$2T\sin\phi = 500 \text{ N} \;\Rightarrow\; T = 2083 \text{ N}$$

The force acting on the car is just the cosine of the angle times this tension, assuming that we want the force directed towards the tree.

$$F = T\cos\phi = 2062 \text{ N}$$

Now we want to find how far the car is shifted. The original right triangle had sides 3 ft, 25 ft, and $\sqrt{9+625}$ =25.18 ft. The rope will not stretch anymore, so the hypotenuse of this triangle will remain the same, 25.18 ft. The rope is pushed an additional 2 ft, so the short side now has length 5 ft. The long side must have length $\sqrt{634-25} = 24.68$ ft. The car has moved twice this distance, since the are two triangles with total hypotenuse ≈ 50 ft; only 0.63 ft of car movement is produced by pushing the rope 2 ft. This isn't a very practical way to get the car out of the ditch.

2. (a.) In this problem we will again use the fact that $\sin\theta \approx \theta$ when $\theta \ll 1$. The angle that the rope makes with the normal force is $\Delta\theta/2$, so the tension in the normal direction is $T\sin(\Delta\theta/2) \approx T\Delta\theta/2$. There are two tensions here, one from each side of the rope, and they must balance the normal force. The normal force on this section of string is then just

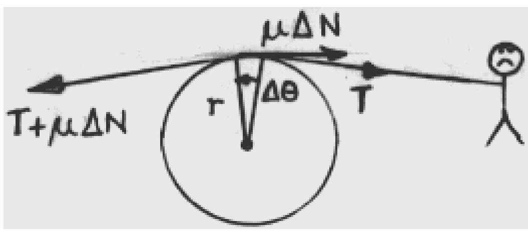
$$\Delta N = T\Delta\theta$$

(b.) The length of rope that covers an angle $\Delta\theta$ on a circular object is just $r\Delta\theta$. θ of course is measured in radians. The normal force per length on the cylinder is $\Delta N/\Delta\ell$. Plugging in the result for ΔN from part (a.), and using $\Delta\ell = r\Delta\theta$, we find

$$\frac{\Delta N}{\Delta\ell} = \frac{T}{r}$$

(c.) When the tension on the rope is not constant, there can be slipping. The force countering this is friction. If the tension changes by an amount ΔT along a small section of rope, the frictional force must be equal to it. The frictional force is $\mu\Delta N$ when it is just about to slip. We thus get $\Delta T = \mu\Delta N$. Plugging in the results of (a.), we get $\Delta T = \mu T\Delta\theta$. We are going to promote this relation between small quantities to a differential relation, so that we can get a differential equation to solve.

$$dT = \mu T d\theta \;\Rightarrow\; \frac{dT}{d\theta} = \mu T$$



This is a differential equation that we can solve by direct integration.

$$\frac{dT}{d\theta} = \mu T \;\Rightarrow\; \frac{dT}{T} = \mu d\theta$$
$$\Rightarrow \int_{T_0}^{T(\theta)} \frac{dT'}{T'} = \int_0^{\theta} \mu d\theta'$$

These integrals are ones that you should memorize if you haven't yet. The result is

$$\ln T(\theta) - \ln T_0 = \mu\theta \;\Rightarrow\; T(\theta) = T_0 e^{\mu\theta}$$

The tension increases exponentially, provided that the rope is about to slip.

(d.) Here we are going to calculate some values for this amplification of force. $\mu = 0.2$ and $T_0 = 100$ lbs. The values of T for 1, 2, 3, and 4 complete turns are as follows. One complete turn has angle 2π. $\text{Exp}(2\pi\mu) = 3.51$, so the tension at the other end is 351 lbs. For two turns, the angle is 4π, so the amplification factor is $\exp(4\pi\mu) = 12.35$ and the tension at the other end is 1235 lbs. For three complete turns, the tension is 4338 lbs. For four complete turns, the tension is 15240 lbs, almost 8 tons!

3. A dish sits in the middle of a square table of side s. The coefficient of friction between dish and tablecloth is μ_1. The coefficient of friction between the dish and table is μ_2. The tablecloth is rapidly pulled out from under the dish. The dish moves a distance x_1 while in contact with the moving tablecloth, and a distance x_2 while in contact with the table. Let the mass of the dish be m, but we will see that this doesn't matter.

(a.) The tablecloth is being pulled out from under the dish, so the dish is sliding on the tablecloth. The frictional force tends to pull the dish to the edge of the table because this opposes the direction of the sliding. The normal force of the dish on the table is just mg, so the force of sliding friction is just $\mu_1 mg$. Newton's second law then tells us that

$$F = ma = m\mu_1 g \;\Rightarrow\; a = \mu_1 g$$

This is a constant acceleration, so we can easily determine the amount of time that the dish is on the tablecloth and its maximum velocity. We know the total distance traveled is x_1, so we get the following equations for the time of contact t_1 and the maximum velocity v:

$$x_1 = \frac{1}{2}\mu_1 g t_1^2 \qquad v = \mu_1 g t_1$$

These equations are easily solved for t and v:

$$t_1 = \sqrt{\frac{2x_1}{\mu_1 g}} \qquad v = \sqrt{2x_1\mu_1 g}$$

(b.) We do the same thing for the period when the dish slides on the table. This time, the frictional force tends to slow the dish down. The frictional force is $-\mu_2 mg$, so the acceleration is $a = -\mu_2 g$. The trip starts at the maximum velocity v, and ends with the dish at rest having moved a distance x_2. We can again solve for the travel time t_2 and the maximum velocity v:

$$x_2 = -\frac{1}{2}\mu_2 g t_2^2 + v t_2 \qquad v = \mu_2 g t_2$$

Again these equations are easily solved

$$t_2 = \frac{v}{\mu_2 g} \qquad v = \sqrt{2x_2\mu_2 g} \qquad t_2 = \sqrt{\frac{2x_2}{\mu_2 g}}$$

(c.) In part (c.), we derived two expressions for the maximum velocity v. If we equate these we can get an expression relating the distances traveled x_1 and x_2.

$$\sqrt{2x_1\mu_1 g} = \sqrt{2x_2\mu_2 g} \;\Rightarrow\; \mu_1 x_1 = \mu_2 x_2$$

For the dish to remain on the table, we need the total distance traveled to be less than half the length of the table, $x_1 + x_2 \leq s/2$. We can combine these equations to solve for the distance x_1 that the dish spends on the tablecloth. We will consider the case where the dish stops at the edge of the table so the total distance traveled is $s/2$. From the previous equation we get a relation between x_1 and x_2

$$x_2 = \frac{\mu_1}{\mu_2} x_1$$

Combining this with the previous equation, we get the final answer

$$x_1 + x_2 = \frac{s}{2} \;\Rightarrow\; \left(1 + \frac{\mu_1}{\mu_2}\right) x_1 = \frac{s}{2}$$
$$\Rightarrow\; x_1 = \left(\frac{s}{2}\right) \frac{\mu_2}{\mu_1 + \mu_2}$$

(d.) To find the amount of time the dish is in contact with the tablecloth, we just use the result of part (a.), $t_1 = \sqrt{2x_1/\mu_1 g}$. Plugging in our result, we get

$$t_1 = \sqrt{\left(\frac{s}{g}\right)\left(\frac{\mu_2}{\mu_1}\right)\frac{1}{\mu_1 + \mu_2}}$$

(e.) It is possible that the dish won't slide at all. This is the case of static friction. This force must be overcome but a sufficiently hard tug. The tablecloth must accelerate at the beginning of the pull to get to its high, but constant velocity. If this acceleration is too low ($a_1 < \mu_1^{\text{static}} g$), the force of static friction will be sufficient to accelerate the dish at the same rate, and the host will not be pleased. If the initial acceleration is high enough, the force of static friction cannot impart the necessary acceleration to the dish, and it begins to slide.

4. A string has length L and can support a tension T. A mass m is spun around on the end of the string.

(a.) The string is spun horizontally. The centripetal acceleration is v^2/L, so the force needed to supply this acceleration is mv^2/L. In this case, the only force to consider is the tension. The maximum velocity is given by

$$\frac{mv^2}{L} = T \;\Rightarrow\; v = \sqrt{\frac{LT}{m}}$$

The above would be correct if gravity could be ignored for part (a.). However, we know this cannot be the case, because then there would be no definition for the word "horizontal". Gravity must be present. If so, the string makes an angle θ with the horizontal, and the radius of the spin is $L\cos\theta$. The centripetal acceleration is $v^2/L\cos\theta$. This must be provided entirely by the tension in the rope in the radial direction, which is $T\cos\theta$. This gives a relationship between v and θ for a given T:

$$v^2 = \frac{TL}{m}\cos^2\theta \;\Rightarrow\; \cos^2\theta = \frac{mv^2}{TL}$$

The tension must also oppose the force of gravity downward, giving a second relation, which will be easier to handle when squared

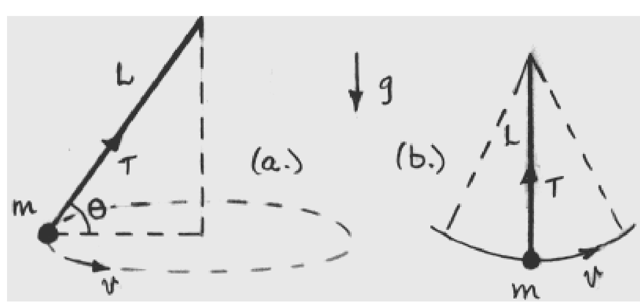
$$T\sin\theta = mg \;\Rightarrow\; \sin^2\theta = \left(\frac{mg}{T}\right)^2$$

Adding these two equations, and using $\cos^2\theta + \sin^2\theta = 1$, we get a condition on the velocity

$$\frac{m}{TL}v^2 + \frac{m^2 g^2}{L^2} = 1 \;\Rightarrow\; v^2 = \frac{LT}{m} - \frac{mg^2 L}{T}$$

The maximum velocity considering gravity is thus

$$v = \sqrt{\frac{LT}{m} - \frac{mg^2 L}{T}}$$



(b.) The string is spun vertically. In this case we must also consider gravity. Gravity directly opposes the tension when the mass is at its low point. At any other point, the tension is less. At the low point, the two forces of tension and gravity oppose each other, so the tension must be higher to provide the centripetal acceleration. When the rope is at an angle θ to the vertical, the centripetal force is provided by two sources, gravity and tension

$$\frac{mv^2}{L} = T(\theta) - mg\cos\theta$$

$T(\theta)$ is largest at the bottom of the path, when $\theta = 0$. Set $T(0) = T$, the largest allowed tension. This gives the final answer

$$\frac{mv^2}{L} = T - mg \;\Rightarrow\; v = \sqrt{\frac{LT}{m} - Lg}$$

5. K&K problem 2.31

Find the effective spring constants of these two spring systems. Remember that the frequency of oscillation is $\omega = \sqrt{k/m}$.

(a.) Consider the point where the springs are attached to each other. There shouldn't be any force acting there because the small point is massless. We can write equations for the displacements of the two springs x_1 and x_2. This spring force acts like a tension in that it pulls from both ends.

$$F_1 = -k_1x_1 = F_2 = -k_2x_2 \;\Rightarrow\; k_1x_1 = k_2x_2$$

The total displacement of the mass is $x_1 + x_2$, and the total force is just the spring force of the bottom spring $F = -k_2x_2$. Applying the general relation $F = -kx$, where k is the spring constant of the combined spring system and $x = x_1 + x_2$, we get

$$F = -k_2x_2 = -k(x_1 + x_2) \;\Rightarrow\; k = \frac{k_2x_2}{x_1 + x_2}$$

Using the fact that x_1 can be written in terms of x_2, we can remove the position dependence from the equation:

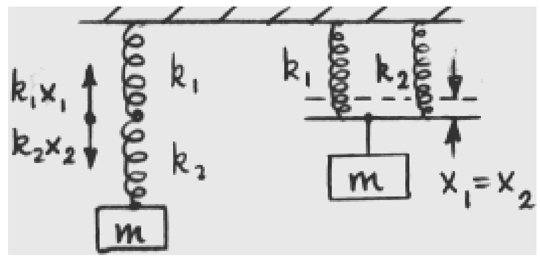
$$k = \frac{k_2}{\frac{k_2}{k_1} + 1} \;\Rightarrow\; k = \frac{k_1k_2}{k_1 + k_2}$$

This relationship is usually written in a different way that is easier to remember. You can check that it is correct:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

The final result for the frequency is

$$\omega_a = \sqrt{\frac{k_1k_2}{m(k_1 + k_2)}}$$



(b.) In this case the thing to notice is that the displacements of the two springs must be equal, otherwise the support would be tilting.

$$F_1 = -k_1x \qquad\qquad F_2 = -k_2x$$

The total force acting on the mass is just the sum of these two forces $F = F_1 + F_2$. We can easily find the effective spring constant of the system:

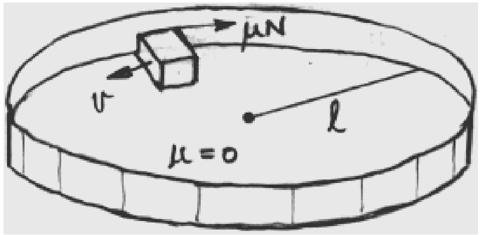
$$F = -kx = -k_1x - k_2x \;\Rightarrow\; k = k_1 + k_2$$

The final result for the frequency is

$$\omega_b = \sqrt{\frac{k_1 + k_2}{m}}$$

6. K&K problem 2.35

(a.) In this problem we need to solve a differential equation. A block slides on a frictionless table inside a fixed ring of radius l. The ring has a coefficient of friction μ. We want to find the velocity as a function of time. At time $t = 0$ the velocity is v_0. We will assume that the block moves in the circular path defined by the ring. This makes it effectively a one dimensional problem.



There are two forces acting on the block in the plane of the table. They are the normal force exerted by the ring and the frictional force. The normal force merely makes the block move in the circular path that we assumed. We do need to know it though, because we want to calculate the frictional force. We find it in the usual way, by requiring that it provide the centripetal acceleration.

$$a_{\text{centripetal}} = \frac{v^2}{l} \;\Rightarrow\; N = \frac{mv^2}{l}$$

We can now write the equation of motion for the particle by Newton's second law

$$m\frac{dv}{dt} = -\mu N = -\frac{\mu m v^2}{l} \;\Rightarrow\; \frac{dv}{dt} = -\frac{\mu}{l}v^2$$

This equation can be solved by direct integration, as you saw in problem 2:

$$\frac{dv}{v^2} = -\frac{\mu}{l} \;\Rightarrow\; \int_{v_0}^{v(t)} \frac{dv}{v^2} = -\int_0^t \frac{\mu\, dt}{l}$$
$$\Rightarrow\; \frac{1}{v_0} - \frac{1}{v(t)} = -\frac{\mu t}{l}$$

This result can be simplified to get K&K's result

$$v(t) = v_0\left(1 + \frac{\mu v_0 t}{l}\right)^{-1}$$

(b.) Now that we know the velocity of the block, finding the position is easy. It is easiest to describe the position in terms of the angle on the circle. We can easily determine the angular velocity as a function of time, because $\omega(t) = v(t)/l$. We know the velocity, so to get the position we just integrate. Assume that $\theta = 0$ at $t = 0$, which also means $x = l,\ y = 0$

$$\theta(t) = \int_0^t \omega(t')dt' = \int_0^t \frac{v_0}{l}\Big(\frac{1}{1+\mu v_0 t'/l}\Big)dt'$$
$$= \frac{1}{\mu}\ln\Big(1 + \frac{\mu v_0}{l}t'\Big)_0^t$$

The final result for the total angle traveled is

$$\theta(t) = \frac{1}{\mu}\ln\Big(1 + \frac{\mu v_0}{l}t\Big)$$

Notice that the total distance traveled is infinite if one waits for an infinite time, even though the velocity approaches zero as time increases. We can of course write the x and y coordinates of the block as functions of time:

$$x(t) = l\cos(\theta(t)) \qquad\qquad y(t) = l\sin(\theta(t))$$

7. The two skaters each have mass 70 kg. Skater A carries a 10 kg bowling ball. Initially each skater is moving at 1 m/sec, and they are approaching each other. They are going to try to avoid collision by throwing the bowling ball back and forth.

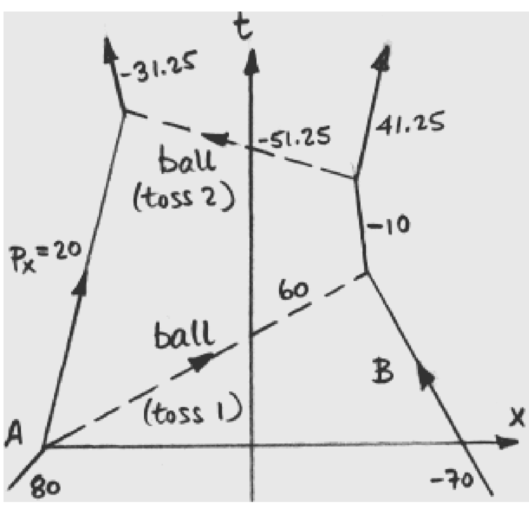
This problem uses conservation of momentum. Skater A starts out with $p_A = (70+10) \times 1 = 80$ kg-m/sec of momentum. Notice that we must include the momentum of the bowling ball in the momentum of skater A when he is carrying it. This adds 10 kg-m/sec to skater A's 70 kg-m/sec, since the bowling ball has a mass of $m = 10$ kg. Skater B is going in the opposite direction, so her momentum is negative, $p_B = -70$ kg-m/sec. Skater A throws the bowling ball to skater B in an attempt to stop the collision. Since there are no external forces on the system consisting of skater A and the bowling ball, the total momentum of these two objects is conserved. Throwing the bowling ball at 5 m/sec relative to the (initial) velocity of skater A gives it a momentum of 60 kg-m/sec. This is because the bowling ball velocity is 6 m/sec when we add the initial velocity of 1 m/sec. Skater A is left with a momentum of $p_A = 20$ kg-m/sec, so he hasn't reversed direction or stopped.

Next we consider the system of skater B and the bowling ball. Again there are no external forces, so the momentum of skater B plus the bowling ball is conserved. The total momentum is 60 kg-m/sec from the ball and -70

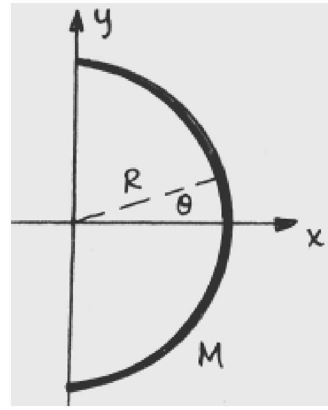
kg-m/sec from skater B. When skater B catches the ball, she will then have all of this momentum, $p_B = -10$ kg-m/sec. After this exchange, skater B has the ball, and the two skaters are still approaching each other. One toss was not enough. To summarize the first toss

$$\begin{aligned} \text{initial } p_A &= 80 \;\; p_B = -70 \\ \text{intermediate } p_A &= 20 \;\; p_B = -70 \;\; p_{\text{ball}} = 60 \\ \text{final } p_A &= 20 \;\; p_B = -10 \end{aligned}$$

The second toss will be enough to stop the collision. We calculate the velocity of skater B (including the bowling ball): $v = p/m = -0.125$ m/sec. Now skater B throws the bowling ball to skater A. The bowling ball's velocity will be -5.125 m/sec, so its momentum will be -51.25 kg-m/sec. This leaves skater B with $p_B = 41.25$ kg-m/sec. This is in the opposite direction to her initial motion. Skater A gets all of the momentum of the ball again, so $p_A = -31.25$ kg-m/sec. This is also opposite to his initial direction. So, after two tosses, the skaters are moving away from each other and the collision is averted. Plotting position versus time for the two skaters, we get a graph like the following:



8. This is an example of a center of mass calculation. Let's put the hoop on a polar coordinate system so that it goes from $\theta = (-\pi/2, \pi/2)$. In cartesian coordinates this is on the right half-plane.



It is fairly obvious that the center of mass lies on the line $Y_{\text{CM}} = 0$, or $\theta = 0$. The center of mass is calculated by the following

$$X_{\text{CM}} = \frac{1}{M} \int x\lambda\, dl$$

where λ is the linear mass density and dl is a differential of length on the hoop. If the hoop has mass M, the linear mass density is $\lambda = M/\pi R$. We can use polar coordinates to integrate this. Remember that $x = R\cos\theta$ for points on the hoop. The differential of length on the hoop is $dl = Rd\theta$, so the integral we need to do is

$$X_{\text{CM}} = \frac{1}{M} \int_{-\pi/2}^{\pi/2} \frac{RM}{\pi} \cos\theta\, d\theta = \frac{2}{\pi} R$$

University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

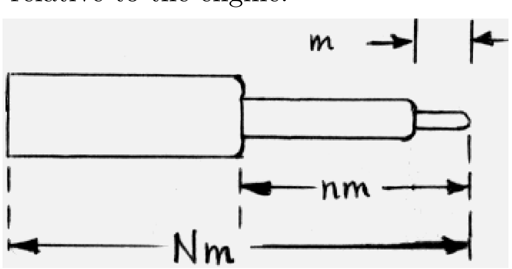
PROBLEM SET 4

1. A very flexible uniform chain of mass M and length L is suspended from one end so that it hangs vertically, the lower end just touching the surface of a table. The upper end is suddenly released so that the chain falls onto the table and coils up in a small heap, each link coming to rest the instant that it strikes the table. Find the force exerted by the table on the chain at any instant, in terms of the weight of the chain already on the table at that moment.

2. A supersonic plane of mass M has an air-speed $v = 1000$ m/sec. Its jet engine takes in 80 kg of air per sec, mixes it with 30 kg of fuel per sec, and compresses the mixture so that it ignites. The resulting hot gasses leave the engine with velocity 3000 m/sec relative to the plane. What thrust (force) does the engine deliver?

3. K&K problem 3.13 "A ski tow consists of...".

4. This problem is designed to illustrate the advantage that can be obtained by the use of multiple-staged instead of single-staged rockets as launching vehicles. Suppose that the payload (*e.g.* a space capsule) has mass m and is mounted on a two-stage rocket (see figure). The total mass – both rockets fully fueled, plus the payload – is Nm. The mass of the second-stage rocket plus the payload, after first-stage burnout and separation, is nm. In each stage the ratio of burnout mass (casing) to initial mass (casing plus fuel) is r, and the exhaust speed is V, constant relative to the engine.



(a.) Show that the velocity v gained from the first-stage burn, starting from rest and ignoring gravity, is given by

$$v = V \ln \frac{N}{rN + n(1-r)}.$$

(b.) Obtain a corresponding expression for the additional velocity u gained from the second stage burn.

(c.) Adding v and u, you have the payload velocity w in terms of N, n, and r. Taking N and r as constants, find the value of n for which w is a maximum.

(d.) Show that the condition for w to be a maximum corresponds to having equal gains of velocity in the two stages. Find the maximum value of w, and verify that it makes sense for the limiting cases described by $r = 0$ and $r = 1$.

(e.) Find an expression for the payload velocity of a single-stage rocket with the same values of N, r, and V.

(f.) Suppose that it is desired to obtain a payload velocity of 10 km/sec, using rockets for which $V = 2.5$ km/sec and $r = 0.1$. Show that the job can be done with a two-stage rocket but is impossible, however large the value of N, with a single-stage rocket.

5. A boat of mass M and length L is floating in the water, stationary; a man of mass m is sitting at the bow. The man stands up, walks to the stern of the boat, and sits down again.

(a.) If the water is assumed to offer no resistance at all to motion of the boat, how far does the boat move as a result of the man's trip from bow to stern?

(b.) More realistically, assume that the water offers a viscous resistive force given by $-kv$, where k is a constant and v is the velocity of the boat. Show that in this case one has the remarkable result that the boat should eventually return to its initial position!

(c.) Consider the paradox presented by the fact that, according to (b.), any nonzero value of k, however, small, implies that the boat ends up at its starting point, but a strictly zero value of k implies that it ends up somewhere else. How do you explain this discontinuous jump in the final position when the variation of k can be imagined as continuous, down to zero? For details, see D. Tilley, *American Journal of Physics* Vol. **35**, p. 546 (1967).

6. The Great Pyramid of Gizeh when first erected (it has since lost a certain amount of its outermost layer) was about 150 m high and had a square base of edge length 230 m. It is effectively a solid block of stone of density about 2.5 g/cc.

(a.) What is the minimum amount of work required to assemble the pyramid, if the stone is initially at ground level?

(b.) Assume that a slave employed in the construction of the pyramid had a food intake of about 1500 Cal/day (1 Cal = 4182 joules). The Greek historian Herodotus reported that the job took 100,000 slaves 20 years. What was the minimum efficiency of a slave (defined as work done divided by energy consumed)?

7. A particle of mass m, at rest at t=0, is subjected to a force $\mathbf{f}(t)$ whose magnitude at t=0 is F. This magnitude decreases linearly with time, becoming zero at time $t = T$. The direction of the force remains unchanged. What is the kinetic energy of the particle at time T?

8. A wooden block of mass M, initially at rest on a table with coefficient of sliding friction μ, is struck by a bullet of mass m and velocity v. The bullet lodges in the center of the block. How far does the block slide?

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SOLUTION TO PROBLEM SET 4

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. A chain of mass M and length L falls onto a table. Initially, the chain is hanging so that its lower end just touches the table. The chain is falling in gravity, so the velocity of a link that is falling is given by $v = gt$. The distance that the chain has fallen is given by $x = gt^2/2$. These two facts tell us how much of the chain is on the table at a given time. The density of the chain is M/L, so the mass of chain on the table is just Mx/L:

$$M(t) = \frac{1}{2}\frac{M}{L}gt^2$$

The rate at which the mass is falling on the table is just

$$\frac{d}{dt}M(t) = \frac{M}{L}gt = \frac{M}{L}v(t)$$

At time t, the free elements of chain are moving with speed $v(t)$. This is the velocity they have when they hit the table. The total rate at which momentum is being transferred is

$$\frac{dp}{dt} = v(t)\frac{d}{dt}M(t) = \frac{M}{L}v^2(t)$$

Writing this in terms of $M(t)$, the mass on the table at time t, we get the following:

$$\frac{dp}{dt} = 2M(t)g$$

The rate of change of momentum should be familiar to you from Newton's second law which states

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

Thus the table must be exerting this force on the chain to slow it down. Remember also that the table exerts a normal force on the chain which is equal to the force of gravity

$$F_N = M(t)g$$

Thus the total force that the table exerts on the chain is three times the weight of the chain on the table:

$$F(t) = F_N(t) + 2M(t)g = 3M(t)g$$

2. The airspeed of a plane is $v = 1000$ m/sec. The engines take in 80 kg of air per second and mix it with 30 kg of fuel. The mixture is expelled after it ignites, and it is moving at a velocity of 3000 m/sec relative to the plane. We can calculate the thrust of this engine by calculating the rate of change of momentum. The fuel is ejected at a rate of 30 kg/sec, and it is given a velocity of 3000 m/sec relative to the plane. It started at rest with respect to the plane, so it need to be given the full velocity. The rate of change of momentum this corresponds to is

$$\frac{dp}{dt} = \frac{dm}{dt}v = 30 \times 3000 = 90,000 \text{ kg} - \text{m/sec}^2$$

The air also contributes to the momentum. It is expelled at a rate of 80 kg/sec. Its velocity is already 1000 m/sec relative to the plane, so it only needs to gain 2000 m/sec of velocity in the engine. The rate of change of momentum that this corresponds to is

$$\frac{dp}{dt} = \frac{dm}{dt}v = 80 \times 2000 = 160,000 \text{ kg} - \text{m/sec}^2$$

The total rate of momentum transferred to the exhaust by the plane's engine is thus

$$\frac{dp}{dt} = 250,000 \text{ kg} - \text{m/sec}^2$$

This rate of momentum transfer is equal to the thrust of the engine:

$$F_{\text{thrust}} = 250,000 \text{ N}$$

3. K&K problem 3.13 This problem concerns the total force that a ski lift must exert to lift skiers to the top of a hill. There will be two parts to the force. The first is just the force necessary to oppose the force of gravity on the skiers. The second is the force required to accelerate the skier at the bottom from rest to the speed of the lift. The rope is 100 meters long, and it is pulled at 1.5 meters per second. On average, one skier uses the tow rope every five seconds. This means the tow rope travels $5 \times 1.5 = 7.5$ meters between skiers, so $100/7.5 = 13\frac{1}{3}$ skiers are on the rope on average. Each skier weighs 70 kg, so the average total weight of the skiers who are on the rope is 933 kg. The component of the force of gravity that must be offset by the rope is determined by the angle of the slope, which is $20°$. The component of the acceleration of gravity that is directed down the slope is just $g \sin 20° = 0.342g$. Therefore the force that the tow rope must exert to offset that component of gravity is $933 \times g \times 0.342 = 3128$ N. In addition, when a skier grabs the rope, he must be accelerated to the speed of the rope, 1.5 m/sec. The change in momentum for the skier is $1.5 \times 70 = 105$ kg-m/sec. This change in momentum must be provided by the motor once every five seconds, which is how often skiers use the lift. On average, this force is 105/5=21 N. Therefore the total force that the lift must provide is, on average, 3128+21=3149 N.

4. A two stage rocket carries a payload of mass m. The total mass of the rocket is Nm, and the mass of the second stage and payload is nm. In each stage, the mass of the fuel is a fraction (1-r) of the total, so the mass of the casing is a fraction r of the total mass. The first stage has a mass $(N-n)m$,which is just the total minus the mass of the second stage.

(a.) Since gravity can be ignored, the equation for rocket motion derived in class reduces to

$$v(t) - v_0 = V \ln \frac{M_0}{M(t)} \ .$$

To determine the velocity gain v from the first burn, we need only to compute the mass of the rocket at the end of the burn. The initial mass is Nm, while the mass of fuel burned by the first stage is $(Nm - nm)$, the mass of the first stage, multiplied by $(1-r)$. The difference $m(n+r(N-n))$ of these two masses is the residual mass after the first burn. So the first burn velocity gain is

$$v = V \ln \frac{N}{n + r(N-n)} = V \ln \frac{N}{Nr + n(1-r)} \ .$$

(b.) The method for this part is the same as for part (a.) because the first equation guarantees that the velocity gain of a rocket is independent of its initial velocity. Here the initial mass is the full mass of the second stage, nm. The final mass is nm minus the mass of fuel consumed in the second burn, which is $(1-r)(nm-m)$. This yields $m(1+r(n-1))$ for the final mass, and a second burn velocity gain of

$$u = V \ln \frac{n}{1 + r(n-1)} = V \ln \frac{n}{nr + (1-r)} \ .$$

(c.) Here we optimize n with all other parameters fixed. We wish to maximize $v+u$. As V is fixed, we choose equivalently to minimize $Q = \ln(V/(v+u))$ in order to simplify the algebra. From (a.) and (b.) we have

$$Q = \frac{Nr + n(1-r)}{N} \frac{nr + (1-r)}{n} \ .$$

Carrying out the division,

$$Q = (r + \frac{n}{N}(1-r))(r + \frac{1}{n}(1-r)) \ .$$

Multiplying,

$$Q = r^2 + \frac{(1-r)^2}{N} + r(1-r)(\frac{n}{N} + \frac{1}{n}) \ .$$

Only the last term depends on n:

$$\frac{d}{dn}\left(\frac{n}{N} + \frac{1}{n}\right) = 0, \quad n = \sqrt{N} \ .$$

(d.) For this value of n, the velocity gains from the first and second burns are equal:

$$v + u = 2u = 2V \ln \frac{\sqrt{N}}{1 + r(\sqrt{N} - 1)} \ .$$

(e.) A single stage rocket has the same values of N, r, and V. The initial mass is Nm, as in part (a.), and the final mass is $m + r(Nm - m)$, in analogy to part (b.) with N substituted for n. The final velocity is

$$v = V \ln \frac{N}{Nr + (1-r)} \ .$$

(f.) We want the final velocity of the payload to be $v = 10$ km/sec, and we have a rocket with exhaust velocity $V = 2.5$ km/sec and $r = 0.1$. First let's see if this can be done with a single stage rocket. Plugging into the result from part (e.), we see that

$$10 = 2.5 \ln \frac{N}{0.9 + 0.1N}$$

We try to solve for the necessary N

$$\begin{aligned} e^4 &= \frac{N}{0.9 + 0.1N} \\ 5.46N + 49.1 &= N \\ N &= -11.0 \ . \end{aligned}$$

This answer doesn't make any sense, which means that a single stage rocket can't do the job. Let's now look at the optimal two stage rocket, using the result from part (d.):

$$10 = 5 \ln \frac{N}{0.1N + 0.9\sqrt{N}}$$

Again we try to solve for N:

$$\begin{aligned} e^2 &= \frac{N}{0.1N + 0.9\sqrt{N}} \\ 0.739N + 6.65\sqrt{N} &= N \\ 0.261\sqrt{N} &= 6.65 \\ N &= 650 \ . \end{aligned}$$

This rocket indeed can be built.

5. A boat of mass M and length L is floating at rest. A man of mass m is sitting at the stern. He stands up, walks to the bow and sits down again.

(a.) There is no force from the water, therefore the net force on the system is zero. The momentum of the system is conserved, and the center of mass remains at the same velocity, in this case zero. Centering the boat at $x = 0$, we can calculate the center of mass

$$X_{CM} = \frac{-(L/2)m}{M+m}$$

After the man is sitting at the bow, the center of the boat will be at some position x, which means that the man will be at a position $x + (L/2)$. However, the center of mass will be in the same place.

$$\begin{aligned} \frac{Mx + m(x + (L/2))}{M+m} &= \frac{-(L/2)m}{M+m} \\ x &= -\frac{mL}{M+m} \ . \end{aligned}$$

The boat has moved from its initial position.

(b.) This time, the water exerts a viscous force $F = -kv$ on the boat. We can show that the boat will always return to its original position. Newton's second law gives the following equation. We want to use the total mass of the boat plus the man, because we don't want the man accelerating relative to the boat

$$(m+M)\dot{v} = -kv \Rightarrow \int_{v_0}^{v(t)} \frac{dv}{v} = -\frac{k}{M+m} \int_{t_0}^{t} dt$$

This gives

$$v(t) = v_0 \exp\left(-\frac{k}{M+m}(t - t_0)\right)$$

The distance traveled in this interval is just the integral of the velocity

$$x(t) = \frac{(M+m)v_0}{k}\left(1 - e^{-\frac{k}{M+m}(t-t_0)}\right)$$

Now we just need to find the initial velocity of the boat. When the man starts moving, say he applies an impulse Δp. This is the same impulse that the boat must receive, but in the opposite direction. Thus, the velocity of the man is $u = \Delta p/m$ and the velocity of the boat is $v = -\Delta p/M$. This means that the velocity of the man relative to the boat is $u - v = (m+M)\Delta p/Mm$. The man is now walking at constant speed relative to the boat. We plug in the initial velocity of the boat

$v = -\Delta p/M$ to the solution of the differential equation and we find the velocity of the boat

$$v(t) = -\frac{\Delta p}{M} \exp\left(-\frac{k}{M+m}(t-t_0)\right)$$

At time $\tau = L/(u-v) = LMm/(M+m)\Delta p$, the man has reached the other end of the boat. The velocity of the boat is

$$v(\tau) = -\frac{\Delta p}{M} \exp\left(-\frac{k}{M+m}\tau\right)$$

and it has traveled a distance

$$x(\tau) = -\frac{(M+m)\Delta p}{kM}\left(1 - \exp\left(-\frac{k}{M+m}\tau\right)\right)$$

He again applies an impulse, but this time it is $-\Delta p$. This gives the boat a change in velocity of $+\Delta p/M$. The total velocity of the boat is now

$$v(\tau) = \frac{\Delta p}{M}\left(1 - \exp\left(-\frac{k}{M+m}\tau\right)\right)$$

Using this as the initial velocity, we again solve the differential equation

$$v(t) = \frac{\Delta p}{M}\left(1 - e^{-\frac{k}{M+m}\tau}\right) e^{-\frac{k}{M+m}t}$$

This is correct for all $t > \tau$. We now calculate the total distance traveled in the second part of the trip. We take the final time to be $t = \infty$.

$$x(\infty) = \frac{(M+m)\Delta p}{kM}\left(1 - \exp\left(-\frac{k}{M+m}\tau\right)\right)$$

This is exactly the opposite of the distance traveled in the first part. Thus the boat will eventually return to its starting point.

(b′.) Here is a quick, elegant way to prove the result of part (b.). It deserves full grading credit. We do not mention only this method because, as seen above, the problem is amenable to solution by systematic calculation as well as brilliant insight.

Consider the impulse applied by the force F_{ext} of the water on the boat. To specify the impulse, which is the time integral of F_{ext}, we must specify the time interval. We choose the interval from $t = 0-$, just before any motion starts, to $t = \infty$, at which time all motion must have stopped due to effects of viscosity. At both of those times the total momentum of the boat+man system, whose rate of change is controlled by F_{ext}, is zero. Therefore the impulse in question, which is equal to the difference $P(\infty) - P(0-)$ of the boat+man system, must vanish.

The same impulse can also be written as

$$\begin{aligned} 0 &= \int_{0-}^{\infty} F_{\text{ext}} dt \\ &= -k \int_{0-}^{\infty} \frac{dx}{dt} dt = -k \int_{0-}^{\infty} dx \\ &= -k(x(\infty) - x(0-)) , \end{aligned}$$

where x is the position of the boat. This proves that the boat returns to its original position.

(c.) The result of part (b.) says that any viscous force, no matter how small, results in the boat returning to its original location. The result of part (a.) says that when there is no viscous force, the boat moves some distance. Mathematically, the difference between the two results is due to the order in which the limits are taken. In part (a.), the first thing done is to take the limit as $k \to 0$, no viscous force. Then the limit as $t \to \infty$ is taken. If we look at the result of part (b.), we first take the limit as $t \to \infty$, then we consider what happens when there is no viscous force. This is an instance in which we cannot reverse the order of taking limits. Denoting the results from part (a.) and (b.) by capital letters, we see that

$$\lim_{t\to\infty} \lim_{k\to 0} A \neq \lim_{k\to 0} \lim_{t\to\infty} B$$

So much for the reason why, mathematically, the results of (a.) and (b.) are not the same. Physically, they are not in conflict. As the coefficient k approaches zero in part (b.), the speed with which the boat ultimately migrates back to its original position approaches zero also. This cannot be distinguished by physical measurement from the limiting case (a).

6. The Great Pyramid at Gizeh is h=150 m high and has a square base of side $s = 230$ m. It has a density $\rho = 2.5$ g/cc.

(a.) If all the stone is initially at ground level, it must be raised to its position in the pyramid. The work required to this is

$$W = Mgh_{cm} = \int \rho g z \, dV$$

The volume element is the area of the square at a height z times dz, the differential of height. The square has side s at $z = 0$ and side 0 at $z = h$. The width of the square decreases linearly with height, so the width and area at height z is given by

$$w(z) = s\left(1 - \frac{z}{h}\right) \qquad A(z) = s^2\left(1 - \frac{z}{h}\right)^2$$

The volume element dV is given by $dV = A(z)\,dz$. We can now perform the integral. Expanding the polynomial in z

$$W = \rho g s^2 \int_0^h \left(z - 2\frac{z^2}{h} + \frac{z^3}{h^2}\right) dz$$

This is a simple integral to perform:

$$W = \rho g s^2 h^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{12}\rho g s^2 h^2$$

Plugging in the values for these constants, we get the amount of work required to erect the pyramid

$$W = 2.43 \times 10^{12} \text{ Joules}$$

(b.) The slaves employed in building this pyramid consumed 1500 Calories per day, which is 6.3×10^6 joules per day. With 100,000 slaves working for 20 years, this is 730 million slave-days of work to build the pyramid. The total energy the slaves spent is thus 4.6×10^{15} joules. The efficiency thus implied is low, $\epsilon = 5.3 \times 10^{-4}$. This does not necessarily reflect a low intrinsic efficiency, since the slaves undoubtedly expended most of their energy on activities other than lifting the stone blocks to their final position.

7. A force $\mathbf{f}$(t) has magnitude F at $t = 0$, magnitude 0 at $t = T$, and it decreases linearly with time. The direction remains the same. The magnitude of the force is thus

$$f(t) = F\left(1 - \frac{t}{T}\right)$$

The force acts on a particle of mass m initially at rest. The kinetic energy at $t = T$ is just the integral

$$K = \int_0^T F\left(1 - \frac{t}{T}\right) dx = \int_0^T F\left(1 - \frac{t}{T}\right) v\, dt$$

We can find v by applying Newton's second law, but once we have it, we don't need to do the integral because we know that $K = mv^2/2$

$$f(t) = m\frac{dv}{dt} = F\left(1 - \frac{t}{T}\right)$$

We can just directly integrate both sides with respect to t, with limits $t = 0$ and $t = T$

$$v(T) = \frac{F}{2m}T$$

We now have the answer

$$K = \frac{1}{8}\frac{F^2T^2}{m}$$

8. Instantaneously after the collision of the bullet and block, after the bullet has come to rest but before the frictional force on the block has had time to slow it down more than an infintesimal amount, we can apply momentum conservation to the bullet-block collision. At that time the total momentum of the block+bullet system is $(M + m)v_0'$, where v_0' is the velocity of the block+bullet system immediately after the collision. Momentum conservation requires that momentum to be equal to the initial momentum mv of the bullet. Thus

$$v_0' = \frac{mv}{M + m} \; .$$

After the collision, the normal force on the block+bullet system from the table is $(M + m)g$, giving rise to a frictional force

$$\mu N = \mu(M + m)g$$

on the sliding block+bullet system. This causes a constant acceleration μg of that system opposite to its velocity.

Take $t = 0$ at the time of collision. Afterward, the block+bullet system's velocity in the horizontal direction will be $v'(t) = v'_0 - \mu g t$. It will continue sliding until $v'(t) = 0$, at which point the frictional force will disappear and it will remain at rest. Solving, the time at which the block-bullet system stops is

$$t = v'_0/(\mu g) \ .$$

The distance traveled in that time is

$$x = v'_0 t - \frac{1}{2}\mu g t^2 = \frac{1}{2} v'_0 t = \frac{(v'_0)^2}{2\mu g} \ .$$

Plugging in the already deduced value for v'_0, this distance is

$$x = \left(\frac{m}{M+m}\right)^2 \frac{v^2}{2\mu g} \ .$$

University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

PROBLEM SET 5

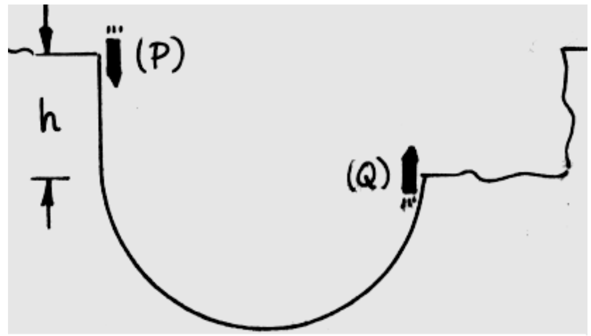
1. Mass M rests on a table, and mass m is supported above it by a massless spring which connects the two masses.

(a.) Find the minimum downward force that must be exerted on m such that the entire assembly will barely leave the table when this force is suddenly removed.

(b.) Consider this problem in the time-reversed situation: Let the assembly be supported above the table by supports attached to m. Lower the system until M barely touches the table and then release the supports. How far will m drop before coming to a stop? Does knowledge of this distance help you solve the original problem?

(c.) Now that you have the answer, check it against your intuition by (1) letting M be zero and (2) letting m be zero. Especially in the second case, does the theoretical result agree with your common sense? If not, discuss possible sources of error.

2. It has been claimed that a rocket would rise to a greater height if, instead of being ignited at ground level (P), it were ignited at a lower level (Q) after it had been allowed to slide from rest along a frictionless chute – see the figure. To analyze this claim, consider a simplified model in which the body of the rocket is represented by a mass M, the fuel is represented by a mass m, and the chemical energy released in the burning of the fuel is represented by a compressed spring between M and m which stores a definite amount of potential energy, U, sufficient to eject m. (This corresponds to instantaneous burning and ejection of all the fuel – *i.e.* an explosion.) Then proceed as follows:



(a.) Assuming a value of g independent of height, calculate how high the rocket would rise if fired directly upward from rest at (P).

(b.) Let (Q) be a distance h vertically lower than (P), and suppose that the rocket is fired at (Q) after sliding down the frictionless chute. What is the velocity of the rocket at (Q) just before the spring is released? Just after the spring is released?

(c.) To what height above (P) will the rocket rise now? Is this higher than the earlier case? By how much?

(d.) Remembering energy conservation, can you answer a skeptic who claims that someone has been cheated of some energy?

3. K&K problem 4.7 "A ring of mass M hangs...".

4. Assume the moon to be a sphere of uniform density with radius 1740 km and mass 7.3×10^{22} kg. Imagine that a straight smooth tunnel is bored though the moon so as to connect *any* two points on its surface. The gravitational force on an object by a uniform sphere is equal to the force that would be exerted by the fraction of the sphere's mass which lies at *smaller* radius than the object, as if that fraction were concentrated at the center of the sphere.

(a.) Show that the motion of objects along this tunnel under the action of gravity would be simple harmonic (neglect friction with the walls of the tunnel).

(b.) Calculate the period of oscillation.

(c.) Compare this with the period of a satellite travelling around the moon in a circular orbit at the moon's surface.

5. K&K problem 4.23 "A small ball of mass m is placed on top...".

6. In an historic piece of research, James Chadwick in 1932 obtained a value for the mass of the neutron by studying elastic collisions of fast neutrons with nuclei of hydrogen and nitrogen. He found that the maximum recoil velocity of hydrogen nuclei (initially stationary) was 3.3×10^7 m/sec, and that the maximum recoil velocity of nitrogen-14 nuclei was 4.7×10^6 m/sec with an uncertainty on the latter of $\pm10\%$. What does this tell you about:

(a.) The mass of the neutron (in amu)?

(b.) The initial velocity of the neutrons used?

(Take the uncertainty of the nitrogen measurement into account. Take the mass of a hydrogen nucleus as 1 amu and the mass of a nitrogen-14 nucleus as 14 amu.)

7. K&K problem 4.13 "A commonly used potential energy function...".

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Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 5

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. A mass M rests on a table, and a mass m is supported on top of it by a massless spring connecting it to M.

(a.) We want to find the force F needed to push down on the spring so that the whole system will barely leave the table. We solve this by conservation of energy. The energy stored in a spring is given by $kx^2/2$ where x is the displacement from equilibrium of the spring. We can measure the gravitational potential of the small mass relative to the equilibrium point of the spring. Initially then, the spring is compressed with a force $F + mg$ which is just the weight of the small mass plus the added force. Hooke's law tells us that $x = -(F + mg)/k$. We can now write the initial energy of the system

$$E_i = -\frac{mg}{k}(F + mg) + \frac{(F + mg)^2}{2k}$$

The first term is the gravitational energy relative to the equilibrium point of the spring, and the second term is the energy stored in the spring. We want the spring to be able to lift the mass M off the table. To do this it must apply a force equal to Mg, its weight. When the spring is released, it will oscillate. At the top of the oscillation, there will be no kinetic energy. The displacement y of the spring must barely provide the force to lift the lower block: $ky = Mg$. The energy here is the following

$$E_f = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

Conservation of energy tells us that these are the same, so now we can solve for F.

$$-\frac{mg}{k}(F + mg) + \frac{(F + mg)^2}{2k} = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

Cancelling k and using the quadratic formula to solve for $F + mg$,

$$F + mg = \left(mg \pm \sqrt{m^2g^2 + 2Mmg^2 + M^2g^2}\right)$$

The expression under the square root is just $(M + m)g$, so the expression simplifies a lot:

$$F = \pm(M + m)g$$

We obviously want the plus sign.

(b.) This is a similar situation. The mass M is dangling and barely touching the table. The displacement of the spring just supports the weight of the block so $kx = Mg$. At the other end, the small mass is momentarily at rest at some distance $-y$ from equilibrium. The energies are

$$E_i = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

$$E_f = -mgy + \frac{ky^2}{2}$$

We equate these energies and solve for y.

$$-mgy + \frac{ky^2}{2} = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

Using the quadratic formula again

$$ky = mg \pm \sqrt{m^2g^2 + 2Mmg^2 + M^2g^2}$$

Again the discriminant is a perfect square, and we want the positive value of y, so

$$y = \frac{(2m + M)g}{k}$$

The total distance that the mass falls is $x+y = d$.

$$d = 2\frac{(M + m)g}{k}$$

This is just twice the displacement caused by the force in part (a.), which makes sense, because the displacement upwards should be the same as the displacement downwards.

(c.) (1) When M is zero, the necessary applied force is mg. This is just the weight of the small mass. The spring will bounce back with the same force, so this is what is needed to lift the whole assembly. The distance fallen makes sense also, because the spring starts at its equilibrium length. The mass wants to sit at mg/k below this, to just support the weight. Thus it will oscillate down to $2mg/k$ below this point.

(2) When m is zero, the force needed to move the assembly is Mg; again this is the total weight. The distance traveled by the end of the spring in the second case is just $2Mg/k$. The end of the spring is Mg/k away from equilibrium when it begins, so the total distance traveled by the end is $2Mg/k$. While this seems to work out, it does not necessarily agree with common sense: a massless spring would not seem to be able to pull a massive block off the table by virtue of its own motion. However, we realize that, as the spring mass approaches zero in this idealization, its maximum velocity approaches infinity. This explains why the spring is still able to pull the block off the table, defying our intuition.

2.

(a.) The rocket is fired directly upward from the ground. The initial energy is just U, the energy in the fuel. After the fuel is spent, the fuel mass m is moving down at speed u and the remaining rocket mass M is moving upwards at speed v. Because momentum is conserved over this very short time, $mu = Mv$. The energy of the system is given by conservation of energy, and at launch, all of the energy is kinetic:

$$U = \frac{1}{2}Mv^2 + \frac{1}{2}mu^2 = \frac{Mv^2}{2}\left(1 + \frac{M}{m}\right)$$

Now we want to consider the motion of the mass M alone. Its kinetic energy is $Mv^2/2$, which we can find from the previous equation.

$$K_M = \frac{U}{1 + M/m}$$

Energy for the mass M is now conserved, so we can just set $K_M = Mgd$, where d is the maximum height achieved by the rocket. This gives the answer to part (a.):

$$d = \frac{U}{Mg}\frac{1}{1 + M/m}$$

(b.) This is a little more involved. The rocket has gone around part of an oval track and is now a distance h below where it started. The gravitational energy $(M + m)gh$ gets converted to kinetic energy, so we get the velocity v_0 of the rocket before the fuel is used:

$$(M + m)gh = \frac{1}{2}(M + m)v_0^2 \;\Rightarrow\; v_0 = \sqrt{2gh}$$

Ignoring for the moment the gravitational energy, the energy of the rocket at this point is

$$E_i = U + \frac{1}{2}(M + m)v_0^2 = U + (M + m)gh$$

The spring (fuel) imparts a change in velocity Δv to M and Δu to m. As in part (a.), instantaneous conservation of momentum gives $M\Delta v = m\Delta u$. After the spring is released, the energy corresponding to E_i is

$$E_f = \frac{1}{2}M(v_0 + \Delta v)^2 + \frac{1}{2}m(v_0 - \Delta u)^2$$

We know by conservation of energy that $E_i = E_f$, and we have $M\Delta v = m\Delta u$, so we can find Δv:

$$\Delta v = \sqrt{\frac{2U}{M(1 + M/m)}}$$

The total velocity of M is now $v = v_0 + \Delta v$:

$$v = \sqrt{2gh} + \sqrt{\frac{2U}{M(1 + M/m)}}$$

(c.) We can easily find the kinetic energy of the remaining rocket, and, using energy conservation, the maximum height H to which it rises above its current position:

$$K = \frac{1}{2}M(v_0 + \Delta v)^2 = MgH$$

Using $v_0 = \sqrt{2gh}$, we can solve for H:

$$H = \frac{2gh + \Delta v\sqrt{2gh} + \Delta v^2}{2g}$$

Plugging in the result for Δv, we arrive at the final answer

$$H = h + \frac{U}{Mg(1+M/m)} + \sqrt{\frac{hU}{Mg(1+M/m)}}$$

This is an interesting result. The first term just gets the rocket back to the height where it started in the first place. The second term gets it to the maximum height of the rocket in part (a.). The fact that the third term is positive means that the rocket actually flies higher in this case. The gain in height is just the third term

$$\Delta H = \sqrt{\frac{hU}{Mg(1+M/m)}}$$

(d.) This result does not conflict with energy conservation, which says only that the total energy of the system is conserved. We have been neglecting what happens to the mass m, which will take away a smaller amount of energy in the second case. If we looked at the total energy of both pieces, it would be conserved.

3. K&K problem 4.7. This problem is one in which both force and energy need to be considered. The forces on the ring are gravity, the tension in the thread T, and the normal forces due to the beads. The forces on the ring in the vertical direction are

$$F_{\text{ring}} = T - Mg - 2N(\theta)\cos\theta$$

where θ is the angle of the bead's position from the top, and $N(\theta)$ is taken to be positive outward. The two beads will move symmetrically. We now need to find the normal force $N(\theta)$. First we determine the velocity of the bead from conservation of energy. It yields the following:

$$E/2 = mgL(\cos\theta - 1) + \frac{1}{2}mv^2(\theta) = 0$$

This gives the velocity, and thus the centripetal acceleration a_c, as a function of θ:

$$v^2(\theta) = 2gL(1-\cos\theta) \;\Rightarrow\; a_c = 2g(1-\cos\theta)$$

The centripetal acceleration is provided by gravity and the normal force. Since a positive normal force is outward, at the top the normal force will be negative. The radial equation of motion will be

$$ma_c = N + mg\cos\theta \Rightarrow N(\theta) = mg(2 - 3\cos\theta)$$

Now we can go back to the force equation for the ring and use this result. The total force on the ring will be zero, but the ring will just start to move upwards when the thread is slack, which is when $T = 0$. Using both of these facts, we get an equation for θ

$$Mg = 2mg\cos\theta(3\cos\theta - 2)$$

This is just a quadratic equation in $\cos\theta$. Multiplying it out, we can apply the quadratic formula to get the answer

$$\cos\theta = \frac{1}{3} \pm \frac{1}{3}\sqrt{1 - \frac{3M}{2m}}$$

There is a small problem here in that the discriminant can be negative, making the cosine of the angle complex. This of course is unphysical. The problem is that for sufficiently small m, the motion of the small masses is never important enough to cause the tension in the rope to vanish, so our calculation is wrong from the start. Insisting that $\cos\theta$ be real, we obtain the condition

$$m > \frac{3}{2}M$$

Taking the positive root, the final answer is

$$\theta = \cos^{-1}\left(\frac{1}{3} + \frac{1}{3}\sqrt{1 - \frac{3M}{2m}}\right)$$

4. We assume that the moon is a uniform sphere of mass $M = 7.3 \times 10^{22}$ kg and radius $R = 1740$ km. A straight, frictionless tunnel connects two points on the surface. Given the mass and radius, the density is just $\rho = 3M/4\pi R^3$. We need to know the acceleration due to gravity at a distance r from the center of the moon. This is also straightforward. Recall that a spherical shell of mass exerts no force on objects inside it, so at a radius r, the only force we need to consider is due to the mass in the moon interior to radius r. This is just the density times the volume interior to r, or $\mathcal{M}(r) = Mr^3/R^3$. The acceleration due to gravity is then just $g(r) = -G\mathcal{M}(r)/r^2 = -GMr/R^3$. Thus the acceleration due to gravity increases linearly as one moves away from the center of a uniform solid sphere.

(a.) In a spherical polar coordinate system with its $\hat{\mathbf{z}}$ axis at the moon's north pole, assume that the tunnel lies in a straight line between $(r, \theta_0, \phi) = (R, \theta_0, 0)$ and (R, θ_0, π), *i.e.* between two points at the same north latitude $\pi/2 - \theta_0$ having the largest possible difference in longitude. This means that the distance along a great circle between the ends of the tunnel is $2\theta_0$, while the distance from the center of the moon to the center of the tunnel is $z_0 = R\cos\theta_0$. Now assume that the mass makes an angle ψ $(-\frac{\pi}{2} < \psi < \frac{\pi}{2})$ with a line connecting the center of the moon and the center of the tunnel, *i.e.* with the $\hat{\mathbf{z}}$ axis. The distance of the mass from the center of the tunnel is then $x = z_0 \tan\psi$, while its distance from the center of the moon is $r = z_0/\cos\psi$. We now need to know the component F_x of the gravitational force $-GMm\mathbf{r}/R^3$ which lies in the $(\hat{\mathbf{x}})$ direction of the tunnel, which makes an angle $\pi/2 - \psi$ with the radial direction. This is

$$\begin{aligned} F_x &= -\frac{GMmr}{R^3}\cos(\pi/2 - \psi) \\ &= -\frac{GMm}{R^3}\frac{z_0}{\cos\psi}\sin\psi \\ &= -\frac{GMm}{R^3}x \end{aligned}$$

This is like the force from a Hooke's law spring with effective spring constant $k_{\text{eff}} = GMm/R^3$, yielding simple harmonic oscillation with resonant angular frequency

$$\omega_0 = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{GM}{R^3}}$$

(b.) Plugging in values of M and R for the moon, and using $T = 2\pi/\omega_0$, we get for the period of oscillation

$$T = 6536 \text{ seconds } = \text{ 109 minutes}$$

(c.) A satellite traveling in a circular orbit must have centripetal acceleration provided by gravity, which means that

$$\frac{v^2}{R} = \frac{GM}{R^2} = \omega^2 R$$

From the last equality we see that the angular frequency ω of a circular orbit of radius R around the moon is the same as ω_0 above. Of course the period is the same as well.

5. K&K problem 4.23. Two balls of masses M and m are dropped from height h and collide elastically. The small ball is on top of the larger ball. Conservation of energy for the system gives its speed v right before the balls hit the ground:

$$(M+m)gh = \frac{1}{2}(M+m)v^2 \;\Rightarrow\; v = \sqrt{2gh}$$

The ball M collides with the ground first. In order to conserve energy, it must still have speed v instantaneously after it bounces from the ground. Now it immediately collides with the small ball. (Think of this problem as if there were a very small gap between the two balls so that the first ball to hit the ground has a chance to bounce before the second one hits it.) We consider the elastic collision between the two balls, each moving at speed v towards the other.

The easiest frame in which to study this collision is a comoving (inertial) frame that is instantaneously at rest with respect to the large ball M immediately after it has rebounded with velocity $v = \sqrt{2gh}$ from its elastic collision with the ground. In this frame, M is instantaneously at rest, and m has (upward) velocity $-2v$. When the collision occurs, if $m \ll M$ as stated in the problem, M seems to m like a "brick wall" from which it bounces back elastically with the same speed. Thus, in the comoving frame immediately after the collision, m has velocity $+2v$. Finally, transforming back to the lab frame, m acquires an extra velocity increment v, for a total of $3v$. Since the height that m reaches is proportional to the square of its velocity, this means that m reaches nine times the height from which it originally was dropped.

A less elegant approach considers the collision between M and m in the lab frame. Here is it essential not to apply the approximation $m \ll M$ until near the end, since cancellations occur which may make nonleading terms more important than would initially be suspected.

In the lab frame, conservation of momentum gives

$$Mv - mv = MV_M + mV_m$$

We also have conservation of energy through this collision. This condition gives

$$\frac{1}{2}(M+m)v^2 = \frac{1}{2}MV_M^2 + \frac{1}{2}mV_m^2$$

These are two equations in the two unknowns V_m and V_M, since we already know $v = \sqrt{2gh}$. We are interested in V_m, which yields the desired final height $V_m^2/2g$ of m, but we are not interested in V_M. So we plan to eliminate V_M by solving for it using the first equation and then substituting for it in the second.

Before proceeding with this algebra, it is convenient to substitute

$$\begin{aligned} \epsilon &= m/M \\ u &= V_m/v \\ U &= V_M/v \end{aligned}$$

so that all terms are dimensionless. The two equations above become

$$\begin{aligned} 1 - \epsilon &= U + \epsilon u \\ 1 + \epsilon &= U^2 + \epsilon u^2 \end{aligned}$$

Solving the first equation for U,

$$U = 1 - \epsilon - \epsilon u$$

Substituting this value for U in the second equation,

$$\begin{aligned} 1 + \epsilon &= 1 - 2\epsilon + \epsilon^2 - 2\epsilon u + 2\epsilon^2 u + \epsilon^2 u^2 + \epsilon u^2 \\ 0 &= \epsilon(1+\epsilon)u^2 - 2\epsilon(1-\epsilon)u - \epsilon(3-\epsilon) \\ 0 &= u^2 - 2\frac{1-\epsilon}{1+\epsilon}u - \frac{3-\epsilon}{1+\epsilon} \end{aligned}$$

Neglecting ϵ with respect to 3 or 1 in both quotients, the polynomial is

$$u^2 - 2u - 3 = (u-3)(u+1)$$

with the physical solution

$$\begin{aligned} u &= 3 \\ V_m &= 3v = \sqrt{18gh} \\ h' &= \frac{V_m^2}{2g} = 9h \end{aligned}$$

as before.

6. This is a collision problem that has different unknown quantities than those to which you are accustomed, but it is still solvable. We have two collisions to study, and the unknowns are the neutron mass and the initial and final speeds of the neutrons. The initial speeds are the same, so there are four unknowns in total. We have two collisions, each of which yields two equations (one for momentum conservation, one for energy conservation since the collisions are elastic). Therefore the system can be solved uniquely. The directions of the scattered neutrons relative to the incident directions do not represent additional unknowns, since the maximum recoil velocities of the target nucleii will occur when the collisions take place head-on, with the incoming neutrons bouncing straight back. Thus we can take this to be a one dimensional problem.

The equations are the following (the energy equations have been multiplied by 2):

$$m_n v = m_n v' + m_H v_H \quad m_n v^2 = m_n v'^2 + m_H v_H^2$$

$$m_n v = m_n v'' + m_N v_N \quad m_n v^2 = m_n v''^2 + m_N v_N^2$$

Solving these equations for m_n and v requires careful algebra. We square the first momentum equation to get a relation between v', m_n, and v

$$v'^2 = \frac{(m_n v - m_H v_H)^2}{m_n^2}$$

Now we plug this into the first energy equation

$$m_n v^2 = \frac{(m_n v - m_H v_H)^2}{m_n} + m_H v_H^2$$

Expanding,

$$m_H^2 v_H^2 - 2m_n m_H v v_H + m_n m_H v_H^2 = 0$$

Writing this as an equation for v, we get

$$v = \frac{1}{2}\left(1 + \frac{m_H}{m_n}\right) v_H$$

This is fairly simple result. If we perform the same manipulations on the nitrogen equations, we will get an analogous result

$$v = \frac{1}{2}\left(1 + \frac{m_N}{m_n}\right) v_N$$

We can now use these to solve for m_n and v. Equating the right hand sides, we get a single equation for the mass.

$$\begin{aligned} m_n v_H + m_H v_H &= m_n v_N + m_N v_N \\ m_n &= \frac{m_N v_N - m_H v_H}{v_H - v_N} \ . \end{aligned}$$

We can now use this to find the initial velocity of the neutrons:

$$\begin{aligned} v &= \frac{v_H}{2}\Big(1 + \frac{m_H(v_H - v_N)}{m_N v_N - m_H v_H}\Big) \\ &= \frac{v_H}{2}\Big(\frac{m_N v_N - m_H v_N}{m_N v_N - m_H v_H}\Big) \\ &= \frac{v_H v_N}{2}\Big(\frac{m_N - m_H}{m_N v_N - m_H v_H}\Big) \ . \end{aligned}$$

We want to know the mass of the neutron in amu, so we plug in $m_H = 1$ and $m_N = 14$ (greater accuracy is unnecessary, since the recoil velocities are measured only to 10%). We also look at both boundaries of the nitrogen velocity, calling these results $m_\pm$ and $v_\pm$. Plugging in numbers, the values of m_n are

$$\begin{aligned} m_n &= 1.159 \text{ amu} \\ m_+ &= 1.415 \text{ amu} \\ m_- &= 0.911 \text{ amu} \ . \end{aligned}$$

Chadwick's experimental work is seen to be reliable; today's accepted value for the neutron mass is 1.008665 amu, or 938.27231 $\pm$ 0.00028 MeV/c^2, well within his experimental range. The range of initial neutron velocity is given by

$$\begin{aligned} v &= 3.07 \times 10^7 \text{ m/sec} \\ v_+ &= 2.82 \times 10^7 \text{ m/sec} \\ v_- &= 4.13 \times 10^7 \text{ m/sec} \ . \end{aligned}$$

7. K&K problem 4.13. The Lennard-Jones potential is given by

$$U = \epsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6\right]$$

(a.) We find the minimum of this potential by differentiating it with respect to r and setting the results equal to zero:

$$\frac{dU}{dr} = -\frac{12\epsilon}{r}\left[\left(\frac{r_0}{r}\right)^{12} - \left(\frac{r_0}{r}\right)^6\right] = 0$$

This is easy to solve:

$$\left(\frac{r_0}{r}\right)^{12} = \left(\frac{r_0}{r}\right)^6 \Rightarrow r = r_0$$

The depth of the potential well is just $U(r_0) = -\epsilon$. Thus the potential well has a depth ϵ.

(b.) We find the frequency of small oscillations by making a Taylor expansion of the potential about $r = r_0$. Read section 4.10 in K&K for more information on this. We can write the potential as follows:

$$\begin{aligned} U(r) = U(r_0) &+ \left(\frac{dU}{dr}\right)_{r=r_0}(r - r_0) \\ &+ \frac{1}{2}\left(\frac{d^2U}{dr^2}\right)_{r=r_0}(r - r_0)^2 + \cdots \end{aligned}$$

We know that $dU/dr = 0$ at $r = r_0$, so we drop the middle term.

$$U(r) \approx -\epsilon + \frac{1}{2}\left(\frac{d^2U}{dr^2}\right)_{r=r_0}(r - r_0)^2$$

This is exactly the form of the potential of a mass on a spring. We only have to identify the spring constant. Remembering that $U_{\text{spring}} = kx^2/2$, we make the identification

$$k = \left(\frac{d^2U}{dr^2}\right)_{r=r_0}$$

For the Lennard-Jones potential, we already know the first derivative, so we need to differentiate once more.

$$\frac{d^2U}{dr^2} = \frac{12\epsilon}{r^2}\left[13\left(\frac{r_0}{r}\right)^{12} - \left(7\frac{r_0}{r}\right)^6\right]$$

Plugging in $r = r_0$, we find the effective spring constant for this potential

$$k = \frac{72\epsilon}{r_0^2}$$

We now consider two identical masses m on the ends of this "spring". Their (coupled) equations of motion are:

$$m\ddot{r_1} = k(r - r_0) \quad m\ddot{r_2} = -k(r - r_0)$$

where $r = r_2 - r_1$ is the distance between the masses. Subtracting these two equations, we get

$$m\ddot{r} = -2k(r - r_0)$$

The frequency of oscillation is then $\omega^2 = 2k/m$. (Note that we could have obtained the same result by considering the two-mass system to be a *single* mass of *reduced mass* $m_{\text{reduced}} = m_1 m_2/(m_1 + m_2)$). Plugging in the above value for the effective spring constant k,

$$\omega = 12\sqrt{\frac{\epsilon}{r_0^2 m}}$$

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PROBLEM SET 6

1. K&K problem 6.1 "Show that if the total linear momentum...".

2. K&K problem 6.3 "A ring of mass M and radius R lies...".

3. Expansion of the previous problem:

(a.) Let the azimuth of the bug on the ring be ϕ – that is, ϕ is zero when the bug starts walking, and 360° when the bug makes one revolution. Assume (for part (a.) only) that the ring is fixed. Calculate the angular momentum l of the bug about the pivot in the previous problem, as a function of ϕ. Check that your result is consistent with what you used in the previous problem when ϕ was 180°.

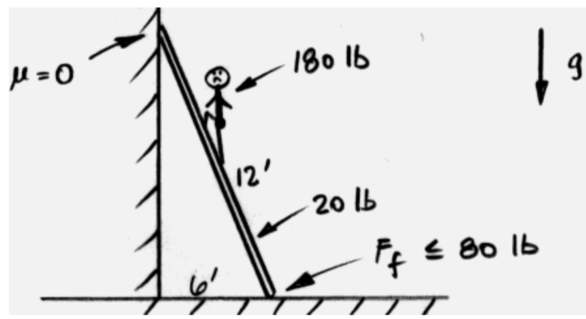
(b.) Now assume that the bug is fixed at some azimuth ϕ on the ring, but that the ring itself is not fixed, having angular velocity Ω about the pivot (opposite to the angular velocity of the bug when the bug was moving). Calculate the angular momentum l' of the bug as a function of Ω and ϕ.

(c.) Now assume that neither the bug nor the ring are fixed. By requiring that the total angular momentum $l + l'$ of the bug about the pivot be balanced by the angular momentum of the ring, obtain an expression for the angular velocity Ω of the ring, as a function of ϕ.

(d.) Get an integral expression for the angle θ through which the ring rotates, as a function of time, assuming that $d\phi/dt = \omega =$ constant. You need not evaluate the integral. Note that the system is "bootstrapping" its way around the pivot!

4. K&K problem 6.5 "A 3,000-lb car is parked on a...".

5. A man begins to climb up a 12-ft ladder (see figure). The man weighs 180 lb, and the ladder 20 lb. The wall against which the ladder rests is very smooth, which means that the tangential (vertical) component of force at the contact between ladder and wall is negligible. The foot of the ladder is placed 6 ft from the wall. The ladder, with the man's weight on it, will slip if the tangential (horizontal) force at the contact between the ladder and ground exceeds 80 lb. How far up the ladder can the man safely climb?



6. K&K problem 6.8 "Find the moment of inertia of a uniform sphere...".

7. K&K problem 6.14 "A uniform stick of mass M and length L is...".

8. K&K problem 6.18 "Find the period of a pendulum...".

University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 6

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. K&K problem 6.1

(a.) We know that the total linear momentum of the system is zero. (This would occur, for example, if we were in the center of mass frame.)

$$\mathbf{P} = \sum_i \mathbf{p}_i = \mathbf{0}$$

Examine the total angular momentum of the system. The vector from the origin to point i is denoted $\mathbf{r}_i$. The angular momentum in general depends on where the origin is:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

We now want to find the angular momentum about a new origin whose position vector is $\mathbf{R}$ in the current coordinate system. In this new system, the position vector of point i becomes $\mathbf{r}_i - \mathbf{R}$. Each point has its position changed by the same amount. The new value of the angular momentum is

$$\mathbf{L}_{\text{new}} = \sum_i (\mathbf{r}_i - \mathbf{R}) \times \mathbf{p}_i$$

Expanding,

$$\mathbf{L}_{\text{new}} = \sum_i \mathbf{r}_i \times \mathbf{p}_i - \sum_i \mathbf{R} \times \mathbf{p}_i$$

Because $\mathbf{R}$ is the same for all points, we can pull it outside of the sum:

$$\mathbf{L}_{\text{new}} = \sum_i \mathbf{r}_i \times \mathbf{p}_i - \mathbf{R} \times \sum_i \mathbf{p}_i$$
$$= \sum_i \mathbf{r}_i \times \mathbf{p}_i - \mathbf{R} \times \mathbf{P}$$

We know that $\mathbf{P} = \mathbf{0}$, so we are done.

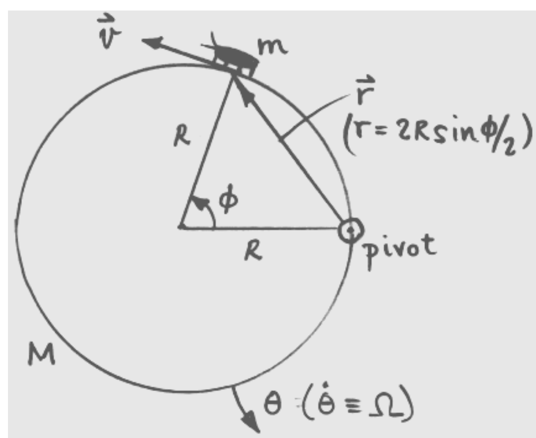
$$\mathbf{L}_{\text{new}} = \mathbf{L}$$

(b.) The proof for this part is identical if angular momentum is replaced by torque and linear momentum is replaced by force.

2. K&K problem 6.3

This problem and the next concern the same system – that of a bug walking along a hoop that is free to pivot around a point on its edge. The hoop lies flat on a frictionless surface. The ring has mass M and radius R, and the bug has mass m and walks on the ring with speed v.

The key idea in this problem is conservation of angular momentum. About the pivot there is no net torque on the system, so the total angular momentum about that point is conserved. The ring starts at rest with the bug on the pivot, and the bug starts walking at speed v. Immediately after the bug starts walking, the total angular momentum measured about the pivot point continues to be zero. The ring is not yet moving, so it has no angular momentum; the bug has begun to move, but it is at $\mathbf{r} = 0$, so it has no angular momentum yet.



We want to find the angular velocity Ω of the ring when the bug is opposite to the pivot. The bug is moving at speed v on the ring, but the ring is also moving. The bug is at a distance $2R$ from the pivot, so the velocity of that portion of the ring which is under the feet of the bug is $2\Omega R$. The total velocity of the bug is thus $v + 2\Omega R$. Next we need to know the moment of inertia of the hoop. A hoop has moment of inertia $I = MR^2$ about its center of mass. We use the parallel axis theorem to find the moment of inertia about a point on the edge.

$$I = I_{\text{CM}} + Md^2$$

The distance d from the center of mass to the desired axis in this case is just R, so the moment of inertia of the hoop about a point on the edge is $I = 2MR^2$. We can now find an expression for the total angular momentum of the system. For the hoop we use $L = I\Omega$ and for the bug we use $L = mvr \sin\theta$. The angle θ between the position vector and the velocity vector of the bug in this case is simply $\pi/2$, so $\sin\theta$ is just 1. We now write the angular momenta of the two pieces

$$L_{\text{bug}} = 2mR(v + 2\Omega R) \qquad L_{\text{hoop}} = 2MR^2\Omega$$

Since angular momentum about the pivot is conserved throughout the motion, We know that $L_{\text{bug}} + L_{\text{hoop}} = 0$. This gives the following expression:

$$2mvR + 4m\Omega R^2 + 2MR^2\Omega = 0$$

We solve this equation for Ω in terms of v and get

$$\Omega = -\frac{mv}{MR + 2mR}$$

Note that the minus sign means that the hoop rotates in a direction opposite to that in which the bug moves. This makes sense because the total angular momentum about the pivot point must vanish.

3. We now study the bug and hoop system in more detail. See the diagram in the previous problem.

(a.) In this part we assume that the ring is fixed. We want to calculate the angular momentum of the bug about the pivot point. The first step is to find the distance r between the bug and the pivot. We can do this using the law of cosines. Consider the triangle made by the line between the bug and the pivot, and the two radial lines extending from the center of the hoop to the bug and pivot, respectively. This is an isosceles triangle, with two equal sides of length R having an angle ϕ between them. If we define the azimuth of the bug on the hoop to be zero at the pivot, the angle ϕ is simply the azimuth of the bug on the hoop. The length r of the third side is found using the law of cosines:

$$r^2 = R^2 + R^2 - 2R^2\cos\phi \;\Rightarrow\; r^2 = 2R^2(1 - \cos\phi)$$

Using the trigonometric identity $1 - \cos\phi = 2\sin^2\frac{\phi}{2}$, we can get a simple result for r:

$$r = 2R\sin\frac{\phi}{2}$$

Now we know v and r. The only thing left to determine is the angle between the position and velocity vectors. The first step is to find the angle between the position vector $\mathbf{r}$ of the bug and the line from the center of the circle to the bug. The isosceles triangle (like any triangle) has a total angle π, and its central angle is ϕ. The remaining two angles are equal, so they must be $(\pi - \phi)/2$ each. Thus the three angles add up to π radians. Because the velocity vector $\mathbf{v}$ is tangent to the circle, the angle between $\mathbf{r}$ and $\mathbf{v}$ is $\pi/2$ minus this angle. Thus the angle between $\mathbf{r}$ and $\mathbf{v}$ is $\phi/2$. We can now get the angular momentum of the bug. Assuming that the ring is fixed, $l = mvr\sin\theta$ from the bug alone, so

$$l = 2mvR\sin^2\frac{\phi}{2}$$

(b.) In this part we assume that the ring is rotating with angular velocity Ω, but that the bug is fixed on the ring. The velocity of the bug is just Ωr, where r is the same as was calculated in the previous part. The velocity in this case is always perpendicular to its position vector. This can be seen by remembering that the bug isn't moving on the ring, so it must be in uniform circular motion about the pivot, with a velocity that is tangent to its present position. Therefore the angular momentum l' of the bug is simply mvr, yielding

$$l' = m\Omega r^2 = 4m\Omega R^2\sin^2\frac{\phi}{2}$$

(c.) We now allow both the bug and the ring to move. The total angular momentum of the bug is $l+l'$ from parts (a.) and (b.) respectively. To this we must add the angular momentum of the ring to get the total angular momentum of the system. From problem **2.** we know that the total angular momentum must be zero. The angular momentum of the ring is $I\Omega$, so we get the following equation.

$$4m\Omega R^2 \sin^2\frac{\phi}{2} + 2mvR\sin^2\frac{\phi}{2} + 2MR^2\Omega = 0$$

We solve this for Ω in terms of ϕ. The result is

$$\Omega = -\frac{mv\sin^2\frac{\phi}{2}}{MR + 2mR\sin^2\frac{\phi}{2}}$$

This agrees with the result of problem **2.** when the bug is at $\phi = \pi$, opposite to the pivot.

(d.) Finally, we want to find an expression for the angle θ through which the ring rotates. We know that θ is related by a simple differential equation to the angular velocity Ω of the hoop

$$\Omega = \frac{d\theta}{dt}$$

but we want to express Ω in terms of ϕ so that we can use the fact that $Rd\phi/dt$, the speed v of the bug with respect to the rim of the hoop, is constant. We apply the chain rule to get

$$\Omega = \frac{d\theta}{dt} = \frac{d\theta}{d\phi}\frac{d\phi}{dt}$$

Substituting $d\phi/dt = v/R$, where v is constant, we can write an integral for θ:

$$\theta = -\frac{R}{v}\int_{\phi_o}^{\phi}\frac{mv\sin^2(\phi'/2)}{MR + 2mR\sin^2(\phi'/2)}d\phi'$$

We can simplify this a little, but doing the integral is hard, which is why you weren't asked to evaluate it. Setting the initial bug azimuth ϕ_0 to zero and using the fact that $d\phi/dt = v/R$ is a constant so that $\phi = vt/R$,

$$\theta(t) = -\int_0^{vt/R}\frac{\sin^2(\phi'/2)}{(M/m) + 2\sin^2(\phi'/2)}d\phi'$$

4. K&K problem 6.5

A car of mass m is parked on a slope of angle θ facing uphill. The center of mass is a distance d above the ground, and it is centered between the wheels, which are a distance l apart. We want to find the normal force exerted by the road on the front and rear tires.

It is easiest to do this problem choosing the origin as the point on the road directly below the center of mass. About this point there are three torques. The normal force on the front (N_f) and rear (N_r) set of wheels provides a torque, and also gravity provides a torque $mgd\sin\theta$ because the car isn't horizontal. However, the forces of friction on the tires don't provide any torque because they are in line with the direction to the origin. The torque from the front wheels and the torque due to gravity tend to want to flip the car over backwards, while the torque on the rear wheels opposes this tendency. We want the sum of the torques to vanish, because the (static) car is not undergoing any acceleration, angular or linear:

$$\begin{aligned} 0 &= N_f\frac{l}{2} + mgd\sin\theta - N_r\frac{l}{2} \\ N_r - N_f &= \frac{2mgd}{l}\sin\theta \end{aligned}$$

We can get one more condition from the fact that the car is not undergoing linear acceleration perpendicular to the road. This means that the normal forces exactly cancel gravity:

$$N_r + N_f = mg\cos\theta$$

We can take the sum and difference of these two equations to get expressions for N_f and N_r. These are

$$\begin{aligned} N_r &= mg\Big(\frac{1}{2}\cos\theta + \frac{d}{l}\sin\theta\Big) \\ N_f &= mg\Big(\frac{1}{2}\cos\theta - \frac{d}{l}\sin\theta\Big) \end{aligned}$$

Plugging in $\theta = 30^\circ$, $mg = 3000$ lb, $d = 2$ ft, and $l = 8$ ft, we get $N_r =$1674 lb and $N_f =$924 lb.

5. We will solve this problem symbolically and wait until the end to plug in numbers. This is always good practice because it makes it a lot easier to check the units of the result and to explore whether the result is reasonable when the inputs have limiting values. We take M to be the mass of the man ($Mg = 180$ lb) and m to be the mass of the ladder ($mg = 20$ lb). The length H of the ladder is 12 ft, and its point of contact with the wall is $d = 6$ ft from the wall. The angle that it makes with the wall is $\theta = \arcsin(d/H) = 30°$. Finally, the force of friction on the ladder from the ground is $F_f \leq F_f^{\max}$, where $F_f^{\max} = 80$ lb.

There are five forces to consider in this problem. They are the two normal forces on the ladder, N_g from the ground and N_w from the wall; the force F_f of friction at the base of the ladder; and the two forces of gravity, Mg on the man and mg on the ladder. This is a torque balance problem, so choosing a good origin makes it a lot easier. With this choice of the point of contact with ground, two of the five forces contribute no torque about that point. Not bad! As a sanity check we evaluate μ, the coefficient of friction between the ladder and the ground. The normal force N_f from the floor is equal and opposite to $(M+m)g$, the sum of the weights of the ladder and the man. We are given the maximum frictional force $F_f^{\max}$, and we know that $F_f^{\max} = \mu N$, so $\mu = F_f^{\max}/((M+m)g) = 80/200 = 0.4$, a reasonable value.

We now calculate the torques. To find the maximum height h to which the man can climb without the ladder slipping, we assume that the ladder is about to slip. This means that the normal force N_w from the wall is equal and opposite to $F_f^{\max}$, exactly countering the maximum force of friction: since these two forces are the only forces in the horizontal direction they must sum to zero. The torque from the wall is then $\tau_w = -F_f^{\max} H \cos\theta$, where the minus sign indicates that this torque pushes clockwise. The torque from the weight of the ladder is exerted at the midpoint of the ladder, its center of mass. The value of this torque is $\tau_m = mg\frac{H}{2}\sin\theta$. Similarly, the torque exerted by the weight of the man, who is a distance h up the ladder, is $\tau_M = Mgh\sin\theta$. Requiring these three torques to sum to zero,

$$\begin{aligned} 0 &= \tau_M + \tau_m + \tau_w \\ &= Mgh\sin\theta + mg\frac{H}{2}\sin\theta - F_f^{\max} H\cos\theta \end{aligned}$$

Solving for h,

$$\begin{aligned} h &= \frac{F_f^{\max} H\cos\theta - mg\frac{H}{2}\sin\theta}{Mg\sin\theta} \\ \frac{h}{H} &= \frac{F_f^{\max}}{Mg}\cot\theta - \frac{m}{2M} \\ &= \frac{\mu(M+m)}{M}\cot\theta - \frac{m}{2M} \\ &= \frac{\mu(M+m)\cot\theta - (m/2)}{M} \\ &= \frac{0.4(200)\sqrt{3} - 10}{180} \\ &= 0.7142 \\ h &= 8.571 \text{ ft} \end{aligned}$$

6. K&K problem 6.8

Because of the spherical symmetry, we work in spherical polar coordinates. To find the moment of inertia we need to evaluate the integral

$$I = \int r_\perp^2 \rho dv$$

where in these coordinates $r_\perp = r\sin\theta$ is the perpendicular distance to the axis and $dv = r^2 dr\, d(\cos\theta)\, d\phi$ is the element of volume. The integral to evaluate is thus

$$I = M\frac{\int_0^R r^4\,dr \int_{-1}^1 \sin^2\theta\, d(\cos\theta) \int_0^{2\pi} d\phi}{\int_0^R r^2\,dr \int_{-1}^1 d(\cos\theta)\int_0^{2\pi} d\phi}$$

where the denominator is the volume V of the sphere, needed to evaluate its density $\rho = M/V$. Substituting $u \equiv r/R$,

$$\frac{I}{MR^2} = \frac{\int_0^1 u^4\,du \int_{-1}^1 \sin^2\theta\, d(\cos\theta)\int_0^{2\pi} d\phi}{\int_0^1 u^2\,du \int_{-1}^1 d(\cos\theta)\int_0^{2\pi} d\phi}$$

In both the numerator and the denominator, all three integrals have limits that do not depend on the other variables, so each integral can be evaluated independently. The ϕ integrals cancel, and the u integrals have the ratio

$$\frac{1/5}{1/3} = \frac{3}{5}$$

The integrand in the $\cos\theta$ integral in the numerator can be rewritten

$$\begin{aligned}\sin^2\theta\, d(\cos\theta) &= (1-\cos^2\theta)d(\cos\theta)\\ &= d(\cos\theta) - d\Big(\frac{1}{3}\cos^3\theta\Big)\end{aligned}$$

Therefore the ratio of the θ integrals is

$$\frac{2-\frac{2}{3}}{2} = \frac{2}{3}$$

Putting it all together,

$$I = MR^2 \times \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}MR^2$$

7. K&K problem 6.14

When the stick is released, there are two forces acting on it, gravity at the midpoint, and the normal force at the point B. We use the point B as the origin, so the only torque about this point is provided by gravity. At the moment of release, the stick is still horizontal, so the torque is

$$\tau_B = -\frac{Mgl}{2}$$

where the minus sign indicates that the torque pulls clockwise. We know that the moment of inertia of a thin stick about its endpoint is $I = Ml^2/3$, so we can easily find the angular acceleration α from $\tau = I\alpha$.

$$-\frac{Mgl}{2} = \frac{1}{3}Ml^2\alpha \;\Rightarrow\; \alpha = -\frac{3g}{2l}$$

The vertical acceleration of the center of mass is given by the simple formula $a = \alpha r$, where r is the distance between the center of mass and point b, about which the stick is (instantaneously) executing circular motion. (This is analogous to the expression $v = \omega r$.) Here this distance is $r = l/2$. This gives the acceleration of the center of mass:

$$a = -\frac{3}{4}g$$

where the minus sign indicates that the acceleration is downward. Finally we use Newton's second law to find the normal force at B. We know the acceleration and we know the force of gravity, so this is a simple equation

$$N - Mg = -\frac{3}{4}Mg \;\Rightarrow\; N = \frac{1}{4}Mg$$

where the positive direction is up, opposite to the force of gravity.

8. K&K problem 6.18

We want to find the equation of motion of the pendulum to determine the frequency. We will use the torque equation $\tau = I\alpha$. If we choose the pivot point of the pendulum as the origin, only one force provides torque, the force of gravity. It acts on the center of mass of the pendulum, a distance l_{cm} from the pivot point. The magnitude of this force is just $(M+m)g$. Thus the total torque is

$$\tau = -(M+m)gl_{cm}\sin\theta$$

where θ is the angular position of the pendulum. Writing the torque equation and approximating $\sin\theta \approx \theta$, we get

$$I\ddot{\theta} = -(M+m)gl_{cm}\theta$$

We recognize that this is the equation for a simple harmonic oscillator. The angular frequency and period are thus

$$\omega = \sqrt{\frac{(M+m)gl_{cm}}{I}} \qquad T = 2\pi\sqrt{\frac{I}{(M+m)gl_{cm}}}$$

All that is left is to evaluate l_{cm} and I. The equation for the center of mass is easy to use. The center of mass of the rod is halfway along its length, and the disk is a distance l from the pivot, so

$$l_{cm} = \frac{ml/2 + Ml}{M+m}$$

This expression simplifies the formula for the period

$$T = 2\pi\sqrt{\frac{I}{(M+m/2)gl}}$$

In the first case, where the disk is tied to the rod, the moment of inertia is determined using the parallel axis theorem. The disk is fixed to the rod, so, as the rod pivots, the disk must rotate at the same angular velocity. The moment of inertia of a stick about its end is $ml^2/3$. The moment of inertia of a disk about its center is $MR^2/2$. Because the center of mass is displaced a distance l from the origin, the parallel axis theorem tells us that the total moment of inertia of the disk is $MR^2/2 + Ml^2$. Thus the total moment of inertia of the pendulum is

$$I = \frac{1}{2}MR^2 + \left(M + \frac{1}{3}m\right) l^2$$

This gives the period of oscillation

$$T = T = 2\pi\sqrt{\frac{MR^2/2 + (M+m/3)l^2}{(M+m/2)gl}}$$

In the second case, where the disk is free to rotate on the rod, the moment of inertia is smaller. Because the disk is not fixed, it has no tendency to rotate. Its effective moment of inertia about the center of mass is thus zero. For our purpose it is the same as a point mass a distance l away from the origin. The new moment of inertia is

$$I = \left(M + \frac{1}{3}m\right) l^2$$

The period of oscillation in this case is also smaller:

$$T = 2\pi\sqrt{\frac{(M+m/3)l}{(M+m/2)g}}$$

This is the same as our answer for the first case in the limit $R \to 0$.

University of California, Berkeley
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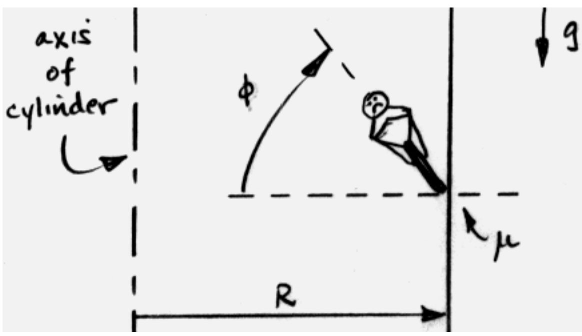
PROBLEM SET 7

1. A trick cyclist rides his bike around a "wall of death" in the form of a vertical cylinder (see figure). The maximum frictional force parallel to the surface of the cylinder is equal to a fraction μ of the normal force exerted on the bike by the wall. Assume that the cyclist and his bike are small relative to the radius of the cylinder.



(a.) At what minimum speed must the cyclist go to avoid slipping down?

(b.) At what angle ϕ to the horizontal must he be inclined at that minimum speed?

(c.) If μ=0.6 (typical of rubber tires on dry roads) and the radius of the cylinder is 5 m, at what minimum speed must the cyclist ride, and what angle does he make with the horizontal?

2. K&K problem 6.24 "Drum A of mass M and radius R...".

3. K&K problem 6.27 "A yo-yo of mass M has an axle...".

4. Two men, each of mass 100 kg, stand at opposite ends of the diameter of a rotating turntable of mass 200 kg and radius 3 m. Initially the turntable makes one revolution every 2 sec. The two men make their way to the middle of the turntable at equal rates.

(a.) Calculate the final rate of revolution and the factor by which the kinetic energy of rotation has been increased.

(b.) Analyze, at least qualitatively, the means by which the increase of rotational kinetic energy occurs.

(c.) At what radial distance from the axis of rotation do the men experience the greatest centrifugal force as they make their way to the center?

5. K&K problem 7.4 "In an old-fashioned rolling mill, grain...".

6. K&K problem 7.5 "When an automobile rounds a curve...".

7. K&K problem 8.2 "A truck...".

8. K&K problem 8.4 "The center of mass...".

University of California, Berkeley
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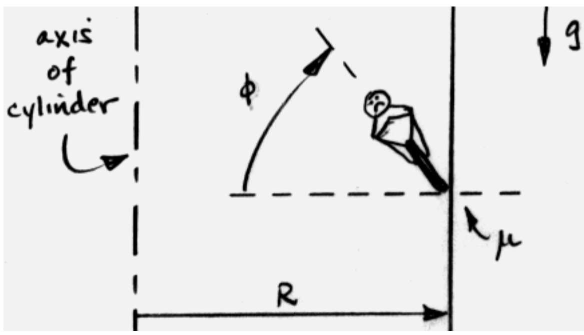
SOLUTION TO PROBLEM SET 7

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. We will need to use fictitious forces to solve this problem easily. Fictitious forces are never necessary, but they often simplify problems greatly.



(a.) We want to know the minimum speed the cyclist needs not to slip down the side. The force of static friction must be mg to hold him up, so we require that $\mu N = mg$. The force of static friction can be less than μN, but we are setting it to the maximum to see what the limit is. Thus we need $N > mg/\mu$. The only force acting horizontally in the system is the normal force, so it must entirely provide the centripetal acceleration, which is v^2/R. We thus obtain

$$\frac{mv^2}{R} \geq \frac{mg}{\mu} \;\Rightarrow\; v \geq \sqrt{\frac{gR}{\mu}}$$

(b.) We now need to consider the fictitious centrifugal force. The cyclist is in an accelerating frame of reference because he is moving in a circle. To correctly apply Newton's second law in the cyclist's frame, we must introduce the centrifugal force, which points outward with magnitude mv^2/R. In the frame of the cyclist there are four forces: the normal force and the centrifugal force cancel each other, and the friction and gravity cancel each other. This is required to insure that, by definition, the cyclist isn't accelerating in his own frame of reference. The problem is now reduced to a torque balance to find the angle at which the cyclist is stable. We choose the point of contact as the origin. There are two torques, τ_g caused by gravity and τ_c caused by the centrifugal force. Both act at the center of mass. (This is important to note: fictitious forces always act at the center of mass!) We are assuming that the size of the cyclist l is very small in comparison with the radius R, so

$$\tau_g = +mgl\cos\phi \qquad \tau_c = -\frac{mlv^2}{R}\sin\phi$$

We want to know the angle where the cyclist is about to slip, so the normal force is mg/μ, equal to the centripetal force mv^2/R. Substituting for mv^2/R in the above equation,

$$mgl\cos\phi = mgl\frac{1}{\mu}\sin\phi \;\Rightarrow\; \tan\phi = \mu$$

So when the cyclist is about to slip, he rides at an angle

$$\phi = \tan^{-1}\mu$$

(c.) Taking $\mu = 0.6$ and $R = 5$ meters, we find that the cyclist must ride at a speed of least 9.0 meters per second, or 29.5 ft/sec, or 20.1 mph. On a road bike this is a mellow cruising speed. At this minimum speed, the angle the cyclist must make with the horizontal is 31 degrees. We caution you not to try this at home; it's tough to get up to speed without crashing!

2. K&K problem 6.24

This problem is similar to many pulley problems that you have seen before. We need to apply both Newton's second law and the torque equation to solve it. We denote the (positive downward) acceleration of the falling mass as a. There are two forces on it in the vertical direction, tension and gravity. Newton's second law requires

$$Mg - T = Ma$$

We now need to apply the torque equation to both drums. For each drum we choose its own center as the origin. Each drum feels only one torque, the torque from the tension. This has a magnitude $\tau = TR$ in both cases. Notice that both of these torques have the same sign, thus the drums will tend to angularly accelerate in the same direction. Writing the torque equation for each drum, with angular accelerations α_1 for the top drum and α_2 for the bottom drum,

$$TR = I\alpha_1 \qquad TR = I\alpha_2$$

where the moments of inertia of both disks are the same, $I = MR^2/2$. From these equations it is easy to see that $\alpha_1 = \alpha_2 \equiv \alpha = TR/I$, so the angular accelerations are both

$$\alpha = \frac{2T}{MR}$$

We now need to find a relation between a and α. The linear acceleration due to each disk is given simply by $a = \alpha R$. There are two disks, both unwinding with the same angular acceleration α, so the linear acceleration of the bottom one is just $a = 2\alpha R$. The previous equation becomes

$$a = \frac{4T}{M} \Rightarrow T = \frac{Ma}{4}$$

Plugging this into the very first equation that we got from Newton's law, we find the initial acceleration of the drum, assuming that it moves straight down, to be

$$a = \frac{4}{5}g$$

Will the drum in fact move straight down? For the moment assuming that the answer is "yes", consider a (downward accelerating but nonrotating) frame with its origin at the (instantaneous) point of tangency between the lower drum and the tape. In this frame, the CM of the drum experiences a downward force mg and an upward fictitious force $\frac{4}{5}mg$ which does not quite compensate it. Therefore it feels a net downward force $\frac{1}{5}mg$. About the chosen origin this force causes a net (clockwise) torque, which causes the lower drum to swing to the left like a pendulum bob in this frame. This contradicts our assumption of a pure downward motion. Therefore the actual motion will be more complicated than this problem asks you to assume.

3. K&K problem 6.27

We need to apply both Newton's law and the torque equation. The forces on the yo-yo horizontally are the force F and the friction f. The vertical forces are the normal force and gravity, which immediately tell us that $N = Mg$. We want to find the maximum force we can apply with the yo-yo not slipping. It is important to note that the force of friction, which stops the disk from slipping, is controlled by the coefficient μ_s of *static* friction because the surface of a rolling wheel is at rest with respect to the ground. Since we are concerned with the maximum allowed force, we will consider the maximum allowed friction, which is $\mu_s N$. Newton's law gives us

$$F - \mu_s Mg = Ma$$

The moment of inertia of the yo-yo is $I = MR^2/2$. Because we want the yo-yo to roll without slipping, we can use $a = \alpha R$. The torque equation gives us

$$\mu_s MgR - Fb = I\alpha = \frac{1}{2}MRa$$

We want to solve these two equations for F, the maximum allowed force. Eliminating a, we get

$$F - \mu_s Mg = 2\mu_s Mg - 2F\frac{b}{R}$$

Solving for F, we get

$$F = \mu_s Mg\frac{3R}{R + 2b}$$

Since $R > b$ the applied force F is always larger than the frictional force, so the yo-yo always accelerates to the right.

4. We will solve this problem symbolically and plug in numbers at the end. This is always a good practice because it makes it a lot easier to go back and check your work for correct dimensions and reasonable results for limiting cases.

(a.) Let the disk have mass M and radius R, and the two men each have mass m. If the men are momentarily at a radius r from the center of the disk, the total moment of inertia is given by

$$I(r) = \frac{1}{2}MR^2 + 2mr^2$$

The initial angular velocity of the disk is ω_0, so, when $r = R$, the initial angular momentum of the system is

$$L = I\omega_0 = \left(\frac{1}{2}M + 2m\right) R^2\omega_0$$

There are no net torques acting on the system, so L is conserved. We can use L conservation, $I(r)\omega(r) = I(R)\omega_0$, to obtain the angular velocity of the system as a function of the radius of the men:

$$\omega(r) = \omega_0 \frac{MR^2 + 4mR^2}{MR^2 + 4mr^2}$$

The final angular velocity ω' is just the angular velocity when the men reach the center, $\omega(r = 0)$.

$$\omega' = \omega_0 \frac{M + 4m}{M} = \left(1 + \frac{4m}{M}\right)\omega_0$$

Plugging in values, we find that the final angular velocity is 1.5 revolutions per second.

The factor by which the kinetic energy has increased is

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}I(0)\omega'^2}{\frac{1}{2}I(R)\omega_0^2}$$

Evaluating this, we find

$$\frac{K_f}{K_i} = \frac{\frac{1}{4}MR^2\left(1 + \frac{4m}{M}\right)^2\omega_0^2}{\frac{1}{4}MR^2\omega_0^2 - mR^2\omega_0^2}$$

Simplifying,

$$\frac{K_f}{K_i} = 1 + \frac{4m}{M}$$
$$\Delta K = \frac{4m}{M}K_i$$

For the masses in the problem, K_f/K_i is equal to 3. The rotational kinetic energy is tripled.

(b.) The extra kinetic energy comes from the work that the men must do against the (ficitious) centrifugal force to make their way from the edge of the turntable to the center. This is the qualitative statement which the problem requests. Optionally, one can perform a quantitative analysis:

Each man pushes against the centrifugal force to get to the center. The work they do is converted to rotational kinetic energy. The centrifugal force on each man is given by

$$F_c = m(\omega(r))^2 r = m\omega_0^2 r\left(\frac{1 + 4m/M}{1 + 4mr^2/(MR^2)}\right)^2$$

The work done is just F_c integrated from zero to R, doubled since each man does the same amount of work.

$$\Delta W = 2m\omega_0^2\left(1 + \frac{4m}{M}\right)^2 \int_0^R \frac{r\,dr}{(1 + \frac{4mr^2}{MR^2})^2}$$

You can look this up in a table, or notice that the top is proportional to the derivative of the bottom, so antidifferentiating is not too hard:

$$\Delta W = -2m\omega_0^2\left(1 + \frac{4m}{M}\right)^2 \left(\frac{MR^2/8m}{(1 + \frac{4mr^2}{MR^2})^2}\right)_0^R$$

Evaluating this, we get

$$\Delta W = \frac{MR^2\omega_0^2}{4}\left\{\left(1 + \frac{4m}{M}\right)^2 - \left(1 + \frac{4m}{M}\right)\right\}$$

Multiplying this out,

$$\Delta W = \frac{MR^2\omega_0^2}{4}\left(\frac{4m}{M} + \frac{16m^2}{M^2}\right)$$

Simplifying,

$$\Delta W = mR^2\omega_0^2 + \frac{4m^2}{M}R^2\omega_0^2$$
$$= \left(1 + \frac{4m}{M}\right)mR^2\omega_0^2$$

This is $4m/M$ times the initial kinetic energy, so, as expected, ΔW is equal to the kinetic energy gain ΔK that we already calculated.

(c.) We want to find where the maximum centrifugal force is felt. This is just a maximization problem. Differentiate F_c with respect to r and set it to zero, and also check the endpoints.

$$\frac{dF_c}{dr} = \frac{d}{dr}\left(m\omega_0^2 r \frac{(1+4m/M)^2}{(1+4mr^2/MR^2)^2}\right) = 0$$

This gives

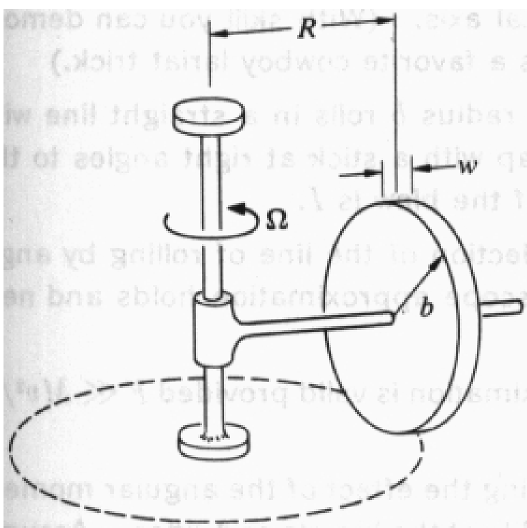
$$1 - \frac{16mr^2}{MR^2}\frac{1}{1+4mr^2/MR^2} = 0$$

We can solve this for r. Set $x = 4mr^2/MR^2$. Then $1-(4x/(1+x)) = 0$ so $x = 1/3$, and

$$\frac{4mr^2}{MR^2} = \frac{1}{3} \Rightarrow r = R\sqrt{\frac{M}{12m}}$$

If we plug in the mass values for this problem, we obtain $r = R/\sqrt{6}$. Since the centrifugal force is everywhere positive, and it is zero at the center, this extremum must in fact be the maximum.

5. K&K problem 7.4



Referring to the diagram, the stone orbits around the vertical shaft with orbiting angular velocity Ω. The velocity of the stone's CM is thus $v = \Omega R$. The stone is rolling without slipping on the flat surface, so its rolling angular velocity is

$$\omega = \frac{v}{b} = \Omega\frac{R}{b}$$

in magnitude. Since angular velocity is a vector, we can add these separate components to obtain the full angular velocity vector. In cylindrical coordinates, with $\hat{\mathbf{z}}$ pointing along the axis of the orbit,

$$\boldsymbol{\omega} = -\frac{R}{b}\Omega\hat{\mathbf{r}} + \Omega\hat{\mathbf{z}}$$

where the leading minus sign tells us that the radial component of $\boldsymbol{\omega}$ is negative, *i.e.* the millstone is rotating clockwise about its horizontal axle. To calculate the angular momentum, we choose as an origin the intersection of the centerline of the vertical shaft and the centerline of the horizontal axle. Both the shaft and the axle are parallel to mirror symmetry axes of the millstone; thus we expect that the component of angular momentum due to $\boldsymbol{\Omega}$ will be parallel to $\boldsymbol{\Omega}$, and the component of angular momentum due to $\boldsymbol{\omega}$ will be parallel to $\boldsymbol{\omega}$. More quantitatively, the component L_z along $\hat{\mathbf{z}}$ is equal to $(I' + MR^2)\Omega$, where $I' = \frac{1}{4}Mb^2$ is the moment of inertia of a disk about a *diameter* and MR^2 is added to I' by use of the parallel axis theorem. Since L_z is constant, no torque is required to maintain it and we don't need to consider it further. To calculate the radial component L_r of the angular momentum, we need $I = Mb^2/2$, the moment of inertia of a disk about its *center*:

$$L_r = -\frac{1}{2}Mb^2\omega$$

where the minus sign again reminds us that the millstone is rolling clockwise about its horizontal axle. Remember that, in cylindrical coordinates, the only unit vector which is constant is $\hat{\mathbf{z}}$; the radial and azimuthal unit vectors depend on θ. Even though the magnitude of L_r is constant, its direction is changing. In a time increment dt, the azimuth θ of the millstone axle with respect to the shaft changes by an angular increment $d\theta = \Omega dt$. This causes $\mathbf{L}$ to change by

$$\begin{aligned} d\mathbf{L} &= \hat{\boldsymbol{\theta}} L_r d\theta \\ &= \hat{\boldsymbol{\theta}} L_r \Omega dt \\ &= -\hat{\boldsymbol{\theta}}\frac{1}{2}Mb^2\omega\Omega dt \end{aligned}$$

The torque is thus

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = -\hat{\boldsymbol{\theta}}\frac{1}{2}Mb^2\omega\Omega$$

Finally we consider the forces on the system. The vertical shaft exerts a force on the horizontal axle, gravity pulls down on the millstone's CM, and the normal force pushes up on the millstone. However, with respect to the chosen origin, the first of these forces can exert no torque because it is applied at $\mathbf{r} = 0$. The torque due to gravity is in the $+\hat{\boldsymbol{\theta}}$ direction, and the torque due to the normal force N of the flat surface on the millstone is in the $-\hat{\boldsymbol{\theta}}$ direction. Thus, in the $+\hat{\boldsymbol{\theta}}$ direction, we have

$$-NR + MgR = \tau = -\frac{1}{2}Mb^2\omega\Omega$$

$$M\Big(g + \frac{1}{2}\frac{b^2}{R}\omega\Omega\Big) = N$$

Substituting $\omega = R\Omega/b$,

$$N = M\left(g + \frac{1}{2}b\Omega^2\right)$$

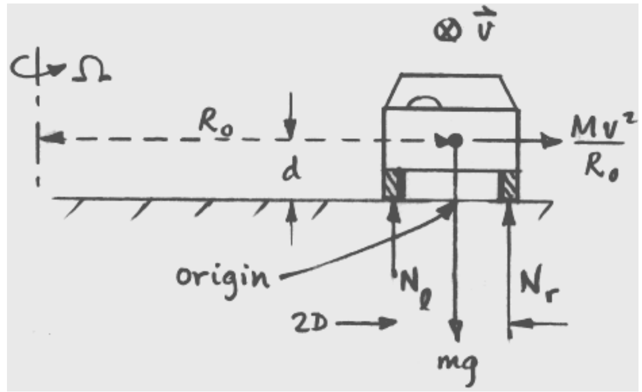
Of course, by Newton's third law, the contact force exerted by the millstone upon the flat surface is equal and opposite to N. As advertised, the effective weight of the millstone for crushing grain is greater than Mg; this increment rises quadratically with the angular velocity.

What keeps the millstone from accelerating upward, since the upward normal force on it is greater than the downward force of gravity? The force of the vertical shaft on the horizontal axle, which we ignored in the torque equation because it is applied at the origin, must push downward, in alignment with gravity, with the value $Mb\Omega^2/2$.

Such millstones must have been in use before the time of Newton, so the benefits of their increased effective weight when rolling in a circle must have been discovered empirically rather than logically.

6. K&K problem 7.5

(a.) If the flywheel were horizontal with its spin axis pointing up, it would have little effect, since the direction of its angular momentum would not change as the car turns left or right. So the flywheel should be vertical with its spin axis pointing either sideways or forward. Deciding between these alternatives requires a more quantitative analysis.



Let's look at the car from the rear while it is in motion with speed v. First we'll consider the car without any flywheel. Suppose that the car is in the process of turning to the left, taking a turn of radius R_0. Adopt a reference frame attached to the car, with an origin halfway between the tires at the level of the road. It is easy to see why the act of turning causes the normal forces on the tires to become unbalanced. The sum of the torques on the car must remain zero if the car (assumed to have no suspension system, so it doesn't lean) keeps all four tires on the road. With respect to the origin chosen, the forces of friction on both tires can exert no torque, because these forces act directly toward or away from the origin. Neither can the force of gravity exert a torque about this origin, for the same reason. That leaves N_l and N_r, the normal forces on the left and right sets of tires, and Mv^2/R_0, the fictitious centrifugal force which pulls the CM to the right in this accelerating frame. Let the CM be a distance d above the road; let the right-left separation of the wheels be $2D$. Along $-\hat{\mathbf{v}}$, the sum of the torques is then

$$-N_l D + N_r D - \frac{Mv^2}{R_0}d = 0$$

Clearly N_r must exceed N_l if this equation is to be satisfied. This is the problem we are trying to solve with the flywheel.

The torques that we just considered were along $-\hat{\mathbf{v}}$. If the flywheel is to help, its angular momentum $\mathbf{L}$ should be directed so that, when the car turns left, the flywheel produces a torque on the car equal to $+Mv^2d/R_0$ along $-\hat{\mathbf{v}}$. By Newton's third law, the torque of the car on the flywheel should correspondingly be equal to $+\hat{\mathbf{v}}Mv^2d/R_0$. So, as the car turns left, the change in $\mathbf{L}$ of the flywheel should be directed along $+\hat{\mathbf{v}}$. This will happen if the angular momentum vector of the (vertical) flywheel is pointing to the right. This means that the flywheel should rotate in the opposite direction as the tires. For simplicity, we'll install it at height d from the road so as not to perturb the CM.

It's not necessary to reverse the flywheel direction for right as opposed to left turns, because both the centrifugal force and the change in $\mathbf{L}$ will correspondingly reverse direction.

(b.) Now that we have determined the flywheel direction, we can calculate the desired magnitude L of the flywheel's angular momentum. We have

$$\frac{dL}{dt} = L\Omega = Mv^2d/R_0$$

where $\Omega = v/R_0$ is the angular velocity of the car around the turn. Solving,

$$L = Mvd$$

For a disk-shaped flywheel of mass m and radius r, $I = mr^2/2$, and the flywheel's angular velocity should be

$$\omega = \frac{2Mvd}{mr^2}$$

This is independent of the turn radius R_0, which is very nice. We've achieved perfectly flat cornering for a turn of any radius! Unfortunately, ω depends linearly on the velocity v of the car. So, unless we can come up with a quick easy way of varying the kinetic energy of a big flywheel in concert with the square of the car's speed, we're not going to get rich installing these devices as high-performance vehicle options.

7. K&K problem 8.2

(a.) The acceleration of the truck is A, and the mass and width of its rear door are M and w. The door starts fully open. The door can be thought of as a series of thin sticks, pivoted about their ends. The moment of inertia of the door is thus

$$I = \frac{1}{3}Mw^2$$

The easiest way to find the angular velocity of the door is to use work and energy. The rotational kinetic energy is given by

$$K = \frac{1}{2}I\omega^2$$

The work done by a torque on a system is given by

$$W = \int \tau \cdot d\theta$$

In this system there is one torque of interest. We use the hinge of the door as the origin, so the only torque comes from the fictitious force of acceleration. When the door has swung through an angle θ, this torque is given by

$$\tau = \frac{1}{2}MAw\cos\theta$$

Note that $w/2$ is just the distance to the center of mass. From this we can easily calculate the work done from 0 to 90 degrees.

$$W = \int_0^{\pi/2} \frac{1}{2}MAw\cos\theta\, d\theta = \frac{1}{2}MAw$$

Substituting the expression for rotational kinetic energy, we find the angular velocity of the door after it has swung through 90°:

$$\frac{1}{2}I\omega^2 = \frac{1}{2}MAw \;\Rightarrow\; \omega = \sqrt{\frac{3A}{w}}$$

(b.) The force on the door needs to do two things. It needs to accelerate the door at a rate A, and it needs to provide the centripetal acceleration to make the door rotate. At θ=90 degrees, the torque is zero, so the angular velocity is not changing. Instantaneously, the door is in uniform circular motion. The force required to accelerate the door is just

$$F_A = MA$$

The force required to provide the centripetal acceleration is

$$F_c = M\omega^2 \frac{w}{2}$$

The total force is the sum of these, and they act in the same direction. We substitute the value for ω from part (a.) to get

$$F = F_A + F_c = MA + \frac{3}{2}MA = \frac{5}{2}MA$$

8. K&K problem 8.4

(a.) This is a torque balance problem. A car of mass m has front and rear wheels separated by a distance l, and its center of mass is midway between the wheels a distance d off the ground. If the car accelerates at a rate A, it feels a fictitious force acting on the center of mass. This tends to lift the front wheels. When the front wheels are about to lift off the ground, the normal force on the front wheels, N_f is zero. This means that the normal force on the back wheels N_b must equal the weight of the car, $N_b = mg$. The simplest origin to use in this problem is the point on the road directly under the center of mass. Here there are only three torques, due to the two normal forces and fictitious force. The torque from N_b exactly balances the torque from the fictitious force when the wheels are about to lift, so we have

$$\frac{1}{2}N_b l = mAd = \frac{1}{2}mgl \;\Rightarrow\; A = \frac{l}{2d}g$$

For the numbers given, $A = 2g = 19.6\ \text{m/sec}^2$.

(b.) For deceleration at a rate g, again we simply apply torque balance about the same origin. We also need the fact that

$$N_f + N_b = mg$$

Torque balance gives

$$\frac{1}{2}N_f l - \frac{1}{2}N_b l - mgd = 0$$

Substituting from the previous equation, we get

$$N_f = \left(\frac{1}{2} + \frac{d}{l}\right) mg$$

$$N_b = \left(\frac{1}{2} - \frac{d}{l}\right) mg$$

Plugging in the numbers, we get $N_f = 3mg/4 =$ 2400 lb, and $N_b = mg/4 = 800$ lb.

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Physics H7A Fall 1998 (*Strovink*)

PROBLEM SET 8

1. K&K problem 8.5 "Many applications...".

2. K&K problem 8.11 "A high speed hydrofoil...".

3. K&K problem 9.3 "A particle moves...".

4. K&K problem 9.4 "For what values of n...".

5. K&K problem 9.6 "A particle of mass m...".

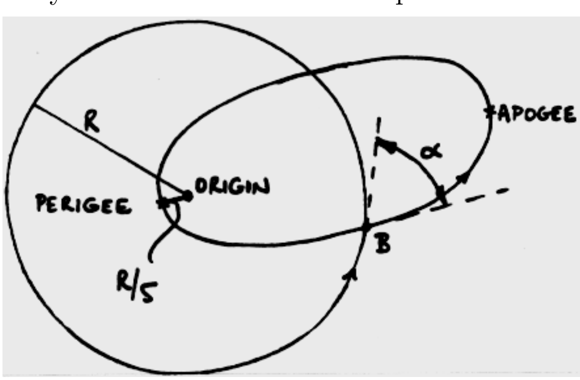
6. K&K problem 9.12 "A space vehicle is in circular orbit...".

7. A satellite of mass m is travelling at speed V in a circular orbit of radius R under the gravitational force of a fixed mass at the origin.

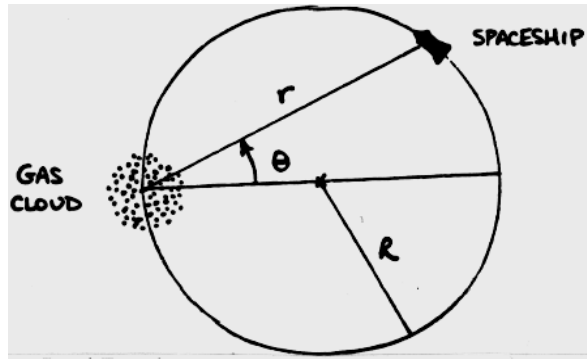
a. Taking the potential energy to be zero at infinite radius, show that the total mechanical energy of the satellite is $-mV^2/2$.

b. At a certain point B in the orbit (see figure), the direction of motion of the satellite is suddenly changed without *any* change in the magnitude of the velocity. As a result, the satellite goes into an elliptical orbit. Its closest approach to the origin is now $R/5$. What is the speed of the satellite at this distance, expressed as a multiple of V?

c. Through what angle α (see figure) was the velocity of the satellite turned at point B?



8. The commander of a spaceship that has shut down its engines and is coasting near a strange-appearing gas cloud notes that the ship is following a path that will take it directly *into* the cloud (see the figure). She also deduces from the ship's motion that its angular momentum with respect to the cloud is not changing. What attractive (central) force could account for such an orbit?



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Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 8

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. K&K problem 8.5

A gyroscope with mass M has angular velocity ω_s and moment of inertia I_s. It pivots at one end, and the center of mass is a distance l from the pivot. The angular momentum of the gyroscope is thus

$$L = I_s\omega_s$$

The gyroscope undergoes an acceleration a perpendicular to the spin axis. The fictitious force will create a torque of magnitude

$$\tau = Mal$$

The direction of this torque is perpendicular to both the acceleration and the gyroscope axis (down in the figure), causing the gyroscope axis to precess in the direction indicated by the angle θ. The magnitude of the angular momentum will not change, but the direction will. Thus the gyroscope axis will rotate around the direction of acceleration. The rate at which this happens is ω, and

$$\frac{dL}{dt} = L\omega = \tau$$

This gives the following relation

$$Mla(t) = I_s\omega_s\omega(t)$$

Both the acceleration and the angular velocity can depend on time. If we integrate both sides of this equation, we can get a relation between the final velocity and the total angle of rotation. The integral of the acceleration is just the velocity and the integral of the angular velocity is just the angle:

$$Mlv = I_s\omega_s\theta \;\Rightarrow\; v = \frac{I_s\omega_s}{Ml}\theta$$

2. K&K problem 8.11

A hydrofoil moves with respect to the earth's surface at the equator with velocity $\mathbf{v} = 200$ mi/hr directed along each of the four points of the compass. At rest with respect to the surface of the earth, the acceleration of gravity is $\mathbf{g}$. We are asked to find the effective gravitational acceleration $\mathbf{g}'$ that is felt by a passenger (of mass m) who is at rest with respect to the hydrofoil.

The (fictitious) Coriolus force on the passenger, which is proportional to the passenger's (vanishing) velocity in the hydrofoil's frame, must be zero if evaulated in this frame. The (fictitious) centrifugal force on the passenger is

$$\begin{aligned}\mathbf{F}_{\text{cent}} &= -m\boldsymbol{\Omega}' \times (\boldsymbol{\Omega}' \times \mathbf{R}) \\ &= -m(\boldsymbol{\Omega}'(\boldsymbol{\Omega}' \cdot \mathbf{R}) - \mathbf{R}(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}')) \\ &= m(-\boldsymbol{\Omega}'(\boldsymbol{\Omega}' \cdot \mathbf{R}) + \mathbf{R}\Omega'^{\,2})\end{aligned}$$

where $\boldsymbol{\Omega}'(\mathbf{v})$ is the *total* angular velocity of the passenger, due *both* to the rotation of the earth and to the motion of the hydrofoil; $\mathbf{R} = R\hat{\mathbf{x}}$ is a vector pointing from the earth's center to the hydrofoil at the equator; and the "bac cab" rule is applied to the first line. The hydrofoil's velocity has one of the four directions (E,W,N,S) $= (\hat{\mathbf{y}}, -\hat{\mathbf{y}}, \hat{\mathbf{z}}, -\hat{\mathbf{z}})$, yielding an angular velocity $\boldsymbol{\omega}$ due to hydrofoil motion relative to the earth's surface:

$$\boldsymbol{\omega} = \frac{v}{R}(\hat{\mathbf{z}}, -\hat{\mathbf{z}}, -\hat{\mathbf{y}}, \hat{\mathbf{y}})$$

To this one must add the earth's angular velocity

$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$$

in order to get the total angular velocity $\boldsymbol{\Omega}' = \boldsymbol{\omega} + \boldsymbol{\Omega}$ of the hydrofoil. Evidently $\boldsymbol{\Omega}'$ is perpendicular to $\mathbf{R}$, so

$$\mathbf{F}_{\text{cent}}(\mathbf{v}) = \hat{\mathbf{x}}mR(\Omega'(\mathbf{v}))^2$$

Therefore $\mathbf{g}'$ points along $-\hat{\mathbf{x}}$, *i.e.* toward the earth's center, for all four directions (E,W,N,S) of $\mathbf{v}$, as does $\mathbf{g}$. Thus

$$\frac{g' - g}{g} \equiv \frac{\Delta g}{g} = \frac{-R((\Omega'^{\,2} - \Omega^2)}{g}$$

For these four directions,

$$\begin{aligned}\Omega'^{\,2} - \Omega^2 &= 2\Omega\omega + \omega^2 \ \text{ (E)} \\ &= -2\Omega\omega + \omega^2 \ \text{ (W)} \\ &= \omega^2 \ \text{ (N and S)}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\Delta g}{g} &= \frac{R}{g}(-2\Omega\omega - \omega^2) \ \text{ (E)} \\ &= \frac{R}{g}(2\Omega\omega - \omega^2 \ \text{ (W)} \\ &= \frac{R}{g}(-\omega^2) \ \text{ (N and S)}\end{aligned}$$

Using $\omega/\Omega = 0.1931$ and $R\Omega^2/g = 0.003432$, we calculate $|\Delta g/g| = 0.001325$ from the $2\Omega\omega$ term and $|\Delta g/g| = 0.000128$ from the ω^2 term. Thus

$$\begin{aligned}\Delta g/g &= -0.001453 \ \text{ (E)} \\ &= +0.001197 \ \text{ (W)} \\ &= -0.000128 \ \text{ (N and S)}\end{aligned}$$

3. K&K problem 9.3

A particle moves in a circle under the influence of an inverse cube law force. This means that the potential is inverse squared and it is attractive. The effective potential is given by

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{A}{r^2}$$

The radial force is zero for a circular orbit, so we can find the radius.

$$0 = \frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{mr^3} + \frac{2A}{r^3}$$

This shows that a circular orbit can have any radius, but there is only one possible magnitude of angular momentum, given by

$$L^2 = 2Am$$

Plugging this value of the angular momentum into the effective potential, we find the peculiar result that

$$U_{\text{eff}} = 0$$

Since U_{eff} is constant, $d^2r/dt^2 = 0$, so if the particle acquires a nonzero radial velocity it will continue with the same radial velocity. If the particle moves with uniform radial velocity v_r, the following equations are satisfied

$$\frac{dr}{dt} = v_r \qquad \frac{d\theta}{dt} = \frac{L}{mr^2(t)}$$

Solving the first is easy: $r(t) = r_0 + v_r t$. Plugging this result into the second equation, we find

$$\frac{d\theta}{dt} = \frac{L}{m(r_0 + v_r t)^2}$$

We can solve this equation by direct integration, assuming that $\theta(0) = 0$:

$$\theta(t) = \int_0^t \frac{L\,dt}{m(r_0 + v_r t)^2} = \frac{L}{mv_r}\left(\frac{1}{r_0} - \frac{1}{r(t)}\right)$$

We replace L with $\sqrt{2mA}$ to get the final answer

$$\theta(t) = \frac{1}{v_r}\sqrt{\frac{2A}{m}}\left(\frac{1}{r_0} - \frac{1}{r(t)}\right)$$

4. K&K problem 9.4

A particle moves in a circular orbit in the potential $U = -A/r^n$. We want to know for which values of n the orbit is stable. The effective potential is given by

$$U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{A}{r^n}$$

To find the circular orbit radius we evaluate $dU_{\text{eff}}/dr = 0$:

$$\frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{mr^3} + \frac{nA}{r^{n+1}}$$

This gives the radius of the circular orbit r_0 when we set it to zero.

$$r_0^{n-2} = \frac{nAm}{L^2}$$

Since r_0^{n-2} must be a positive quantity for any value of n, and $A > 0$, this equation requires

$$n > 0$$

We now look at the second derivative of the effective potential at $r = r_0$. If it is positive, then it is a potential minimum and the orbit is stable.

$$\begin{aligned}\frac{d^2U_{\text{eff}}}{dr^2} &= \frac{3L^2}{mr^4} - \frac{n(n+1)A}{r^{n+2}} \\ &= \frac{1}{r^4}\left(\frac{3L^2}{m} - \frac{n(n+1)A}{r^{n-2}}\right) > 0\end{aligned}$$

Since r is always greater than zero we can divide it away. Substituting for r^{n-2} at $r = r_0$,

$$\frac{3L^2}{m} - \frac{(n+1)L^2}{m} > 0 \;\Rightarrow\; n < 2$$

Putting both inequalities together,

$$0 < n < 2$$

Recall from the previous problem that when $n = 2$ the motion is barely unstable. When $n = 0$, U is constant, so there is no attractive force, therefore no circular orbit: this case is also unstable.

5. K&K problem 9.6

A particle moves in an attractive central force Kr^4 with angular momentum l. If it moves in a circular orbit with radius r_0, the central force must provide exactly the necessary centripetal acceleration:

$$\frac{mv^2}{r_0} = Kr_0^4 = \frac{l^2}{mr_0^3} \;\Rightarrow\; r_0^7 = \frac{l^2}{mK}$$

Relative to $r = 0$, the energy of the orbit is

$$E = \frac{1}{2}mv^2 + \frac{1}{5}Kr_0^5$$

where we have integrated the force to get the potential. Plugging in $v^2 = Kr_0^5/m$, we get

$$E = \frac{1}{2}Kr_0^5 + \frac{1}{5}Kr_0^5 = \frac{7}{10}Kr_0^5$$

Substituting the above value for r_0, we get the final result for the energy (relative to $r = 0$):

$$E = \frac{7}{10}K\left(\frac{l^2}{mK}\right)^{5/7}$$

To find the frequency of small radial oscillations, we must evaluate the second derivative of the effective potential. Remember that for small oscillations the effective spring constant k for radial motion is

$$k = \left.\frac{d^2U_{\text{eff}}}{dr^2}\right|_{r_0}$$

The effective potential is

$$U_{\text{eff}} = \frac{l^2}{2mr^2} + \frac{1}{5}Kr^5$$

The second derivative is easily found

$$\frac{d^2U}{dr^2} = \frac{3l^2}{mr^4} + 4Kr^3$$

Plugging in $l^2 = m^2v^2r^2 = mKr_0^7$, we get

$$\left.\frac{d^2U}{dr^2}\right|_{r_0} = 7Kr_0^3$$

Substituting the value of r_0, we find the effective spring constant k and the angular frequency ω of radial oscillation about the stable circular orbit:

$$\begin{aligned}k &= m\omega^2 = 7K\left(\frac{l^2}{mK}\right)^{3/7} \\ \omega &= \sqrt{\frac{7K}{m}}\left(\frac{l^2}{mK}\right)^{3/14}\end{aligned}$$

6. K&K problem 9.12

A spacecraft of mass m orbits the earth at a radius $r = 2R_e$. It will transfer to another circular orbit with radius $r = 4R_e$.

(a.) We know the radius of each orbit, so we can easily find the energies of the two orbits. The energy of a bound orbit in a $1/r$ potential is given by

$$E = -\frac{GMm}{A}$$

where A is the major axis of the elliptical (here the diameter of the circular) orbit. We can use this to find the energies of the two orbits. The values of A are simply $4R_e$ for the first and $8R_e$

for the second. The energy input needed to go from one orbit to the other is at least

$$\Delta E = -\frac{GMm}{R_e}\left(\frac{1}{8} - \frac{1}{4}\right) = \frac{GMm}{8R_e}$$

Plugging in the values given,

$$\Delta E = 2.34 \times 10^{10} \text{ joules}$$

(b.) At point A, the rocket is fired, putting the spacecraft in an elliptical orbit. The major axis of this orbit is $A = 6R_e$. To find the initial speed, we use the energy equation. The energy is partly gravitational potential energy and partly kinetic energy:

$$E = \frac{1}{2}mv_0^2 - \frac{GMm}{2R_e} = -\frac{GMm}{4R_e}$$

Solving this equation for v_0, we find the orbital speed

$$v_0 = \sqrt{\frac{GM}{2R_e}}$$

The energy of the elliptical orbit is given by

$$E = -\frac{GMm}{6R_e} \;\Rightarrow\; \Delta E = \frac{GMm}{12R_e}$$

The change in energy at point A is entirely due to a change in speed.

$$\Delta E = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 \;\Rightarrow\; v_1 = \sqrt{\frac{2GM}{3R_e}}$$

The change in speed required at point A is thus

$$\Delta v_A = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right)\sqrt{\frac{GM}{R_e}} = 865 \text{ m/sec}$$

We repeat this analysis at point B. Conservation of angular momentum gives

$$2R_e v_1 = 4R_e v_2 \;\Rightarrow\; v_2 = \sqrt{\frac{GM}{6R_e}}$$

The energy of the new circular orbit is given by

$$E = -\frac{GMm}{8R_e} \;\Rightarrow\; \Delta E = \frac{GMm}{24R_e}$$

Again this is due to the change in speed.

$$\Delta E = \frac{1}{2}mv_3^2 - \frac{1}{2}mv_2^2 \;\Rightarrow\; v_3 = \sqrt{\frac{GM}{4R_e}}$$

Finally we obtain the change in speed at point B

$$\Delta v_B = \left(\frac{1}{2} - \sqrt{\frac{1}{6}}\right)\sqrt{\frac{GM}{R_e}} = 726 \text{ m/sec}$$

Since the two velocities at point A are both tangent to each other, and similarly for point B, the only changes in the velocities at either point are the changes in their magnitudes.

7. A satellite of mass m moves in a circular orbit of radius R at speed v. It is influenced by the gravity of a fixed mass at the origin.

(a.) The mechanical energy of the satellite is given by

$$E = \frac{1}{2}mV^2 - \frac{GMm}{R}$$

We know that gravity exactly provides the centripetal acceleration.

$$\frac{GMm}{R^2} = \frac{mV^2}{R} \;\Rightarrow\; \frac{GM}{R} = V^2$$

The total energy is thus

$$E = \frac{1}{2}mV^2 - mV^2 = -\frac{1}{2}mV^2$$

(b.) At a certain point on the orbit, the direction of travel of the satellite changes. The magnitude of the velocity does not change, so the total energy of the orbit doesn't change. We can now find the kinetic energy at closest approach:

$$\begin{aligned} E = -\frac{1}{2}mV^2 &= \frac{1}{2}mv^2 - \frac{5GMm}{R} \\ &= \frac{1}{2}mv^2 - 5mV^2 \\ \Rightarrow\; v &= 3V \end{aligned}$$

(c.) The circular orbit has angular momentum $L_1 = mRV$, while the elliptical orbit's angular momentum, evaluated at the perigee, is

$$L_2 = m\frac{R}{5}3V = \frac{3}{5}L_1$$

Therefore, just after the transition from circular to elliptical orbit, a fraction $\frac{3}{5}$ of the original velocity must remain tangential, while $\frac{4}{5}$ of it becomes radial (the squares of the two fractions must add to unity according to Pythagoras). Therefore the satellite turns through an angle

$$\alpha = \arctan \frac{4/5}{3/5} = 53.1^\circ$$

8. A spaceship is moving on a circular path that will take it directly through a gas cloud. The angular momentum with respect to the gas cloud is measured to be constant. We want to know what attractive central force causes this. Immediately we notice that as the ship passes through the center of the cloud, it velocity must become infinite, because the angular momentum $l = mvr$ is conserved. If l were zero, the ship could only fall straight into the cloud.

We can express the circular trajectory of the ship as a function of θ by inspection:

$$r(\theta) = 2R\cos\theta \quad (-\pi/2 < \theta < \pi/2)$$

Take ϕ to be the azimuth of the spaceship on the circle ($-\pi < \phi < \pi$), with $\phi \equiv 0$ when $\theta = 0$. Consider the isosceles triangle with sides r, r, and R. Requiring its angles to add up to π, it is easy to see that $\phi = 2\theta$.

We are given two definite facts. One is that the ship's angular momentum about the center of the cloud

$$L = mr^2\dot{\theta} = 4mR^2\dot{\theta}\cos^2\theta$$

is constant. (This expression confirms our previous observation that the ship's velocity $2R\dot{\theta}$ must be infinite at the center of the cloud, where $\cos\theta = \cos\pi/2 = 0$.) The second fact is that the spaceship moves in a circle of radius R. The centripetal force $mR\dot{\phi}^2$ required to keep it in circular motion must be supplied by the component along $\mathbf{R}$ of the unknown attractive force F:

$$F\cos\theta = mR\dot{\phi}^2 = 4mR\dot{\theta}^2$$

Using the previous equation for L to eliminate $\dot{\theta}$ from this equation,

$$F\cos\theta = 4mR\frac{L^2}{16m^2R^4\cos^4\theta}$$
$$F = \frac{L^2}{4mR^3\cos^5\theta}$$

Finally, using the first equation to eliminate $\cos\theta$,

$$F = \frac{L^2 32R^5}{4mR^3r^5}$$
$$F = \frac{8L^2R^2}{mr^5}$$

Since L and R are constant, the unknown attractive force depends on the inverse fifth power of the spaceship's separation from the center of the cloud, for this particular spaceship trajectory. The last equation is the desired result. However, this is no simple force field: its coupling to the spaceship is contrived to depend quadratically both upon the spaceship's angular momentum about the cloud's center and upon the radius of its circular orbit.

As an alternative to considering the centripetal force that must be supplied by F, one can hypothesize that F is a conservative as well as a central force. (At least it is clear from the fact that the spaceship's orbit is closed that there can be no *monotonic* decrease or increase in the total energy E). From the above equation for L, one readily sees that the ship's speed $v = R\dot{\phi} = 2R\dot{\theta}$ is proportional to r^{-2}. Therefore the ship's kinetic energy K is proportional to r^{-4}. If E is to be conserved, the potential energy U also must be proportional to r^{-4} so that it can cancel the r dependence of K; its radial derivative $-dU/dr = F_r$ must then be proportional to r^{-5}. The constant of proportionality is easily verified to be the same as is given above.

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PROBLEM SET 9

1.

a. Expand in Taylor series: $f(x) = \ln(1-x)$; $f(x) = 1/(1+x)$.

b. Given two functions $s(x)$ and $c(x)$ such that $ds/dx = c$ and $dc/dx = s$, prove that $s(x) + c(x) = [s(0) + c(0)]e^x$.

2.

a. French problem 1-4(b).

b. French problem 1-9.

c. Prove DeMoivre's theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

3. French problem 3-15.

4. At $t = 0$ a bullet of mass m and velocity v_0 strikes a motionless block of mass M which is connected to a wall by a spring of constant k. The block moves with coordinate $x(t)$ (along the direction of the bullet) on a frictionless table next to the wall. The bullet embeds itself in the block. If $x(t) = \Re[\mathcal{A}\exp(i(\omega t + \phi))]$, with $\mathcal{A}$ real, evaluate ω, $\mathcal{A}$, and ϕ.

5. French problem 4-5.

6. French problem 4-8.

7. A piano has middle C = 256 Hz, and C-above-middle-C = 512 Hz. The white keys of its middle octave consist of middle C; D (1 step above middle C); E (2 steps); F (2.5); G (3.5); A (4.5); B (5.5); and C-above-middle-C (6 steps). Each step causes the frequency to be multiplied by a fixed factor.

a. Find the frequencies of D, E, F, G, A, and B.

b. If G were tuned to a "perfect fifth", its second harmonic ($2\times$ the fundamental frequency) would be the same as middle C's third harmonic. Find the *beat frequency* between the second harmonic of G and the third harmonic of middle C. (Pianos are tuned by listening for such beats.) Stringed instruments are tuned (for compositions in the key of C) so that this beat frequency is zero, producing a smoother tone. When a composition in a different key is played, the stringed instrument can be retuned for that key, which would be impossible for the piano.

8. For an underdamped undriven harmonic oscillator with $\omega_0/\gamma \equiv Q = 100$, find the number of oscillations required to reduce the amplitude of oscillation by the factor $e^\pi \approx 23.1$.

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SOLUTION TO PROBLEM SET 9

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1.

(a.) The Taylor series for $\ln(1-x)$ is found as follows:

$$f(x) = \ln(1-x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}$ denotes the nth derivative of f.

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots$$
$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

We do the same for $f(x) = 1/(1+x)$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

(b.) We have two functions $c(x)$ and $s(x)$ related as follows:

$$\frac{ds}{dx} = c \qquad \frac{dc}{dx} = s$$

An easy way to approach this problem is to solve these differential equations simultaneously. However, as is the case for most "easy" ways to do things, the mathematics leading up to the solution is somewhat advanced. Instead we will use the Taylor series to solve it. Expanding around $x = 0$,

$$s(x) = s(0) + c(0)x + \frac{1}{2}s(0)x^2 + \cdots$$

$$c(x) = c(0) + s(0)x + \frac{1}{2}c(0)x^2 + \cdots$$

Adding these two equations, we see that

$$s(x) + c(x) = [s(0) + c(0)] \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We recognize the sum as the Taylor series of e^x.

$$s(x) + c(x) = [s(0) + c(0)]e^x$$

2.

(a.) French problem 1-4(b).
The magnitude of a complex number $a + ib$, where a and b are real, is just $\sqrt{a^2+b^2}$. The phase angle θ is equal to $\tan^{-1}(b/a)$, where the quadrant is determined by the signs of both a and b. The first vector $(2 + i\sqrt{3})$ has length $\sqrt{7}$ and phase $\theta = \tan^{-1}(\sqrt{3}/2) = 40.9°$. The second vector $(2 - i\sqrt{3})^2$ is merely the square of the complex conjugate of the first vector. Therefore it has length 7 and $-2\times$ the phase, or $-81.8°$.

(b.) French problem 1-9.
The value of i^i is a little odd, but here goes. We need to know how to find the log of a complex number:

$$\ln z = \ln |z| e^{i\theta} = \ln |z| + i\theta$$

There is an ambiguity here, because we can always add an integer multiple of 2π to θ. Here we will choose not to do so, but simply take the value of θ to be between $-\pi$ and π. Thus we obtain

$$i^i = e^{i \ln i} = e^{i^2 \pi/2} = e^{-\pi/2} = 0.2079$$

This means that paying 20 cents is bargain, but a very small one.

(c.) Prove $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i \sin n\theta)$. This is pretty trivial when we remember

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta}$$
$$= \cos n\theta + i \sin n\theta$$

3. French problem 3-15.
An oscillatory system loses energy according

to $E = E_0 e^{-\gamma t}$. We define the Q value as $Q \equiv \omega_0/\gamma$.

(a.) Middle C on a piano is played, and the energy decreases to half of its initial value in one second. The frequency is 256 Hz. The angular frequency is this times 2π, so ω_0 =1608.5/sec. We find γ from

$$\frac{1}{2} = (e^{-\gamma/2})^2 \;\Rightarrow\; \gamma = 0.693$$

Lastly , the Q of the oscillator is

$$Q = 1608.5/0.693 = 2321$$

(b.) The note one octave above is struck (512 Hz). The decay time is the same, so the Q value is simply doubled: Q=4642.

(c.) A damped harmonic oscillator has mass $m = 0.1$ kg, spring constant $k = 0.9$ N/m, and a damping constant b. The energy decays to $1/e$ in 4 seconds. This means that

$$\frac{1}{e} = e^{-4\gamma} \;\Rightarrow\; \gamma = 0.25 \text{ sec}^{-1}$$
$$\Rightarrow\; b = m\gamma = 0.025 \text{ kg sec}^{-1}$$

The natural frequency $\omega_0 = \sqrt{k/m} = 3$ Hz. Finally, the Q of the oscillator is $Q = \omega_0/\gamma = 12$.

4. At $t = 0$, the bullet collides inelastically with the block, so only the momentum is conserved. The final velocity of the block and bullet is given by

$$mv_0 = (M+m)v \;\Rightarrow\; v = \frac{mv_0}{M+m}$$

We now have the initial conditions for the oscillation. The initial position is $x(0) = 0$ and the initial velocity is $v(0) = v$. The frequency ω is given as usual by $\sqrt{k/\text{mass}}$, but the mass in question is the total mass of the system:

$$\omega = \sqrt{\frac{k}{M+m}}$$

The solution is given by $x(t) = \Re[A\exp(i(\omega t + \phi))]$. The initial position tells us that $\cos\phi = 0$. This is ambiguous because the cosine is zero in two places in one oscillation. We want a place where it is zero and rising, because we know that x is increasing at the instant of contact. This is

$$\phi = -\pi/2$$

The solution is now

$$x(t) = \Re[A\exp(i(\omega t - \pi/2)])$$

We differentiate to get the velocity, and evaluate this at $t = 0$.

$$v(0) = v = -A\omega\sin(-\pi/2) = A\omega \;\Rightarrow\; A = \frac{v}{\omega}$$

This gives the final result for the amplitude

$$A = \frac{mv_0}{\sqrt{k(M+m)}}$$

5. French 4-5.

(a.) A pendulum is forced by moving the point of support. The coordinate x gives the location of the pendulum bob, and ξ gives the location of the point of support. The forces on the pendulum are the damping force, which we must assume to be proportional to the absolute velocity of the pendulum, and the force of gravity. The force of gravity depends on the angle by which the pendulum is raised. This is proportional to the distance that the pendulum bob is displaced from the point of support, $x - \xi$. This gives the equation of motion

$$m\frac{d^2x}{d^t 2} = -b\frac{dx}{dt} - \frac{mg}{l}(x - \xi)$$

Using $\omega_0^2 = g/l$ and $\gamma = b/m$, we put this in the standard form with ξ as a forcing term.

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \omega_0^2 \xi$$

(b.) The motion of the point of support is given by $\xi(t) = \xi_0 \cos\omega t$. We use the formula for the amplitude of forced oscillation, but we note that in this case, the equation is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \omega_0^2 \xi_0 \cos\omega t$$

The constant $\omega_0^2\xi_0$ takes the place of the F_0/m we normally see in this type of equation. The amplitude of the oscillation is thus given by

$$A(\omega) = \frac{\omega_0^2\xi_0}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$$

At exact resonance, $\omega = \omega_0$ and the amplitude is

$$A(\omega_0) = \xi_0\frac{\omega_0}{\gamma} = Q\xi_0$$

Now we want to find Q. We are given that the forcing amplitude is $\xi_0 = 1$ mm. The length of the pendulum is $l = 1$ m, so this gives $\omega_0 = 3.13$ sec^{-1}. We know that the amplitude falls off by a factor of e after 50 swings, or 50 periods. We know that $A = A_0\exp(-\gamma t/2)$ so $\gamma t = 2$. t is 50 periods, or $100\pi/\Omega$, where Ω is the frequency of free oscillation

$$\Omega = \sqrt{\omega_0^2+\gamma^2/4}$$

Then

$$\begin{aligned} \gamma t &= 2 \\ \gamma\frac{100\pi}{\Omega} &= 2 \\ 50\pi\gamma &= \sqrt{\omega_0^2+\gamma^2/4} \\ (50\pi)^2\gamma^2 &= \omega_0^2+\gamma^2/4 \end{aligned}$$

Plugging in the numbers, we get $\gamma = 0.0199$. This gives us $Q = 157$, and $A = 15.7$ cm.

(c.) We want to find the frequencies where the amplitude is half of the resonant value. We merely solve

$$\frac{\omega_0^2\xi_0}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = \xi_0\frac{\omega_0}{2\gamma}$$

This gives

$$4\omega_0^2\gamma^2 = (\omega_0^2-\omega^2)^2+\gamma^2\omega^2$$

Turning this into a quadratic equation for ω^2, we get

$$0 = \omega^4 + (\gamma^2-2\omega_0^2)\omega^2 + \omega_0^4 - 4\omega_0^2\gamma^2$$

The solutions to this are

$$\omega^2 = \frac{1}{2}\left(2\omega_0^2-\gamma^2 \pm \sqrt{\gamma^4+12\gamma^2\omega_0^2}\right)$$

Plugging in the numbers, the two frequencies are $\omega = 3.147$ sec^{-1} and $\omega = 3.113$ sec^{-1}.

6. French 4-8.

(a.) A mass is under the influence of a viscous force $F = -bv$. Let $\gamma = b/m$ as usual. The equation of motion is

$$\frac{dv}{dt} + \gamma v = 0$$

We can easily solve this equation by direct integration.

$$v(t) = v_0 e^{-\gamma t}$$

We simply integrate this equation with respect to t to get the position.

$$x(t) = C - \frac{v_0}{\gamma}e^{-\gamma t}$$

C is the integration constant that will allow us to fit an initial condition.

(b.) A driving force $F = F_0\cos\omega t$ is turned on. We want to find the steady state motion. We will use a complex exponential for the forcing term, with the understanding that we take the real part when we're done. The new equation of motion is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} = \frac{F_0}{m}e^{i\omega t}$$

Assume a solution of the form

$$x(t) = Ae^{i\omega t}$$

where A is a complex number. Plugging this into the equation of motion, we see that

$$-\omega^2 A + i\omega\gamma A = \frac{F_0}{m} \;\Rightarrow\; A = \frac{F_0/m}{-\omega^2+i\omega\gamma}$$

We write the denominator as the product of a magnitude and a phase. The magnitude is $\sqrt{\omega^4+\omega^2\gamma^2}$. The denominator has a negative real part and a positive imaginary part, so it is in the second quadrant with phase $\pi - \arctan(\gamma/\omega)$. Since the numerator is real, the phase of A is minus the phase of its denominator, or $\arctan(\gamma/\omega) - \pi$. According to the notation of the problem, the phase of the oscillation is $-\delta$, so we find

$$\delta = \pi - \arctan(\gamma/\omega)$$

The amplitude of the oscillation is just the magnitude of A, given by

$$|A| = \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}$$

The general solution to the problem is

$$x(t) = C - \frac{v_0}{\gamma}e^{-\gamma t} - \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\cos(\omega t + \tan^{-1}(\gamma/\omega))$$

At $t = 0$, we want $x = 0$. At $t = 0$, the last term B in the general solution is

$$\begin{aligned} B(0) &= -\frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\cos(\tan^{-1}(\gamma/\omega)) \\ &= -\frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\frac{\omega}{\sqrt{\omega^2 + \gamma^2}} \\ &= -\frac{F_0/m}{\omega^2 + \gamma^2} \end{aligned}$$

Thus the condition $x(0) = 0$ gives us one equation for C and v_0:

$$C = \frac{v_0}{\gamma} + \frac{F_0/m}{\omega^2 + \gamma^2}$$

The first time derivative of B, evaluated at $t = 0$, is

$$\begin{aligned} \dot{B}(0) &= \frac{F_0/m}{\sqrt{\omega^2 + \gamma^2}}\sin(\tan^{-1}(\gamma/\omega)) \\ &= \frac{F_0\gamma/m}{\omega^2 + \gamma^2} \end{aligned}$$

Then, requiring the first time derivative of the general solution to vanish at $t = 0$, the second equation for v_0 and C is

$$\begin{aligned} 0 &= 0 + v_0 + \frac{F_0\gamma/m}{\omega^2 + \gamma^2} \\ v_0 &= -\frac{F_0\gamma/m}{\omega^2 + \gamma^2} \end{aligned}$$

Plugging this value of v_0 into the first equation,

$$C = -\frac{F_0/m}{\omega^2 + \gamma^2} + \frac{F_0/m}{\omega^2 + \gamma^2} = 0$$

Collecting these results, the solution satisfying both boundary conditions is

$$x(t) = \frac{F_0/m}{\omega^2 + \gamma^2}e^{-\gamma t} - \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\cos(\omega t - \tan^{-1}(\gamma/\omega))$$

7. Middle C is 256 Hz, and C above it is double that frequency, or 512 Hz. The scale is divided into 6 whole steps, or 12 half steps. The note after each half step is a constant multiple f of the frequency of the previous note. When we have gone up twelve half steps, the frequency will have doubled. The constant factor f is thus given by $f^{12} = 2$ or $f = 2^{1/12}$.

(a.) The frequencies in the scale are thus C=256, D=287.4, E=322.5, F=341.7, G=383.6, A=430.5, B=483.3, C=512 Hz. These are zero, two, four, five, seven, nine, eleven, and twelve half steps above middle C, respectively.

(b.) Middle C's third harmonic is three times its fundamental frequency, or 768 Hz. The second harmonic of G is 767.133 Hz. The beat frequency is always just the difference between the two frequencies. In this case the beat frequency is 0.867 Hz, which is easily audible to the piano tuner.

8. An undriven oscillator that is underdamped has a Q of 100. We want to know how many oscillations it takes to damp by a factor of e^{π}. This just means that $\gamma t/2 = \pi$. Now we want to write t in terms of the number of oscillations n. This takes n times the period, or $t = 2\pi n/\omega_0$. This gives $\gamma n/\omega_0 = 1$. Remember that $Q = \omega_0/\gamma$, so $n = Q = 100$.

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PROBLEM SET 10

1. French problem 5-6.

2. French problem 5-10.

3. French problem 5-14.

4. A transverse wave on a string has:

- at x=0, a time variation of the form $\cos\omega t + \sin\omega t$ with a period of 10^{-2} sec.
- a wavespeed of 10 m/sec.
- the appearance of propagating to the left (toward smaller x).
- an amplitude of 0.01 m.

Suppose that the string is described by

$$y(x,t) = \Re(\mathcal{A}_+ \exp i(\omega t - kx) + \mathcal{A}_- \exp i(\omega t + kx))$$

where $\mathcal{A}_\pm$ are complex constants.

a. Substitute numbers in place of all the variables (except x and t) in the above equation.

b. Compute the maximum transverse speed and the maximum slope of the string.

5. If surface tension is included, the phase velocity of a surface wave on a liquid of mass density ρ and surface tension T is

$$v_{\text{ph}} = \sqrt{\frac{2\pi T}{\lambda\rho} + \frac{g\lambda}{2\pi}}$$

where λ is the wavelength and g is the acceleration due to earth's gravity.

a. Find the wavelength, frequency, and phase velocity of ripples on water which advance with *minimum* speed.

b. Find the group velocity of such a wave.

6. A non-dispersive string (ω/k = constant), initially at rest, has length L ($0 \le x \le L$), and mass per unit length μ. At t=0, its shape is

$$y(x) = 3\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}$$

and it is under tension S.

a. What is its period of oscillation T?

b. At $t = T/2$, what is its shape?

7. A guitar string tuned to middle C (256 Hz) is "plucked" at exact center by giving it nonzero transverse velocity in a very small region. What tones (in Hz) will be heard in addition to 256 Hz?

University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 10

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. French 5-6.

(a.) All three springs are identical, with constant k. The equations of motion are

$$\frac{d^2x_A}{dt^2} + 2\omega_0^2 x_A - \omega_0^2 x_B = 0$$

$$\frac{d^2x_B}{dt^2} + 2\omega_0^2 x_B - \omega_0^2 x_A = 0$$

We plug in a guessed solution, where the two masses oscillate at the same frequency, but with different amplitudes A and B. This gives

$$-\omega^2 A + 2\omega_0^2 A - \omega_0^2 B = 0 \;\Rightarrow\; B = A\frac{2\omega_0^2 - \omega^2}{\omega_0^2}$$

$$-\omega^2 B + 2\omega_0^2 B - \omega_0^2 A = 0 \;\Rightarrow\; B = A\frac{\omega_0^2}{2\omega_0^2 - \omega^2}$$

Equating these we see that

$$(2\omega_0^2 - \omega^2)^2 - \omega_0^4 = 0$$

Solving this quadratic equation, we find that the two frequencies are $\omega^2 = \omega_0^2$ and $\omega^2 = 3\omega_0^2$.

(b.) One mass is displaced by 5 cm. This excites each normal mode equally, with amplitude 2.5 cm. To see this, first excite the first normal mode with amplitude 2.5 cm. Now both masses are +2.5 cm from equilibrium. Now excite the second normal mode, also with amplitude 2.5 cm. This moves one mass forward 2.5 cm, and the other back 2.5 cm. One is now 5cm from equilibrium, and the other is at its equilibrium position. It is mass B that is displaced, so the masses obey

$$x_A = 2.5\cos\omega_0 t - 2.5\cos\sqrt{3}\omega_0 t$$

$$x_B = 2.5\cos\omega_0 t + 2.5\cos\sqrt{3}\omega_0 t$$

(c.) After a time τ such that $\cos\omega_0\tau = \cos\sqrt{3}\omega_0\tau$, mass A returns to its equilibrium position $x_A = 0$. This happens when $\omega_0\tau$ is in the second quadrant and $\sqrt{3}\omega_0\tau$ is in the third:

$$\pi - \omega_0\tau = \sqrt{3}\omega_0\tau - \pi$$
$$\omega_0\tau = \frac{2\pi}{1+\sqrt{3}}$$

However, at $t = \tau$, mass B will not have returned to its full original |displacement| since $|\cos\tau| < 1$. Thus, even though mass A will be back in place, mass B will not, and the system will not have returned to (plus or minus) its original state. In fact, because the ratio of the two normal frequencies is irrational, once both normal modes are excited the system can never return to its original state.

2. French 5-10.
The equations of motion for this double spring system are as follows. The coordinate of the top mass is x_A and the coordinate of the bottom mass is x_B.

$$\frac{d^2x_A}{dt^2} + 2\omega_0^2 x_A - \omega_0^2 x_B = 0$$

$$\frac{d^2x_B}{dt^2} + \omega_0^2 x_B - \omega_0^2 x_A = 0$$

Plugging in the standard guess that both masses oscillate at the same frequency, but at amplitudes A and B, we get the following equations.

$$(2\omega_0^2 - \omega^2)A = \omega_0^2 B$$

$$(\omega_0^2 - \omega^2)B = \omega_0^2 A$$

Equating these, we get the quadratic equation

$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

The solutions to this equation are

$$\omega_\pm^2 = \omega_0^2\frac{3\pm\sqrt{5}}{2}$$

The amplitudes in these modes are easily found. For ω_+, we have

$$\left(2\omega_0^2 - \frac{3}{2}\omega_0^2 - \frac{\sqrt{5}}{2}\omega_0^2\right)A_+ = \omega_0^2 B_+$$
$$B_+ = \frac{1-\sqrt{5}}{2}A_+$$

Likewise for ω_-,

$$B_- = \frac{1+\sqrt{5}}{2}A_-$$

3. French 5-14.
In the first normal mode, the three particles have an amplitude ratio $\sqrt{2}/2 : 1 : \sqrt{2}/2$. The second normal mode has amplitude ratios $1 : 0 : -1$. The third normal mode has amplitude ratios $\sqrt{2}/2 : -1 : \sqrt{2}/2$.

4. A wave is described by

$$y(x,t) = \Re\left[A_+ e^{i(\omega t - kx)} + A_- e^{i(\omega t + kx)}\right]$$

(a.) We know that the wave is moving to the left. This corresponds to the second exponential. To see this, we note that a specific place on the wave train always has the same value of $\omega t \pm kx$. We want to see what happens when t increases. For the solution $\exp(i(\omega t + kx))$, we see that as t increases, x must decrease to stay on the same place in the wave. This is a left moving wave. We thus note that $A_+ = 0$. We can now write the complex constant $A_- = Ae^{i\delta}$, where A and δ are real.

$$y(x,t) = \Re\left[Ae^{i(\omega t + kx + \delta)}\right]$$

At $x = 0$ we know that the time dependence in proportional to $\cos\omega t + \sin\omega t$. This tells us that

$$\cos(\omega t + \delta) \propto \cos\omega t + \sin\omega t$$

Using the formula for the cosine of a sum

$$\cos(\omega t + \delta) = \cos\omega t\cos\delta - \sin\omega t\sin\delta$$

For this to work we see that

$$\cos\delta = -\sin\delta \;\Rightarrow\; \delta = -\frac{\pi}{4}$$

Now we can find the amplitude and frequency. The solution is

$$y(x,t) = A\cos\left(\omega t + kx - \frac{\pi}{4}\right)$$

The amplitude is given as 0.01 m, and the period is 10^{-2} sec. The frequency $\omega = 2\pi/T = 200\pi$ sec^{-1}. The speed of waves on the string is $c =$10 m/sec, and we know that $\omega = ck$, so this tells us that $k = 20\pi$ m^{-1}. We now have the final result

$$y(x,t) = \Re\left[0.01e^{i(200\pi t + 20\pi x - \pi/4)}\right]$$
$$= 0.01\cos(200\pi t + 20\pi x - \pi/4)$$

(b.) We can now compute the maximum transverse speed and maximum slope. The transverse speed is

$$\frac{dy}{dt} = -2\pi\sin(200\pi t + 20\pi x - \pi/4)$$

The maximum value that the sine takes is 1, so the maximum transverse speed is $2\pi = 6.28$ m/sec. We can likewise find the maximum slope

$$\frac{dy}{dx} = -0.2\pi\sin(200\pi t + 20\pi x - \pi/4)$$

The maximum slope is thus $\pi/5 = 0.628$.

5. The phase velocity of surface waves is given by

$$v_{\text{ph}} = \sqrt{\frac{2\pi T}{\lambda\rho} + \frac{g\lambda}{2\pi}} = \sqrt{\frac{kT}{\rho} + \frac{g}{k}}$$

(a.) Notice that at both zero and infinite wavenumber, the phase velocity is infinite. To find the minimum phase velocity, we differentiate v_{ph} with respect to k and set to zero.

$$\frac{dv_{\text{ph}}}{dk} = \frac{T/\rho - g/k^2}{2\sqrt{kT/\rho + g/k}} = 0$$
$$k = \sqrt{\frac{\rho g}{T}} \;\Rightarrow\; \lambda = 2\pi\sqrt{\frac{T}{\rho g}}$$

The phase velocity at this wavenumber is

$$v_{\rm ph} = \left(\frac{4gT}{\rho}\right)^{1/4}$$

The frequency $\omega = v_{\rm ph}k$, which gives

$$\omega = \left(\frac{4gT}{\rho}\right)^{1/4}\sqrt{\frac{\rho g}{T}} = \left(\frac{4\rho g^3}{T}\right)^{1/4}$$

(b.) The group velocity of this wave is given by

$$v_{\rm gr} = \left.\frac{d\omega}{dk}\right|_{k=\sqrt{\rho g/T}}$$

We know that $\omega = v_{\rm ph}k$, so

$$\omega = \sqrt{\frac{Tk^3}{\rho} + gk}$$

Taking the derivative with respect to k and evaluating at the wavenumber we found before

$$\begin{aligned}\left.\frac{d\omega}{dk}\right|_{k=\sqrt{\rho g/T}} &= \left.\frac{3Tk^2/\rho + g}{2\sqrt{Tk^3/\rho + gk}}\right|_{k=\sqrt{\rho g/T}} \\ &= \sqrt{\frac{2g}{k}} = \left(\frac{4gT}{\rho}\right)^{1/4}\end{aligned}$$

The slowest waves have the same phase and group velocities. This is a general result. Look at the equation for the frequency, and differentiate it to get the group velocity

$$\omega = v_{\rm ph}k \;\Rightarrow\; \frac{d\omega}{dk} = v_{\rm gr} = v_{\rm ph} + k\frac{dv_{\rm ph}}{dk}$$

We chose the phase velocity to be a minimum, so $v_{\rm gr} = v_{\rm ph}$.

6. A string has tension S and linear mass density μ. This tells us the phase velocity of waves on it, $c = \sqrt{S/\mu}$. The string has length L. At $t = 0$, the string's shape is

$$y(x, 0) = 3\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}$$

(a.) The frequencies of each of these is $\omega = ck$, so the first term has $\omega_1 = c\pi/L$ and the second term has frequency $\omega_2 = 3c\pi/L$. The periods of these two oscillations are $T_1 = 2L/c$ and $T_2 = 2L/3c$. The period of the total oscillation is the longer period, T_1. In one long period the fast oscillation has had exactly three periods. Thus, the period is

$$T = \frac{2L}{c} = 2L\sqrt{\frac{\mu}{S}}$$

(b.) After a time $T/2$, the first term has gone through a half period, and the second term has gone through one and a half periods. In both cases, this just means that there is a minus sign out front.

$$y(x, T/2) = -3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$$

7. A string of frequency 256 Hz is plucked in the exact center. This means that the even numbered modes are not exited at all. This is because the initial condition is symmetric around the middle of the string, and the even numbered modes are antisymmetric around the middle of the string. The odd numbered modes are also symmetric around the center of the string, so they survive. These frequencies are

$$f_n = (2n+1)256 \text{ Hz},\; n = 0\ldots\infty$$

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PROBLEM SET 11

1. French problem 6-9.

2. French problem 6-15(a).

3. French problem 8-9. Note that $\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT$ for the sodium atoms in the vapor, where T is the temperature in degrees Kelvin (°K), and k is Boltzmann's constant, 1.38×10^{-23} J/°K.

4. French problem 8-12.

5. A tank is filled with water to a height H. A hole is punched in its wall a depth h below the surface of the water.

a. Find the horizontal distance from the bottom of the tank that the stream of water hits the ground.

b. Could a hole punched at a different depth produce a stream with the same horizontal range? If so, at what depth?

6. Consider the stagnant air at the front edge of an airplane wing and the air rushing over a wing surface at speed v. Find the greatest possible value for v in streamline flow, using Bernoulli's equation and assuming that air is incompressible. Take the density of air to be 1.2×10^{-3} g/cm^3. Compare this numerical result with the speed of sound, 340 m/sec.

7. Verify by explicit computation in a Cartesian coordinate system that

$$\nabla \times (\nabla f(x,y,z)) = 0$$

8.

(a.) Consider the function $f(x,y,z) \equiv x^2 + y^2 - z^2$. At the point $(x,y,z) = (3,4,5)$, find the *direction* of a vector $d\mathbf{s}$ (of small fixed length) such that $df/|d\mathbf{s}|$ is a maximum.

(b.) Consider the surface $z(x,y) = \sqrt{x^2+y^2}$. At the point $(x,y,z) = (3,4,5)$, find the *direction* of a vector $d\mathbf{u}$ (of small fixed length) which is normal to this surface.

9. A fluid has a velocity field

$$\mathbf{v}(x,y,z,t) = (\hat{\mathbf{y}}x - \hat{\mathbf{x}}y)\omega(t)$$

where ω is some function of time t.

(a.) Prove that the fluid density $\rho(x,y,z,t)$ satisfies

$$y\frac{\partial\rho}{\partial x} - x\frac{\partial\rho}{\partial y} = \frac{1}{\omega}\frac{\partial\rho}{\partial t}$$

(b.) Show that the angular velocity of the fluid about the origin, evaluated at an arbitrary point, is half of $\nabla \times \mathbf{v}$ evaluated at the same point.

(c.) If $\omega(t) = \omega_0 =$ constant, prove that

$$\frac{d\mathbf{v}}{dt} = -\mathbf{r}\omega_0^2$$

where $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$, and $d\mathbf{v}/dt$ is the time rate of change of the velocity of an element of fluid that is *temporarily* at (x,y,z) at time t.

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SOLUTION TO PROBLEM SET 11

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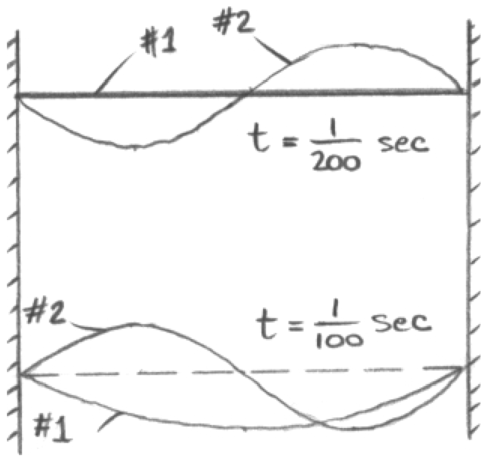
1. French 6-9.
(a.) The lowest resonant frequency of a room is 50 Hz. All integer multiples of this frequency are also resonant. The lowest two modes are excited. These are 50 Hz (the fundamental) and 100 Hz (the [first] harmonic). The amplitude is maximum at $t = 0$. The time interval $t = 1/200$ sec is one fourth of a period for the fundamental and half a period for the harmonic. $t = 1/100 sec$ is one half of a period for the fundamental and a full period for the harmonic. These modes look like



(b.) We can write the total displacement as

$$\xi(x) = A_1 \sin(\pi x/L) \cos(100\pi t) + \\ + A_2 \sin(2\pi x/L) \cos(200\pi t)$$

where A_1 and A_2 are the unknown amplitudes for the fundamental and harmonic mode, respectively. In particular

$$\xi(L/2) = A_1 \cos(100\pi t) + 0$$
$$\xi(L/4) = \frac{A_1}{\sqrt{2}} \cos(100\pi t) + A_2 \cos(200\pi t)$$
$$\xi(3L/4) = \frac{A_1}{\sqrt{2}} \cos(100\pi t) - A_2 \cos(200\pi t)$$

Since the amplitude at $L/2$ is due only to the fundamental, and is equal to 10 μ, we know that $A_1 = 10\ \mu$. The last two equations become

$$\xi(L/4) = 5\sqrt{2} \cos(100\pi t) + A_2 \cos(200\pi t)$$
$$\xi(3L/4) = 5\sqrt{2} \cos(100\pi t) - A_2 \cos(200\pi t)$$

As a trial solution, we assume that A_2 is positive. Then the maximum displacement of 10 μ at $L/4$ is reached at $t = 0$; therefore $A_2 = (10 - 5\sqrt{2})$ μ. The maximum |displacement| at $3L/4$, a negative displacement in this case, is reached at $t = 1/100$ sec, when the harmonic has changed phase by a full 2π, but the fundamental has changed phase by only π and has therefore become negative. With the above values of A_1 and A_2, the displacement at $3L/4$ is equal to -10 μ, in agreement with the problem.

2. French 6-15(a).
This is a bit messy, but bear with it. The string has length L. Its initial conditions are $y(x, 0) = Ax(L - x)$ and $(\partial y/\partial t)_{t=0} = 0$. We write the solution as a Fourier series

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \delta_n)$$

We know the solution at $t = 0$. Define $B_n = A_n \cos\delta_n$.

$$y(x, 0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

We can now solve for the B_n using Fourier's trick:

$$B_n = \frac{2}{L} \int_0^L Ax(L - x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Integrating by parts once, notice that the surface term vanishes:

$$B_n = \frac{2A}{n\pi} \int_0^L (L - 2x) \cos\left(\frac{n\pi x}{L}\right) dx$$

The L term integrates now. It always integrates to zero.

$$B_n = -\frac{4A}{n\pi}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

We integrate by parts again, and again the surface term vanishes.

$$B_n = \frac{4AL}{n^2\pi^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

This integrates easily.

$$B_n = -\frac{4AL^2}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)_0^L = \left.\frac{8AL^2}{n^3\pi^3}\right|_{n\text{ odd}}$$

We see that B_n is zero for all even n. We expected this because the initial condition is symmetric around the center of the string. Now we tackle the velocities.

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\omega_n \sin(\delta_n) = 0$$

This equation is satisfied if we simply set all of the $\delta_n = 0$, so that $\sin\delta_n = 0$ and $\cos\delta_n = 1$. This also means that $A_n = B_n$. We now have the full solution. We have rewritten the sum to only include odd n.

$$y(x,t) = \sum_{m=0}^{\infty}\frac{8AL^2}{(2m+1)^3\pi^3}\times$$
$$\times\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos(\omega_{2m+1}t)$$

3. French 8-9.
This problem concerns a very important effect called Doppler broadening. Sodium atoms emit light of 6000Å. The observed light varies in a small frequency range of (6000 ± 0.02)Å. This is caused by the thermal motion of the sodium atoms. The Doppler effect tells us that

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{0.02}{6000} = 3.33\times 10^{-6}$$

This gives the maximum velocity of the atoms $v_{\max} = 1000$ m/sec. The thermal velocity is given by

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT \Rightarrow T = \frac{m\langle v^2\rangle}{3k}$$

The constant $k = 1.38\times 10^{-23}$ J/Kelvin is Boltzmann's constant, and the appropriate mass $m \approx 23m_p$ is just the mass of the sodium atom. Approximating $\sqrt{\langle v^2\rangle} \approx v_{\max}$, this gives $T \approx 900$ Kelvin. This effect can be used to measure the temperatures of objects in astronomy.

4. French 8-12.
The Doppler effect for a moving source and fixed observer is given by

$$\nu(\theta) = \frac{\nu_0}{1 - \frac{u}{v}\cos\theta}$$

(a.) We want to find the Doppler effect for a fixed source and a moving observer at velocity $-u$. We are going to do this differently than it was done in French, in order to provide an alternative and perhaps easier way to think about it. We are going to go into the rest frame of the wave crests, so the only thing moving will be the observer and the source. The velocities of the wave and the observer are

$$V_w = (v, 0) \qquad V_o = (-u\cos\theta, u\sin\theta)$$

Transforming into the rest frame of the wave by subtracting $(v, 0)$, the velocity of the observer is

$$V_o = (-v - u\cos\theta, u\sin\theta)$$

The distance between wave crests is just $L = v/\nu_0$, so the rate at which the observer crosses the wave crests is just the x velocity divided by this distance. This is the shifted frequency.

$$\nu'(\theta) = \nu_0\left(1 + \frac{u}{v}\cos\theta\right)$$

This trick doesn't work for light because light doesn't have a rest frame. In relativity, there is a different Doppler formula that applies to both situations. It uses only the relative velocity of the source and the observer.

(b.) We want to know the approximate difference between the two formulas. We Taylor expand the first formula, assuming that the speed is much less than the sound speed, $u \ll v$.

$$\nu(\theta) \approx \nu_0 \left(1 + \frac{u}{v}\cos\theta + \frac{u^2}{v^2}\cos^2\theta + \cdots\right)$$

This tells us the approximate difference between the two Doppler shifted frequencies

$$\nu(\theta) - \nu'(\theta) \approx \nu_0 \frac{u^2}{v^2}\cos^2\theta$$

5. We can use Bernoulli's equation to find the velocities of these two flows.

(a.) If we look right as the flow is leaving the tank, we see that it must be at atmospheric pressure. The stream is arbitrarily thin, so this is the only possibility. We then just use the gravitational potential to find the velocities. If the hole were at the top of the tank, the velocity would be zero, so we use this as the constant.

$$\begin{aligned} P_0 + \rho g(H-h) + \frac{1}{2}\rho v^2 &= P_0 + \rho g H \\ v(h) &= \sqrt{2gh} \end{aligned}$$

The time it takes to hit the ground is given by the formula for constant acceleration.

$$H - h = \frac{1}{2}gt^2 \;\Rightarrow\; t = \sqrt{\frac{2(H-h)}{g}}$$

In this time, the water travels a horizontal distance vt, so the distance from the tank is

$$d(h) = \sqrt{4h(H-h)}$$

(b.) We notice that the previous formula says that water leaking from a depth h travels the same distance as water leaking from a depth $H - h$.

6. We again apply Bernoulli's equation. The air at the leading edge is stagnant, and we will assume that it is at atmospheric pressure. Bernoulli's equation gives

$$P_0 = P + \frac{1}{2}\rho v^2$$

The maximum velocity occurs when the pressure is zero. Plugging in P_0=10^5 N/m^2 and $\rho = 1.2$ kg/m^3, we find that v =408 m/sec. This is larger than the speed of sound, $v_s = 340$ m/sec.

7. The gradient ∇f is equal to

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

The x component of $\nabla \times (\nabla f)$ is the difference between the y and z partial derivatives, respectively, of the z and y components of ∇f:

$$(\nabla \times (\nabla f))_x = \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}$$

Interchanging the order of differentiation in either of the terms, this expression is seen to vanish for well-behaved f. By cyclic permutation, the y and z components of $\nabla \times (\nabla f)$ vanish as well.

8. (a.) When the point of observation (x, y, z) is displaced incrementally by $d\mathbf{s}$, where

$$d\mathbf{s} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$$

points in an arbitrary direction, the change df in $f(x, y, z)$ is given by the chain rule:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

The right-hand side can be rewritten as the dot product of

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

and $d\mathbf{s}$ above:

$$df = \nabla f \cdot d\mathbf{s}$$

For a fixed length $|d\mathbf{s}|$, this dot product is greatest when $d\mathbf{s}$ is parallel to ∇f. Therefore, when $df/|d\mathbf{s}|$ is a maximum, the direction of $d\mathbf{s}$ will be along ∇f. With $f = x^2 + y^2 - z^2$, this direction $\hat{\mathbf{n}}$ is

$$\begin{aligned}\hat{\mathbf{n}} &= \frac{\hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}}{|\nabla f|} \\ &= \frac{\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2y - \hat{\mathbf{z}}2z}{2\sqrt{x^2+y^2+z^2}} \\ &= \frac{\hat{\mathbf{x}}6 + \hat{\mathbf{y}}8 - \hat{\mathbf{z}}10}{2\sqrt{9+16+25}} \\ &= \frac{\hat{\mathbf{x}}3 + \hat{\mathbf{y}}4 - \hat{\mathbf{z}}5}{5\sqrt{2}}\end{aligned}$$

(b.) The surface $z(x, y) = \sqrt{x^2 + y^2}$ can be described as

$$0 = f(x, y, z) = x^2 + y^2 - z^2$$

This is the same $f(x, y, z)$ as in part (a.). Suppose the point of observation (x, y, z) is displaced infinitesimally by $d\mathbf{v}$, where $d\mathbf{v}$ is *on the surface* $f = 0$. Then we would expect f not to change at all. However, according to the results of part (a.),

$$df = \nabla f \cdot d\mathbf{v}$$

Therefore df can vanish only if $d\mathbf{v}$ is perpendicular to ∇f. Since $d\mathbf{v}$ can be any displacement which lies on the surface, this requires ∇f to be perpendicular to the surface. Therefore, the direction of the normal to the surface $d\mathbf{u}$ in part (b.) is the same as the direction $\hat{\mathbf{n}}$ of $d\mathbf{s}$ in part (a.), the direction of maximum change in f.

9. The fluid velocity field is

$$\mathbf{v}(x, y, z, t) = (\hat{\mathbf{y}}x - \hat{\mathbf{x}}y)\omega(t)$$

(a.) The equation of continuity (conservation of fluid molecules) requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Therefore

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\omega \nabla \cdot (\hat{\mathbf{y}}x\rho - \hat{\mathbf{x}}y\rho) \\ \frac{1}{\omega}\frac{\partial \rho}{\partial t} &= y\frac{\partial \rho}{\partial x} - x\frac{\partial \rho}{\partial y}\end{aligned}$$

(b.) An element of fluid at $\mathbf{r}_\perp = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ has a velocity $\mathbf{v} = (\hat{\mathbf{y}}x - \hat{\mathbf{x}}y)\omega(t)$ that is always in the xy plane and orthogonal to $\mathbf{r}_\perp$. Thus the element is in circular motion about the z axis (to which $\mathbf{r}_\perp$ is perpendicular), with angular velocity

$$\boldsymbol{\Omega} = \hat{\mathbf{z}}\frac{|\mathbf{v}|}{|\mathbf{r}_\perp|} = \hat{\mathbf{z}}\omega(t)$$

On the other hand

$$\begin{aligned}\nabla \times \mathbf{v} &= \omega(t)\,\hat{\mathbf{z}}\Big(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\Big) \\ &= 2\hat{\mathbf{z}}\omega(t)\end{aligned}$$

(c.) Suppose that the independent variables (x, y, z, t) upon which a vector $\mathbf{A}(x, y, z, t)$ depends change infinitesimally, by (dx, dy, dz, dt). Then, by the chain rule, a component of $\mathbf{A}$, *e.g.* A_x, changes by an amount

$$\begin{aligned}dA_x &= \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz \\ \frac{dA_z}{dt} &= \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z \\ &= \frac{\partial A_x}{\partial t} + \Big(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\Big)A_x \\ \frac{dA_x}{dt} &= \Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)A_x \\ \Rightarrow \frac{d\mathbf{A}}{dt} &= \Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{A}\end{aligned}$$

(This is the *convective derivative*, yielding the total time rate of change of $\mathbf{A}$.) In this problem $\mathbf{A} = \mathbf{v}$ itself, so

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{v} \\ &= 0 + \omega_0\Big(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y}\Big)(\hat{\mathbf{y}}x - \hat{\mathbf{x}}y) \\ &= \omega_0^2\Big(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\Big)(\hat{\mathbf{y}}x - \hat{\mathbf{x}}y) \\ &= \omega_0^2(-y\hat{\mathbf{y}} - x\hat{\mathbf{x}}) \\ &= -\omega_0^2\mathbf{r}_\perp\end{aligned}$$

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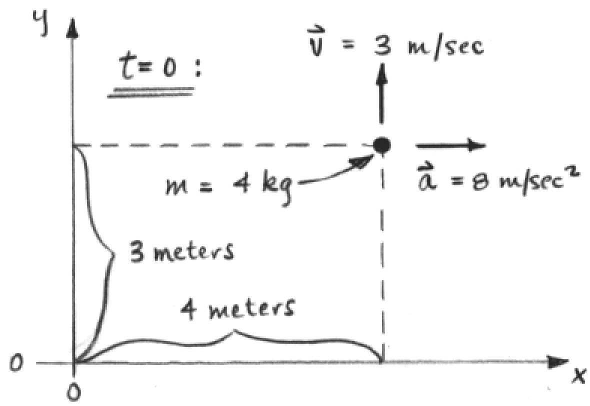
PRACTICE EXAMINATION 1

Directions. Do all three problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. You may use a calculator but you do not need one – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (25 points). A 4 kg mass moves in the x-y plane under the influence of a *constant* force. At t=0, the particle's *position*, *velocity*, and *acceleration* are shown in the diagram.



This problem will be graded on *answers only* – no part credit will be given. The answer to each question has three parts – *value* (3 points), *unit* (1 point), and *direction* (1 point). If for a particular question you believe the *direction* to be undefined or irrelevant, please leave it blank.

Organize your answers in columns as shown:

part	value	unit	direction
(a)			
(b)			
(c)			
(d)			
(e)			

a. What is the particle's *momentum* at t=0?

b. What is the *force* acting on the particle?

c. What is the particle's *momentum* at t=0.375 sec?

d. What is the particle's *position* at t=1 sec?

e. What is the particle's *velocity* when its absolute value (its *speed*) is at a minimum?

2. (40 points) A Millikan oil drop of mass m and charge q moves between two horizontal capacitor plates separated by a distance d. A battery of voltage V is applied to the plates, so that the electrical force on the drop is upward, of magnitude qV/d. When it is moving, the drop experiences an opposing drag force $\mathbf{F} = -k\mathbf{v}$, where $\mathbf{v}$ is its velocity and k is a constant.

a. For $t < 0$ the drop is observed to be exactly stationary, despite the gravitational force that is exerted upon it. What is the voltage V in terms of the other constants?

b. At $t = 0$ the plates are shorted out (V=0), and remain shorted thereafter. Calculate a, the *downward* acceleration of the drop immediately after the plates are shorted.

c. As $t \to \infty$ the acceleration of the drop becomes essentially zero. What is its *downward* velocity v then?

d. For any time $t > 0$, write a differential equation containing v, its first time derivative dv/dt, and constants.

e. Find a solution for the *downward* acceleration $a(t)$, valid for all $t > 0$. [*Hint*: Differentiate the answer to part (d.) with respect to time to get a simple differential equation for $a(t)$. Solve it by integration.

Use the result of part (b.) to determine the constant of integration.]

3. (35 points) A wooden block of mass M, initially at rest on a horizontal table with coefficient of sliding friction μ, is struck by a bullet of mass m and velocity v. The bullet lodges in the center of the block. How far does the block slide?

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SOLUTION TO PRACTICE EXAMINATION 1

Composed and formatted by E.A. Baltz and M. Strovink

1. For each of parts (a.) through (e.), specify magnitude, unit, and direction. Giving vector components is sufficient for magnitude and direction. The mass is 4 kg, the initial velocity is 3 meters per second, and the initial acceleration is 8 meters per second^2. The force acting on the particle is constant, so the acceleration is constant.

(a.) The initial momentum is $\mathbf{p} = m\mathbf{v} = 12$ kg m/sec, in the $+\hat{\mathbf{y}}$ direction.

(b.) The force acting on the particle can be found by $F = ma$, which is 32 kg m/sec^2, or newtons, in the $+\hat{\mathbf{x}}$ direction.

(c.) At $t = 0.375 = 3/8$ sec, the momentum can be found easily because the acceleration is constant. The velocity in the $+\hat{\mathbf{y}}$ direction isn't changing because there is no force, but the $+\hat{\mathbf{x}}$ velocity is given by $v_x = at = 3$ m/sec. The momentum vector is thus (12,12) kg-m/sec. Alternatively, we can write this as a magnitude and a unit vector. The magnitude of this vector is $12\sqrt{2}$. The unit vector must have equal x and y components, and its length must be one. This gives $\hat{\mathbf{p}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$.

(d.) The position at $t = 1$ sec is given by the formulas for constant acceleration. In the y direction, there is no acceleration, so y is given simply by $y = y_0 + v_{0y}t = 6$ m. In the x direction there is a constant acceleration, so the position is given by $x = x_0 + v_{0x}t + at^2/2 = 8$ m. The position vector is thus (8,6) m. Alternatively, the length of this vector is 10 meters; its direction is given by the unit vector $\hat{\mathbf{r}} = (0.8\hat{\mathbf{x}} + 0.6\hat{\mathbf{y}})$.

(e.) The y component of the velocity is constant because there is no force in the y direction. The x component of the velocity vanishes at $t = 0$. Therefore $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$ can never be smaller than it is at $t = 0$, when it is 3 m/sec in the $\hat{\mathbf{y}}$ direction.

2. An oil drop has mass m and charge q. It moves between plates a distance d apart, and voltage V is applied. The electrical force on the drop is qV/d upwards. When the drop is moving, it encounters a drag force $\mathbf{F} = -k\mathbf{v}$.

(a.) At $t < 0$ the drop is stationary. The electrical force must balance the gravitational force, so $mg = qV/d$, giving the potential V

$$V = \frac{mgd}{q}$$

(b.) The plates are shorted at $t = 0$, so there is no more electrical force. At this instant, the drop isn't moving, so it feels only the gravitational force. Its acceleration is just the acceleration of gravity, $a = g$ downwards.

(c.) As $t \to \infty$, the acceleration goes to zero. The velocity can be found by balancing the drag force with the gravitational force, $kv = mg$, so the terminal velocity is

$$v(t \to \infty) = \frac{mg}{k}$$

(d.) For $t > 0$ we can find a differential equation for the velocity. Newton's second law states that $F = dp/dt$ which in this case can be written $F = m\,dv/dt$. There are two forces, gravity and the drag force. The equation we get is

$$m\frac{dv}{dt} = mg - kv$$

(e.) Following the hint, we take d/dt of the answer for (d.):

$$m\frac{d^2v}{dt^2} = -k\frac{dv}{dt}$$

Substituting $a = dv/dt$:

$$m\frac{da}{dt} = -ka$$

Rearranging:

$$\frac{da}{a} = -\frac{k}{m}dt$$

Integrating from 0 to t:

$$\ln a(t) - \ln a(0) = -\frac{k}{m}t$$

Exponentiating:

$$\frac{a(t)}{a(0)} = e^{-\frac{k}{m}t}$$

From part (b.), $a(0) = g$, so

$$a(t) = g\, e^{-\frac{k}{m}t}$$

The acceleration begins with value g and decreases exponentially with time constant equal to m/k.

3. Instantaneously after the collision of the bullet and block, after the bullet has come to rest but before the frictional force on the block has had time to slow it down more than an infintesimal amount, we can apply momentum conservation to the bullet-block collision. At that time the total momentum of the block+bullet system is $(M+m)v_0'$, where v_0' is the velocity of the block+bullet system immediately after the collision. Momentum conservation requires that momentum to be equal to the initial momentum mv of the bullet. Thus

$$v_0' = \frac{mv}{M+m} \ .$$

After the collision, the normal force on the block+bullet system from the table is $(M+m)g$, giving rise to a frictional force

$$\mu N = \mu(M+m)g$$

on the sliding block+bullet system. This causes a constant acceleration μg of that system opposite to its velocity.

Take $t = 0$ at the time of collision. Afterward, the block+bullet system's velocity in the horizontal direction will be $v'(t) = v_0' - \mu g t$. It will continue sliding until $v'(t) = 0$, at which point the frictional force will disappear and it will remain at rest. Solving, the time at which the block-bullet system stops is

$$t = v_0'/(\mu g) \ .$$

The distance traveled in that time is

$$x = v_0' t - \frac{1}{2}\mu g t^2 = \frac{1}{2} v_0' t = \frac{(v_0')^2}{2\mu g} \ .$$

Plugging in the already deduced value for v_0', this distance is

$$x = \left(\frac{m}{M+m}\right)^2 \frac{v^2}{2\mu g} \ .$$

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EXAMINATION 1

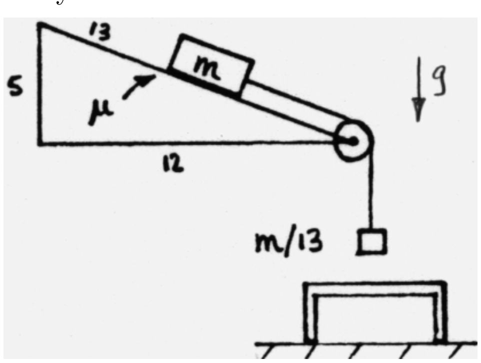
Directions. Do all four problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (20 points) You wish to fly (at low altitude) from Sapporo, Japan to Portland, OR. These cities are both at 45° north latitude (halfway between the equator and the north pole), and they are separated by 90° in longitude (azimuth). Consider the earth to be a sphere of radius R.

a. (10 points) You fly a course that keeps you at constant latitude, *i.e.* you fly due east. Over what distance do you travel?

b. (10 points) You fly a "great circle" course that takes you from Sapporo to Portland in the shortest possible distance (without penetrating the earth, of course). What is that distance?

2. (30 points) A block of mass m slides on an inclined plane with a slope of 5/12 (*i.e.* the slope of the hypotenuse of a 5-12-13 triangle). A massless rope, guided by a massless pulley, connects the block to a second block of mass $m/13$, which is hanging freely above a lower table.

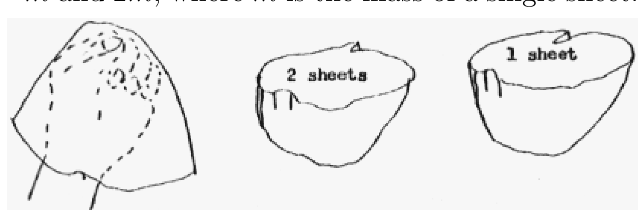


a. (15 points) The two-block system is observed to be moving with a constant velocity v_0. What is the coefficient μ of sliding friction between block and plane?

b. (15 points) After the hanging block hits the table, what is the distance s along the surface of the plane along which the top block continues to slide? (Assume that the plane is long enough that the block does not fall off. If you are unsure of your answer to part (a.), you may leave μ as an undetermined constant.)

3. (35 points) To determine the dependence of the force of air resistance upon the speed of a slowly moving object, one starts with three sheets of paper stacked together as they come out of the package. One then crumples them against one's fist as shown in the sketch.

Next one separates a single sheet from the other two without changing their crumpled (pseudo-conical) shapes. This yields two objects with the same shape, but with different masses m and $2m$, where m is the mass of a single sheet.



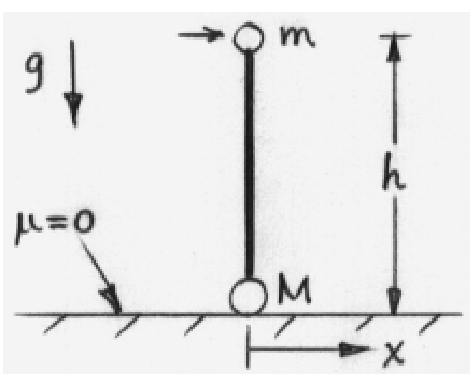
Finally one releases these two objects simultaneously, and compares their motion in still air under the influence of gravity.

a. (5 points) Instantaneously after the two objects are released, what is the ratio R of the (downward) acceleration of the heavier object to that of the lighter object?

b. (15 points) Very soon after being released, the two objects reach terminal (constant) velocity due to the effects of air resistance. After a long time, one observes that the object of mass $2m$ has dropped $\sqrt{2}$ times farther than the object of mass m. Assuming that the force of air resistance on these objects is proportional to v^{α}, where v is the velocity and α is a constant exponent, what is the value of α?

c. (15 points) Suppose that Mother Nature were to turn gravity off while these objects are falling. If one were willing to wait an arbitrarily long time, would they fall an arbitrarily long distance, or would that distance be bounded? Explain.

4. (15 points). An asymmetric barbell stands vertically at rest on frictionless horizontal ice. Mass M rests on the ice, and mass m is a distance h above it; the mass of the bar that rigidly connects these two masses is negligible. The dimensions of the masses can be neglected in comparison to their separation h. Take $x = 0$ to be the initial position of mass M.



Mass m is given an infinitesimal tap in the $+\hat{\mathbf{x}}$ direction that produces a negligible momentum, but does eventually cause the barbell to topple over. You may assume that mass M remains in contact with the ice throughout the motion. When mass m hits the ice, at what horizontal coordinate x_M will mass M be located?

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SOLUTION TO EXAMINATION 1

1. In spherical polar coordinates, take Sapporo and Portland to be at $(r,\theta,\phi) = (R,\pi/4,0) = \mathbf{r}_S$ and $(R,\pi/4,\pi/2) = \mathbf{r}_P$ respectively.

(a.) Here the course is one quarter of a circle with its center on the earth's axis of rotation at a point above the earth's center. This circle has radius $R\sin\theta = R\sqrt{2}/2$. The distance traveled, s, is one quarter of its circumference:

$$s = \frac{1}{4}2\pi\frac{R\sqrt{2}}{2} = \frac{\pi R\sqrt{2}}{4}\ .$$

(b.) The Cartesian coordinates of $\mathbf{r}_S$ and $\mathbf{r}_P$ are

$$\begin{aligned}\mathbf{r}_S &= R(\sqrt{2}/2, 0, \sqrt{2}/2)\\ \mathbf{r}_P &= R(0, \sqrt{2}/2, \sqrt{2}/2)\ .\end{aligned}$$

The angle between $\mathbf{r}_S$ and $\mathbf{r}_P$ is

$$\begin{aligned}\psi_{SP} &= \frac{\arccos\left(\mathbf{r}_S\cdot\mathbf{r}_P\right)}{R^2}\\ &= \arccos\left(1/2\right) = \pi/3\ .\end{aligned}$$

To calculate the minimum distance between Sapporo and Portland along the surface of the earth, we bisect the earth using a plane that contains these two cities as well as the earth's center. The intersection of the earth with the bisecting plane is a circle with its origin at the center of the earth, having a circumference $2\pi R$. Since both $\mathbf{r}_S$ and $\mathbf{r}_P$ lie in this plane, the course consists of the fraction $\psi_{SP}/2\pi$ of this circumference. Therefore the distance traveled is

$$s = \frac{\psi_{SP}}{2\pi}2\pi R = \frac{\pi/3}{2\pi}2\pi R = \frac{\pi R}{3}\ .$$

(This is $0.333/0.354 \approx 94\%$ of the length of the "due east" course.)

2. Let $\theta = \arctan\left(5/12\right)$ be the angle with which the plane is inclined. Since there is no acceleration (or motion) perpendicular to that plane, the normal force N on block m must be equal to the normal component of the gravitational force on block m, or

$$N = mg\cos\theta\ .$$

The frictional force F_f then will be

$$F_f = \mu N = \mu mg\cos\theta$$

opposite to the motion.

(a.) Along the motion, the system consisting of mass m plus mass $m/13$ experiences a force due to gravity, consisting of the sum of the force $mg\sin\theta$ on mass m and $mg/13$ on mass $m/13$. If the velocity of the system is constant, *i.e.* there is no acceleration, the gravitational and frictional forces must balance:

$$\begin{aligned}\mu mg\cos\theta &= mg(\sin\theta + 1/13)\\ \mu &= \frac{5/13 + 1/13}{12/13} = 1/2\ .\end{aligned}$$

(b.) After the hanging block hits the table, the (massless) rope goes limp and has no further effect on the sliding block. *Opposite* to the direction of motion, the net force on m is

$$\begin{aligned}f &= \mu mg\cos\theta - mg\sin\theta\\ &= mg\Big(\frac{1}{2}\frac{12}{13} - \frac{5}{13}\Big) = \frac{mg}{13}\ ,\end{aligned}$$

producing a constant acceleration $a = g/13$ opposite to the motion. The block decelerates for a time $t = v_0/a$ until it comes to rest. During that time, the distance traveled is

$$s = v_0t - \frac{1}{2}at^2 = \frac{v_0^2}{a} - \frac{1}{2}\frac{v_0^2}{a} = \frac{1}{2}\frac{v_0^2}{a} = \frac{13}{2}\frac{v_0^2}{g}\ .$$

3. Two crumpled paper objects have the same force (proportional to v^α) due to air resistance, but different masses $2m$ and m.

(a.) Immediately after the two objects are released from rest, their velocity must still be negligible; otherwise they would have experienced infinite acceleration. Likewise, the force of air resistance, proportional to v^α, is negligible at that time. So the only nonnegligible force acting on them is the force of gravity, $2mg$ and mg respectively, yielding an acceleration g in either case. So the ratio of accelerations is $R = 1$.

(b.) After the objects reach terminal velocity (v_2 and v_1 respectively), and they no longer are accelerating, the forces due to air resistance and gravity must cancel:

$$\begin{aligned} 2mg &= Kv_2^\alpha \\ mg &= Kv_1^\alpha \ , \end{aligned}$$

where K is the unknown constant of proportionality. Taking the ratio of these two equations,

$$v_2 = 2^{1/\alpha} v_1 \ .$$

The "long time" after the objects are dropped is very large compared to the "very soon" time at which they reach terminal velocity. So, to an excellent approximation, the distance they drop during the "long time" is proportional to the terminal velocity. Since mass $2m$ drops a factor $\sqrt{2}$ further,

$$v_2 = \sqrt{2} v_1 \ .$$

Comparing this to the previous equation,

$$\alpha = 2 \ .$$

(c.) After gravity is turned off, the force of air resistance Kv^2 accelerates the falling object opposite to its direction of motion:

$$m\frac{dv}{dt} = -Kv^2 \ .$$

Dividing through by v^2, multiplying through by dt, and integrating,

$$\frac{1}{v} = \frac{K}{m}t + C$$

where C is a constant of integration that will be negligible with respect to Kt/m when t is sufficiently large. Therefore, asymptotically as $t \to \infty$, the downward velocity will be inversely proportional to t. Integrating this statement, the asymptotic distance traveled will increase as $\ln t$, which is arbitrarily large when t is arbitrarily large. Therefore the distance fallen will be *unbounded*. [This result is similar to that obtained in Problem Set 3, Problem 6 (K&K 2.35).]

4. Since the ice is horizontal and frictionless, it cannot exert any force on the barbell in the $\hat{\mathbf{x}}$ direction, either as the result of a contact force or a frictional force. Therefore the x coordinate x_{CM} of the barbell's center of mass, initially at rest at $x = 0$, must remain at $x = 0$. When mass m hits the ice and the barbell is horizontal,

$$\begin{aligned} 0 = X_{CM} &= \frac{Mx_M + m(x_M + h)}{M + m} \\ \Rightarrow \quad x_M &= -h\frac{m}{M + m} \ . \end{aligned}$$

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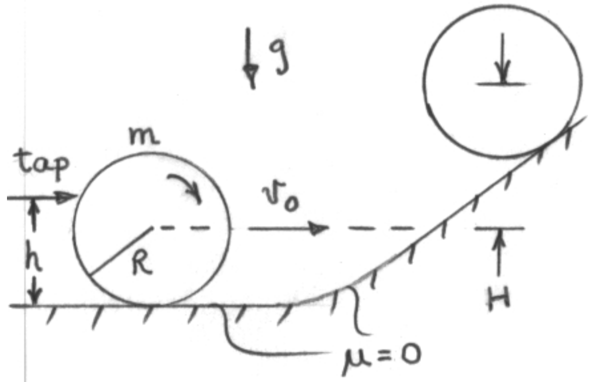
EXAMINATION 2

Directions. Do all three problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

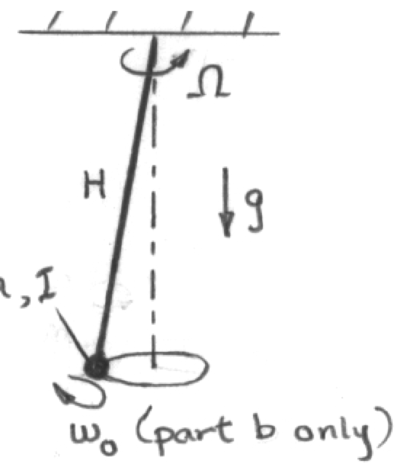
1. (30 points) A cylinder of mass m and radius R initially is at rest on *frictionless* ice. About its C.M., the cylinder's moment of inertia is $mR^2/2$.



a. (15 points) The cylinder receives an instantaneous horizontal tap at a point on its circumference that is a vertical distance h above the ice. Immediately thereafter, it is observed to roll without slipping at a velocity v_0, even though the ice is frictionless. Calculate the value of h.

b. (15 points) The cylinder continues to roll without slipping as long as the frictionless ice remains flat. Eventually the ice slopes upward to form a hill as shown, all the while staying frictionless. To what maximum height H will the cylinder rise?

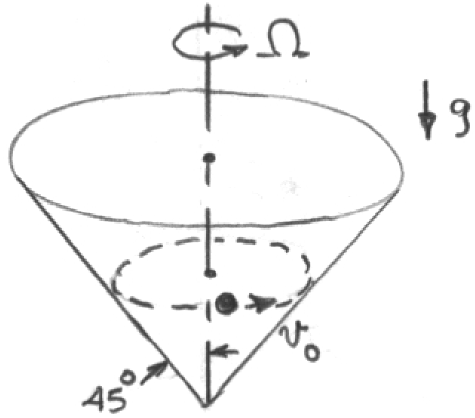
2. (30 points) A thin massless rod of length H hangs freely pivoted from the ceiling. At its end is a bead of mass m, negligible in size compared to H, acted upon by gravity.



a. (15 points) For this part, consider the bead and the rod to be glued together. The C.M. of the bead is observed to travel uniformly around a horizontal circle that is centered below the rod's pivot. The circle's radius is much smaller than H but much larger than the bead. With what angular velocity Ω does the bead move on this path?

b. (15 points) For this part, the bead is no longer glued to the rod; instead it spins about the long axis of the *rod* with a constant, large angular velocity ω_0. About this axis, the moment of inertia of the bead is I. Again, the C.M. of the bead is observed to travel uniformly around a horizontal circle centered below the pivot. What restriction(s) can be placed on the radius r of this circle?

3. (40 points) A tiny pebble moves on the frictionless inner surface of a vertical cone that has a half-angle of 45°. It is observed to be in uniform circular motion with constant velocity v_0. (You are *not* given the radius of this circle!)



a. (15 points) what is the angular frequency Ω of this uniform circular motion?

b. (15 points) The orbiting pebble is now nudged so that its new orbit differs very slightly from the original circle. The nudge causes the orbit radius (the perpendicular distance from the pebble to the cone axis) to oscillate sinusoidally about its mean value. (All the while the pebble remains in contact with the inside surface of the cone.) Calculate the angular frequency ω_r of this small radial oscillation.

c. (10 points) Is the perturbed orbit "closed" – that is, does the perturbed orbit ever repeat itself? Explain.

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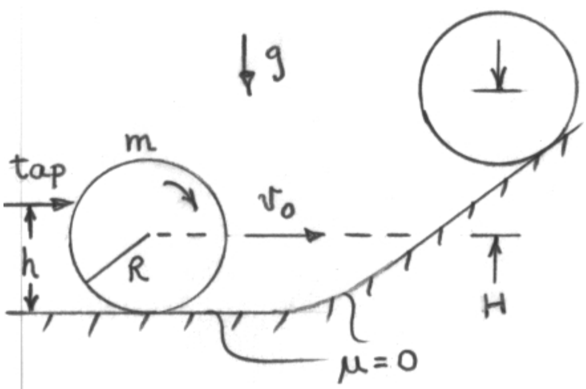
SOLUTION TO EXAMINATION 2

1.



(a.) The horizontal tap produces a (horizontal) *linear* impulse $\int F\,dt \equiv J$. With respect to the center of the cylinder, it also produces a (clockwise) *angular* impulse $\int \tau\,dt = Jb$, where $b = h - R$ is the impact parameter of the horizontal tap. Then, in terms of J, since the cylinder is initially at rest,

$$mv_0 = J$$
$$I\omega_0 = J(h - R)$$

Substituting $I = \frac{1}{2}mR^2$, and imposing the condition $v_0 = R\omega_0$ that the cylinder rolls without slipping, these equations become

$$mv_0 = J$$
$$\frac{1}{2}mR^2\frac{v_0}{R} = J(h - R)$$

These equations are mutually consistent only if $h - R = R/2$, or

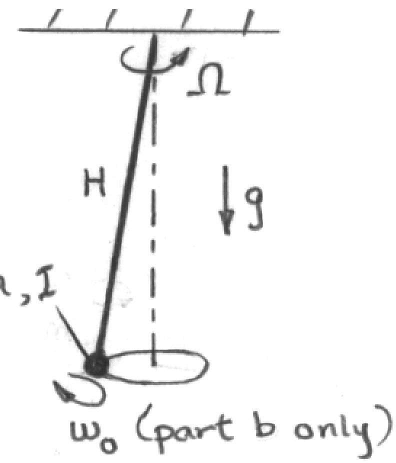
$$h = \frac{3}{2}R$$

(b.) Since the ice is frictionless, and the force of gravity acts on the C.M. of the cylinder, no torques about its axis can be exerted on the cylinder. Therefore its angular momentum and angular velocity $\omega = \omega_0$ remain constant even on the hill. Considering that the kinetic energy of a rigid body decomposes into $K_{\rm trans} + K_{\rm rot}$, we conclude that only $K_{\rm trans} = \frac{1}{2}mv_0^2$ is available to be converted into potential energy mgH. Therefore the maximum height is

$$H = \frac{v_0^2}{2g}$$

.

2.



(a.) Let β be the angle between the stick and the vertical. This part can easily be worked in the C.M. of the bead, where a centrifugal force $m\Omega^2 r = m\Omega^2 H \sin\beta$ balances the horizontal component $mg\tan\beta$ of the tension $mg/\cos\beta$ in the stick. Or it can be worked in the lab, where the horizontal component of the tension supplies the necessary centripetal acceleration. Or, elegantly, the circular motion of the bead can be considered to be the superposition of an x pendulum and, delayed by one-quarter of a period, a y pendulum with the same amplitude. When the approximation $\beta \ll 1$, valid for part (a.), is applied, β cancels out, and any of these approaches yields the usual result

$$\Omega = \sqrt{\frac{g}{H}}$$

(b.) Take the origin to be the pivot point. If the bead is spinning with constant angular velocity ω_0 about the stick's axis, the vertical component

$I\omega_0 \cos\beta$ of the spin angular momentum L remains constant. But the horizontal component $L_h = I\omega_0 \sin\beta$ of L precesses with angular velocity Ω, as in a gyroscope. The torque ΩL_h that is required to maintain this precession is the torque due to gravity, $mgr = mgH \sin\beta$; the stick can't supply this torque because its end coincides with the origin. Then

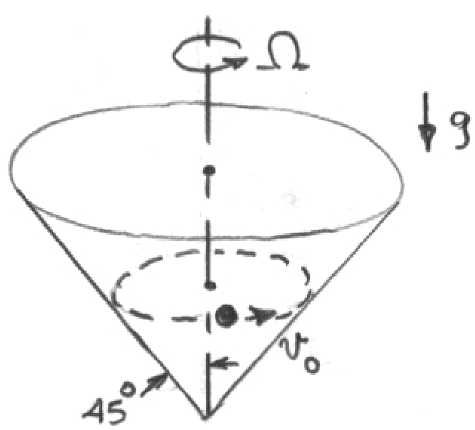
$$\Omega I\omega_0 \sin\beta = mgH \sin\beta$$

The angular velocity of precession is

$$\Omega = \frac{mgH}{I\omega_0}$$

independent of β. Because $\sin\beta$ cancels out of the previous equation, there is no restriction on it and therefore no restriction on the orbit radius $r = H\sin\beta$; this circular motion can occur for any orbit radius $r \leq H$ provided that that ω_0 is large enough to allow the spin angular momentum to dominate the other angular momenta in the problem.

3.



(a.) This part of the problem can be done by balancing forces; the normal force of the 45° cone on the pebble has equal horizontal and vertical components mg. When the horizontal component is equated to the centripetal acceleration $mv_0\Omega$, we obtain immediately

$$\Omega = \frac{g}{v_0}$$

Anticipating what will be needed for part (b.), we can also solve part (a.) using the effective potential

$$U_{\text{eff}} = \frac{l^2}{2mr^2} + mgr$$

where r is the perpendicular distance of the pebble to the cone axis. (In the second term, we are using the fact that, for a 45° cone with $z = r$, an increase Δr causes an increase $mg\Delta z = mg\Delta r$ in the true potential energy.) Then a circular orbit occurs when

$$0 = \frac{dU_{\text{eff}}}{dr} = -\frac{2l^2}{2mr^3} + mg$$

Substituting $l^2 = mv_0 r \times m\Omega r^2$ causes the r dependence to cancel:

$$0 = -\frac{2m^2 v_0 \Omega}{2m} + mg$$
$$\Omega = \frac{g}{v_0}$$

as before.

(b.) Proceeding with the effective potential method, we obtain the effective spring constant k_{eff} for radial motion by differentiating U_{eff} again with respect to r:

$$\begin{aligned} k_{\text{eff}} &= \frac{d}{dr}\Big(-\frac{l^2}{mr^3} + mg\Big) \\ &= \frac{3l^2}{mr^4} \\ &= \frac{3m^2 r^4 \Omega^2}{mr^4} \\ &= 3m\Omega^2 \end{aligned}$$

Thus

$$\omega_r = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{3}\Omega = \sqrt{3}\frac{g}{v_0}$$

(c.) Since the ratio of ω_r to Ω is irrational, an integer number of orbital cycles cannot occur in the same time interval as an integer number of radial cycles. Therefore the orbit never repeats itself and so is not closed.

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FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

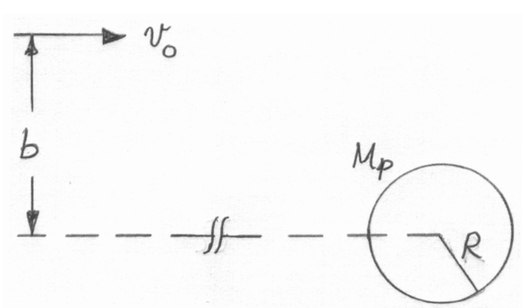
Problem 1. (30 points)
Northern Canada has two peculiar features: owing to lack of roads, most surface transportation occurs by train; and the principal fauna are tiny black flies.

Consider the *elastic* (kinetic energy conserving) head-on collision of a locomotive of mass M and velocity V with a stationary black fly of mass m. You may make any reasonable approximation concerning the relative magnitude of M and m.

a. (15 points)
With what velocity v does the fly recoil from the locomotive?

b. (15 points)
Assuming that the (coasting) locomotive has flat frontal area A, and there are N black flies per cubic meter hovering over the track, apply the results of part (**a.**) to obtain a differential equation for V (neglect air resistance). Solve it to obtain $V(t)$.

Problem 2. (20 points)



A space probe is launched with initial velocity v_0 and impact parameter b toward a very distant planet of radius R and very large mass M_p (see figure). Find the maximum value of b for which the rocket will hit the planet.

Problem 3. (30 points)
For decades, scientists have been designing a "space colony" in which thousands of people could exist while orbiting the sun. People would live on the inside curved surface of a large air-filled cylinder (length of order 10 km, radius R of order 1 km). The cylinder would rotate about its axis with an angular velocity ω such that earth's gravitational acceleration g would be simulated by the centrifugal force acting near that surface. The curved surface would have dirt for farming, and also housing, factories, parks, hills, streams, a lake, etc. Sunlight would enter through one end; it would be controlled by mirrors and shutters to simulate day and night. There would be clouds and weather, etc.

a. (5 points)
Find the angular frequency ω of rotation.

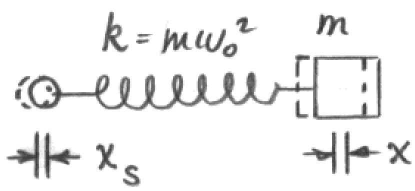
b. (10 points)
Although many aspects of life in this colony would resemble life on earth, one peculiar feature would be the large Coriolis acceleration. When $\mathbf{v}$ (as seen by a colony inhabitant) is perpendicular to $\boldsymbol{\omega}$, the magnitude of the Coriolis acceleration a_C can be expressed as

$$a_C = g\frac{v}{v_C}$$

where v_C is a characteristic velocity. Find v_C appropriate to the surface inhabited by the colonists.

c. (15 points)
For residents in the colony, north is defined to be in the direction of $\boldsymbol{\omega}$; if a resident faces north her right hand points east. A baseball pitcher, new to the colony, fires a ball toward the west with velocity v at his target a distance D away. If he were on the surface of the earth (where the Coriolis force is negligible), the ball would hit its target. In what direction (high, low, north, or south) does the ball miss its target? By what distance d does it miss (you may assume $d \ll D$)?

Problem 4. (30 points)
A mass m is connected by a massless spring of stiffness $k = m\omega_0^2$ to a point of support x_s. When the spring is relaxed, and $x_s = 0$, the mass is at its equilibrium position $x = 0$. The mass moves only in the x direction, without friction.



Suppose that the point of support is constrained by an external force to obey the following motion: $x_s = mA \sin \omega t$, where A and ω are constants, and ω is not necessarily equal to ω_0. The external force does not act directly on the mass, but it nevertheless influences the mass because of the spring.

a. (15 points)
Find the particular solution $x_p(t)$ which would vanish if A were zero.

b. (15 points)
Find the solution $x_0(t)$ which would be correct if the mass were fixed at its equilibrium position and released at $t = 0$.

Problem 5. (30 points)
Consider a thin cylindrical pipe of length L, closed at both ends. The air inside the pipe can support longitudinal (sound) waves that propagate along the axis of the pipe. Let $\xi(x,t)$ be the displacement (along the axis of the pipe) of an air molecule whose equilibrium coordinate (along the same axis) is x. As usual, ξ satisfies the wave equation

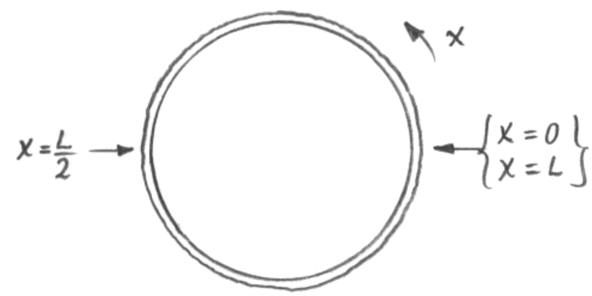
$$\frac{\partial^2 \xi}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} = 0$$

where c is the (phase and group) velocity of sound waves in air.

a. (3 points)
Keeping in mind that the pipe is closed at both ends, write down the boundary conditions on $\xi(0,t)$ and $\xi(L,t)$.

b. (12 points)
The air inside the straight closed pipe is observed to carry a *standing* sinusoidal sound wave. What is the lowest angular frequency ω_s with which this wave can vibrate?

c. (3 points)
The ends of the pipe are now opened and the pipe is bent into a hoop. The end at $x = 0$ is welded to the end at $x = L$, so that the pipe forms a continuously hollow circular torus (like a hula hoop) with a circumference equal to L.



Continue to consider sound waves that propagate along the (bent) axis of the pipe. As long as the circumference of the hoop is much larger than the pipe thickness, which is the case here, $\xi(x,t)$ satisfies the same wave equation as before. However, since the pipe is now bent into a continuously hollow torus, $x = 0$ and $x = L$ now describe the *same* coordinate along the pipe's axis. More generally, $\xi(x,t)$ and $\xi(x+L,t)$ describe the displacement from equilibrium of the *same* molecule.

In light of the above, write down the relationship between $\xi(x,t)$ and $\xi(x+L,t)$.

d. (12 points)
The air inside the bent pipe is observed to carry a *travelling* sinusoidal sound wave. Keeping in mind the result of part (**c**.), what is the lowest angular frequency ω_t that can characterize this wave? What is the ratio of ω_t to the result ω_s of part (**b**.)?

Problem 6. (30 points)
Nonviscous fluid matter is in spherically symmetric, nonrelativistic flow toward a black hole of mass M. Only the gravitational attraction of the black hole itself (as opposed to the gravitational attraction of other fluid elements) is important to the fluid motion. M is growing slowly enough to be taken as constant.

a. (10 points)
Consider Φ, the potential energy per unit mass of fluid due to the gravitational attraction of the black hole. Starting from the standard formula for the gravitational force between two point objects, show that

$$\Phi(r) = -\frac{GM}{r}$$

where r is the distance from the black hole. Note that in spherical polar coordinates

$$\nabla u = \hat{\mathbf{r}}\frac{\partial u}{\partial r} + \ldots$$

(You may use this result for Φ in subsequent parts of the problem.)

b. (10 points)
Because M is taken as constant, the fluid flow is *steady*:

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \rho}{\partial t} = 0$$

where ρ is the mass density. Also, the fluid *pressure* p is known not to vary either with position or time.

Away from the black hole, determine the dependence of fluid |velocity| v upon r.

c. (10 points)
Away from the black hole, determine the dependence of fluid mass density ρ upon r.

Note that, in spherical polar coordinates,

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \ldots$$

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SOLUTION TO FINAL EXAMINATION

Problem 1.

a.
We consider this head-on collision in the center of mass. The center of mass velocity is

$$V^* = V\frac{M}{M+m} \approx V$$

Using this approximation, in the C.M. the fly approaches the locomotive with speed V. Since the collision is elastic, it bounces back with the same speed. Transforming back to the lab, the fly has velocity

$$v \approx V + V = 2V$$

b.
In each collision, the momentum $2mV$ that is gained by the fly is lost by the locomotive:

$$\begin{aligned}\Delta P = M\Delta V &= -2mV \\ \frac{\Delta V}{V} &= -2\frac{m}{M}\end{aligned}$$

In a time interval Δt, the volume swept out by the front of the train is $AV\Delta t$; this volume contains $NAV\Delta t$ flies. So, for $NAV\Delta t$ collisions,

$$\begin{aligned}\frac{\Delta V}{V} &= -2\frac{m}{M}NAV\Delta t \\ \frac{\Delta V}{V^2} &= -2NA\frac{m}{M}\Delta t \\ \int\frac{dV}{V^2} &= -2NA\frac{m}{M}\int dt \\ \frac{1}{V} - \frac{1}{V_0} &= 2NA\frac{m}{M}t \\ V(t) &= \frac{1}{2NA\frac{m}{M}t + \frac{1}{V_0}} \\ V(t) &= \frac{V_0}{1 + 2NAV_0\frac{m}{M}t}\end{aligned}$$

where V_0 is the velocity at $t = 0$.

Problem 2.
At the instant that the probe barely grazes the planet, it will have radius R and velocity $\mathbf{v}_f$ directed tangentially to the planet. Angular momentum conservation requires

$$\begin{aligned}mv_0b &= mv_fR \\ v_f &= v_0\frac{b}{R}\end{aligned}$$

Substituting for v_f in the equation for energy conservation, we obtain

$$\begin{aligned}\frac{1}{2}mv_0^2 &= \frac{1}{2}mv_f^2 - \frac{GM_pm}{R} \\ \frac{1}{2}v_0^2 &= \frac{1}{2}v_0^2\frac{b^2}{R^2} - \frac{GM_p}{R} \\ \frac{1}{2}v_0^2\left(\frac{b^2}{R^2} - 1\right) &= \frac{GM_p}{R} \\ \frac{b^2}{R^2} - 1 &= \frac{2GM_p}{v_0^2R} \\ b &= R\sqrt{1 + \frac{2GM_p}{v_0^2R}}\end{aligned}$$

Problem 3.

a.

$$\begin{aligned}mR\omega^2 &= mg \\ \omega &= \sqrt{\frac{g}{R}}\end{aligned}$$

b.

$$\begin{aligned}2\omega v = a_C &= g\frac{v}{v_C} \\ v_C &= \frac{g}{2\omega} \\ v_C &= \frac{g}{2}\sqrt{\frac{R}{g}} \\ v_C &= \frac{1}{2}\sqrt{gR}\end{aligned}$$

c.

$$\mathbf{F}_C = -2m(\boldsymbol{\omega} \times \mathbf{v})$$

$\boldsymbol{\omega}$ is north, and $-\mathbf{v}$ is east; $\vec{\text{north}} \times \vec{\text{east}}$ is *down*. This is the direction in which the ball misses.

$$\begin{aligned} a_C &= 2\omega v = 2v\sqrt{\frac{g}{R}} \\ d &= \frac{1}{2}a_C t^2 \\ &= \frac{1}{2}2v\sqrt{\frac{g}{R}}t^2 \\ t &= \frac{D}{v} \\ d &= v\sqrt{\frac{g}{R}}\frac{D^2}{v^2} \\ d &= \frac{D^2}{v}\sqrt{\frac{g}{R}} \end{aligned}$$

(We ignore the centrifugal force on the ball, because it is the same on the colony as on earth, and the pitcher already compensates for it.) As a sanity check, if $D = 20$ m and $v = 40$ m/sec (appropriate to baseball), and $R = 1000$ m, we obtain $d \approx 1$ m. Indeed d is much smaller than D. Nevertheless, from the standpoint of the pitcher, the Coriolis force has a big effect on his control.

Problem 4.
The equation of motion for $x(t)$ is

$$\begin{aligned} m\ddot{x} = -k(x - x_s) &= -m\omega_0^2(x - x_s) \\ \ddot{x} &= -\omega_0^2 x + \omega_0^2 mA\sin\omega t \\ \ddot{x} + \omega_0^2 x &= kA\sin\omega t \end{aligned}$$

a.

$$\begin{aligned} \text{try } x_p(t) &= B\sin\omega t \\ (-\omega^2 + \omega_0^2)B\sin\omega t &= kA\sin\omega t \\ B &= \frac{kA}{\omega_0^2 - \omega^2} \\ x_p(t) &= \frac{kA\sin\omega t}{\omega_0^2 - \omega^2} \end{aligned}$$

b.
Because an infinite force from the spring would be required otherwise, $\dot{x}_0(0) = 0$ as well as $x_0(0) = 0$. The general solution to the homogeneous equation of motion ($A = 0$) is

$$x_h(t) = C\cos\omega_0 t + D\sin\omega_0 t$$

The general solution to the full equation is obtained by adding x_h to x_p. Applying initial conditions,

$$\begin{aligned} x_0(t) &= \frac{kA\sin\omega t}{\omega_0^2 - \omega^2} + C\cos\omega_0 t + D\sin\omega_0 t \\ x_0(0) &= 0 \;\Rightarrow\; C = 0 \\ \dot{x}_0(0) &= 0 \;\Rightarrow\; 0 = \frac{\omega kA}{\omega_0^2 - \omega^2} + \omega_0 D \\ D &= -\frac{\omega}{\omega_0}\frac{kA}{\omega_0^2 - \omega^2} \\ x_0(t) &= kA\frac{\omega_0\sin\omega t - \omega\sin\omega_0 t}{\omega_0(\omega_0^2 - \omega^2)} \end{aligned}$$

Problem 5.

a.

$$\xi(x = 0, t) = \xi(x = L, t) = 0$$

b.

$$\begin{aligned} \xi(x,t) &= \sin kx\,\Re\big(\xi_0\exp(-i\omega t)\big) \\ \sin kL &= 0 \\ kL &= n\pi,\; n = 1, 2, \ldots \\ \omega &\equiv ck \\ \omega_s &= \frac{\pi c}{L} \end{aligned}$$

c.

$$\xi(x,t) = \xi(x + L, t)$$

d.

$$\xi(x,t) = \Re\big(\xi_0 \exp\left(i(kx - \omega t)\right)\big)$$
$$\exp(ikx) = \exp(ik(x+L))$$
$$1 = \exp(ikL)$$
$$kL = 2n\pi,\ n = 1,2,\ldots$$
$$\omega \equiv ck$$
$$\omega_t = \frac{2\pi c}{L}$$
$$\omega_t = 2\omega_s$$

Problem 6.

a.
Per unit mass of fluid, the force $\mathbf{f}$ is

$$\mathbf{f} = -\hat{\mathbf{r}}\frac{GM}{r^2}$$

We seek a function $\Phi(r)$ such that

$$-\nabla\Phi = \mathbf{f}$$

or equivalently, using spherical symmetry,

$$\Phi = -\int f_r dr$$

Clearly
$$\Phi(r) = -\frac{GM}{r}$$
satisfies either of these conditions.

b.
Since the flow is steady, we can use Bernoulli's equation (either along a streamline at constant (θ,ϕ), or, since the flow is irrotational, anywhere outside the black hole):

$$\frac{1}{2}\rho v^2 + p + \rho\Phi = \text{constant}$$

Only the first and third terms are not constant, so they must have the same r dependence. Therefore v^2 and Φ have the same r dependence. So
$$v \propto r^{-1/2}$$

c.
In steady flow there can be no buildup of mass density ρ. Therefore the mass flow

$$\rho v\ (\text{kg/m}^2\text{sec}) \times 4\pi r^2\ (\text{m}^2)$$

through a spherical surface of radius r must be independent of r. So, using the result of part (**a**.),
$$\rho v \propto r^{-2}$$
$$\rho \propto r^{-3/2}$$

More formally, but equally acceptably, one can reach the same conclusion by applying the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

and using the fact that for steady flow the first term vanishes.

University of California, Berkeley
Physics H7B, Spring 1999 (Strovink)

General Information

Web site for this class: First link on `http://d0lbln.lbl.gov/` .

Instructors: Prof. Mark Strovink, 437 LeConte; (LBL) 486-7087; (home, before 10) 486-8079; (UC) 642-9685. Email: `strovink@lbl.gov`. Web: `http://d0lbln.lbl.gov/`. Office hours: M 3:15-4:15, Th 10-11. Mr. Robin Blume-Kohout, 208 LeConte; (UC) 642-5430. Email: `rbk@socrates.berkeley.edu`. Office hours (to be held initially in 208 LeConte): M 1-2, W 3-4. You may also get help in the 7B Course Center, 206 LeConte.

Lectures: Tu-Th 11:10-12:30, 2 LeConte. Lecture attendance is essential, since not all of the course content can be found in the course text or handouts.

Labs: In the second week, in 262 LeConte, please enroll in one of *only 2* special H7B lab sections [(A) #401, F 2-4; (B) #402, F 4-6]. Both sections are taught by Mr. Blume-Kohout. If you can make both of these lab (and section, see below) slots, please attempt to enroll in the earlier of these lab slots. Depending on crowding, you may be asked to move to the later lab. During "off" weeks not requiring lab apparatus, your lab section will still meet in the same room, 262 LeConte.

Discussion Sections: Beginning in the second week, please enroll in the 1 hr H7B discussion section corresponding to your H7B lab section: (A) #401, M 2-3, 385 LeConte; (B) #402, W 4-5, 343 LeConte. You are especially encouraged to attend discussion section regularly. There you will learn techniques of problem solving, with particular application to the assigned exercises.

Text (required): E.M. Purcell, **Electricity and Magnetism** (Berkeley Physics Course Volume 2), *Second Edition* (McGraw-Hill, 1985). At the beginning of the semester we will also use Chapters 22 through 26 (pp 493-592) of Resnick, Halliday and Krane, **Physics** (Volume 1), *Fourth Edition* (Wiley, 1992). The publisher has granted permission to make it possible for students to purchase a Xerox copy of these pages from Copy Central.

Problem Sets: Thirteen problem sets are assigned and graded, with solutions provided on the web and at Copy Central. They are due on Thursday at 5 PM on weeks in which there is no exam, beginning in week 2. Deposit problem sets in the box labeled "H7B" outside 208 LeConte. You are encouraged to attempt all the problems. Students who do not do them find it almost impossible to learn the material and to succeed on the examinations. Discuss these problems with your classmates as well as with the teaching staff; however, when the time comes to write up your solutions, *work independently*. Credit for collective writeups, which are easy to identify, will be divided among the collectivists. Late papers will not be graded. Your lowest problem set score will be dropped, in lieu of due date extensions for any reason.

Syllabus: H7B has one syllabus card, which is mandatory. It will be collected at the time of the midterm examination. This card pays for the experiment descriptions and instructions that you will receive from your GSI at the beginning of each laboratory. Also, we expect you also to have the opportunity to purchase at Copy Central a copy of the above mentioned 100 pages of Resnick, Halliday and Krane. Finally, copies of solutions to the problem sets will also be available for purchase at Copy Central. These solutions will also be available on the Web.

Exams: There will be one 80-minute midterm examination and one 3-hour final examination. Before confirming your enrollment in this class, please check that its final Exam Group 12 does not conflict with the Exam Group for any other class in which you are enrolled. Please verify that you will be available for the midterm examination (Th 4 Mar, 11:10-12:30), and for the final examination, W 19 May, 8-11 AM. Except for unforeseeable emergencies, it will not be possible for the midterm or final exams to be rescheduled. Passing H7B requires passing the final exam.

Grading: 25% midterm; 25% problem sets; 45% final exam; 5% lab. Grading is not "curved" -- it does not depend on your performance relative to that of your H7B classmates. Rather it is based on comparing your work to that of a generation of earlier lower division Berkeley physics students, with due allowance for educational trends.

COURSE OUTLINE

Week No.	Week of...	Lecture chapter	Topic (**RHK** = Resnick/Halliday/Krane, *Physics Vol. 1*) (**Purcell** = *Electricity and Magnetism*)	Problem Set No.	Due 5 PM on...	Lab
					(do experiment in lab="expt") (have discussion in lab="disc")	
		MARTIN LUTHER KING HOLIDAY (18-Jan)				
1	18-Jan	**RHK** (22), 25.7	Thermal expansion; heat transfer			no lab
		23.1-23.4	Kinetic definition of temperature			
2	25-Jan	23.5-23.6	Energetics of an ideal gas; equipartition			disc
		24.1-24.4	Maxwellian distribution	1	Th 28-Jan	
3	1-Feb	25.4-25.6	Heat capacities of an ideal gas; first law			expt
		26.1-26.4	Second law of thermodynamics	2	4-Feb	
4	8-Feb	**RHK** 26.5-26.9	Entropy			disc
		Purcell 1.1-1.8	Electric charge	3	11-Feb	
		PRESIDENTS' DAY HOLIDAY (15-Feb)				
5	15-Feb	1.9-1.15	Electric fields			disc
		2.1-2.6	Electric potential	4	18-Feb	
6	22-Feb	2.7-2.13	Gauss' law, Laplace's equation			expt
		3.1-3.4	Electic fields around conductors	5	25-Feb	
7	1-Mar	3.5-3.8	Systems of conductors; capacitors			disc
	4-Mar	*11:10 AM - 12:30 PM*	MIDTERM EXAMINATION (covers PS 1-5)			
8	8-Mar	4.1-4.11	Electric currents			expt
		Appendix A	Special relativity	6	11-Mar	
9	15-Mar	Appendix A	Special relativity			disc
		5.1-5.5	Electric field in different frames of reference	7	18-Mar	
	22-Mar	SPRING RECESS (22-26 Mar)				disc
					25-Mar	
10	29-Mar	5.6-5.9	Fields of moving charges			expt
		6.1-6.2, 6.4-6.5	Magnetic fields	8	1-Apr	
11	5-Apr	6.3, 6.6-6.9	Vector potential; magnetic field transformation			disc
		7.1-7.5	Faraday's law	9	8-Apr	
12	12-Apr	7.6-7.10	Inductance			expt
		8.1-8.5	AC circuits	10	15-Apr	
13	19-Apr	9.1-9.4; 9.6	Maxwell's equations			disc
		10.1-10.6	Electric dipoles	11	22-Apr	
14	26-Apr	10.7-10.12	Electric fields in dielectric media			disc
		11.1-11.6	Magnetic dipoles (but not monopoles)	12	29-Apr	
15	3-May	11.7-11.11	Magnetization			disc
			LAST LECTURE (review)	13	6-May	
	10-May	INSTRUCTION ENDS (10-May)				makeup
		FINAL EXAMS BEGIN (14-May)				
	17-May					
	19-May	*8-11 AM*	FINAL EXAM (Group 12) (covers PS 1-13)			

Physics H7C

University of California, Berkeley

F'99 (Strovink)

RELATIVITY NOTES

1. SPECIAL RELATIVITY

1.1 SPACETIME

Because c = speed of light in vacuum is the same in all reference frames according to Maxwell's equations, we can imagine considering

$$ct = (m/sec)(sec) = (m)$$

to be the 0th dimension in spacetime.

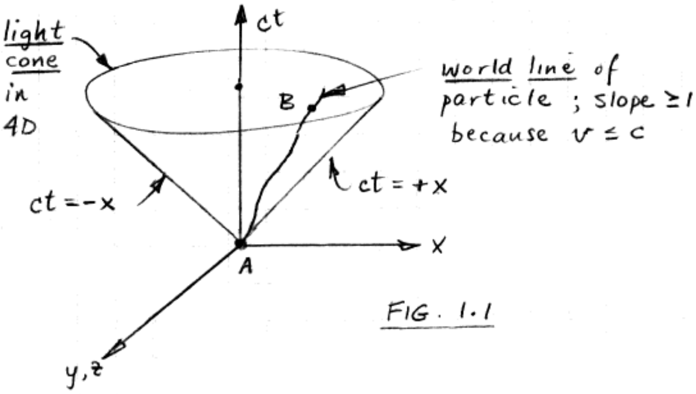


FIG. 1.1

An event is described by $r = (\overset{0}{ct}, \overset{1}{x}, \overset{2}{y}, \overset{3}{z})$. Because information travels at $\le c$, if event B is causally connected to event A, at the origin, event B must be within the light cone.

1.2 DISTANCE IN SPACETIME

What is r^2? Consider 2 non-accelerating ("inertial") reference frames

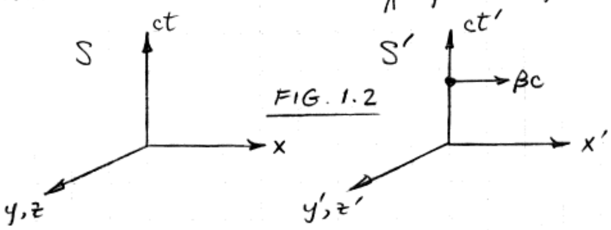


FIG. 1.2

We choose the origins to be the same, i.e. $x = y = z = 0$ is the same point as $x' = y' = z' = 0$ when $ct = ct' = 0$. Frame S′ is moving in the $(x = x')$ direction with respect to S with velocity βc.

A pulse of EM radiation is emitted at $(ct, x, y, z) = (ct', x', y', z') = 0$. In either frame it is a bubble expanding from the 3D origin:

$$x^2 + y^2 + z^2 = c^2t^2$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2 .$$

$$\therefore\ c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \qquad (1.1)$$

for the bubble.

So we define the distance Δr between 2 events r_A and r_B to be

$$\Delta r^2 = c^2(t_B - t_A)^2 \ominus (x_B - x_A)^2 \ominus (y_B - y_A)^2 \ominus (z_B - z_A)^2 \qquad (1.2)$$

Had we used $\oplus$ instead of $\ominus$, (1.1) would have forced the distance between 2 events to be different when viewed in different frames.

Distances between events are called
- timelike if $\Delta r^2 > 0$ $(c^2\Delta t^2 > |\Delta\vec{r}|^2)$
- lightlike $\quad = \quad\quad =$
- spacelike $\quad < \quad\quad <$.

Except for quantum mechanical effects, pairs of events can be causally connected only if the interval between them is timelike (within the light cone) or lightlike (on the light cone).

1.3 ROTATION IN 2D SPACE

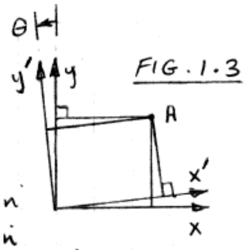


FIG. 1.3

$r = (x, y)$ and $r' = (x', y')$ are the coordinates of point A as viewed in S or S′. From the diagram, when $\theta \ll 1$ we obtain the infinitesimal transformation

$$x' = x + \theta y \qquad\qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (1.3)$$
$$y' = -\theta x + y \qquad \text{or}$$

The distance between point A and the origin is

$$r^2 = x^2 + y^2$$
$$r'^2 = x'^2 + y'^2 = (x + \theta y)^2 + (y - \theta x)^2 \quad \text{neglect}$$
$$= x^2 \cancel{+ 2\theta xy} + y^2 \cancel{- 2\theta xy} + \cancel{\mathcal{O}(\theta^2)}$$
$$= r^2 \ \checkmark$$

When θ is not $\ll 1$, (1.3) becomes

$$x' = \cos\theta\, x + \sin\theta\, y$$
$$y' = -\sin\theta\, x + \cos\theta\, y$$
or
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (1.4)$$

For non infinitesimal rotations it is still true that $r'^2 = r^2$ because $\sin^2\theta + \cos^2\theta = 1$.

1.4 INFINITESIMAL TRANSFORMATION IN 2D SPACETIME

$r = (ct, x)$ and $r' = (ct', x')$ are the coordinates of point A as viewed in S or S'. From the diagram, when $\beta \ll 1$,

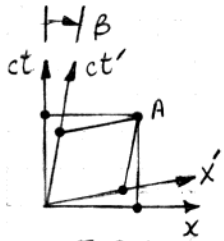


FIG 1.4

$$x' = x \ominus \beta ct$$
$$ct' = -\beta x + ct$$
or
$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \quad (1.5)$$

Why did we draw the diagram in this peculiar way, requiring a $\ominus$ rather than the usual $\oplus$ sign where indicated? The distance between event A and the origin is

$$r^2 = c^2t^2 - x^2$$
$$r'^2 = c^2t'^2 - x'^2 = (ct - \beta x)^2 - (x - \beta ct)^2$$
$$= (ct)^2 - \cancel{2\beta ctx} - x^2 \cancel{+ 2\beta ctx} + \underbrace{O(\beta^2)}_{\text{neglect}}$$
$$= r^2 \ \checkmark$$

The peculiar diagram is <u>necessary to force $r'^2 = r^2$</u>.

The nonrelativistic <u>Galilei transformation</u> is obtained from Eq. (1.5) by ignoring $-\beta x$ with respect to ct :

$$x' = x - \beta ct = x - vt$$
$$t' \approx t \quad (1.6)$$

You used this transformation (perhaps without realizing it) to solve distance = rate × time problems in high school.

The form of Eq. (1.5) which <u>exactly preserves distances in spacetime</u> is

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \quad (1.7)$$

but this discussion so far pertains only to infinitesimal transformations.

1.5. FINITE TRANSFORMATION IN 2D SPACETIME

When β in Fig. 1.4 is not $\ll 1$, we call the rotation parameter η ("eta") rather than β. η is called the "rapidity" or "boost" and, in general, is a function of β.
When the spacetime transformation is no longer infinitesimal, Eq. (1.5) becomes

$$x' = (\cosh\eta)x - (\sinh\eta)ct$$
$$ct' = -(\sinh\eta)x + (\cosh\eta)ct$$
or
$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \quad (1.8)$$

using the <u>hyperbolic functions</u>

$$\cosh a \equiv \frac{e^a + e^{-a}}{2} \qquad \sinh a \equiv \frac{e^a - e^{-a}}{2}$$
$$\tanh a \equiv \sinh a / \cosh a$$
$$\sinh(0) = 0, \ \cosh(0) = 1, \ \tanh(0) = 0$$
$$\sinh(\infty) = \infty, \ \cosh(\infty) = \infty, \ \tanh(\infty) = 1$$
$$\cosh^2 a - \sinh^2 a = 1$$

It is the <u>last</u> property which guarantees $r'^2 = r^2$ for finite transformations in spacetime.

Rewrite Eq. (1.8) as

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \cosh\eta \begin{bmatrix} 1 & -\tanh\eta \\ -\tanh\eta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \quad (1.9)$$

and compare to Eq. (1.7) making use of

$$\cosh^2\eta = \frac{\cosh^2\eta}{\cosh^2\eta - \sinh^2\eta} = \frac{1}{1 - \tanh^2\eta}.$$

Eqs. (1.7) and (1.9) are consistent if

$$\beta = \tanh\eta \ (\le 1), \quad \eta = \tanh^{-1}\beta \quad (1.10)$$

$\Rightarrow \exists$ <u>no</u> faster-than-light particles (tachyons).

Defining $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$,
Eq. (1.9) becomes the <u>Lorentz transformation</u>

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$
or
$$x' = \gamma x - \gamma\beta ct$$
$$ct' = -\gamma\beta x + \gamma ct \quad (1.11)$$

1.6. GENERALIZATIONS OF LORENTZ TRANSFORMATION

- 2D → 4D, $\vec{\beta}$ still along $\hat{x} = \hat{x}'$:

$$\underbrace{\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}}_{r'} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{call this } \Lambda} \underbrace{\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}}_{r} \qquad (1.12)$$

or $r' = \Lambda r$.

- If $\vec{\beta}$ is along $\hat{n}$ rather than $\hat{x}$:

$$r' = \Lambda_R^{-1} \Lambda \Lambda_R r \qquad (1.13)$$

where Λ_R is a <u>3D spatial rotation</u> which transforms the $\hat{n}$ direction to the $\hat{x}$ direction :

$$\Lambda_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ 0 & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ 0 & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{bmatrix} \qquad (1.14)$$

[for a rotation $(\Lambda_R^{-1})_{ji} = (\Lambda_R)_{ij}$, i.e. Λ_R is <u>orthogonal</u>].

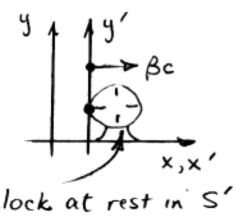
- If $\vec{\beta}$ is along $-\hat{x}$ instead of x, change the sign of β in (1.12). That is,

if $r' = \Lambda(\beta) r$ ← <u>direct</u> L.T.
(Λ is a function of β)

then $r = \Lambda(-\beta) r'$.
↑ <u>inverse</u> Lorentz transformation.

1.7. TIME DILATION

As usual, S and S′ have the same spatial origin at $t = t' = 0$



clock at rest in S′

$$\Delta t' \equiv \underset{\substack{\uparrow \\ 2^{nd}\text{ ring}}}{t_2'} - \underset{\substack{\uparrow \\ 1^{st}\text{ ring}}}{t_1'}$$

as observed at <u>fixed x'</u>.

Using <u>inverse</u> Lorentz transformation,

$$\begin{aligned} ct_2 &= \gamma ct_2' + \gamma\beta x_2' \\ ct_1 &= \gamma ct_1' + \gamma\beta x_1' \end{aligned} \qquad \text{but } x_2' = x_1' \text{ (clock fixed in S′)}$$

$$c\Delta t = \gamma c \Delta t', \qquad \boxed{\Delta t = \gamma \Delta t'} \qquad (1.15)$$

∴ Since γ always ≥ 1 the interval between rings is always <u>longer</u> when measured in a frame which is moving with respect to S′, where the two events occur at the <u>same place</u>.

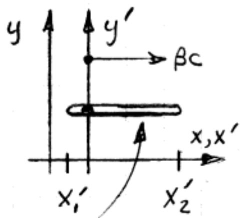
Since S′ is a <u>unique</u> frame in which to observe the time interval between these two events, we assign a unique name to this time interval :

$\Delta t' \equiv \Delta\tau \equiv$ "<u>proper time</u>" interval

$$\left.\begin{aligned} \Delta t &= \gamma \Delta\tau \\ dt &= \gamma d\tau \end{aligned}\right. \quad \text{"time dilation"} \qquad (1.16)$$

(Note that the observer in S uses his/her <u>own</u> fine grid of clocks and data loggers to measure Δt.)

1.8. SPACE CONTRACTION



rod at rest in S′

$\Delta x' \equiv x_2' - x_1'$, measured at any t'.

The observer in S, with his/her <u>own</u> fine grid of clocks, rulers, and data loggers, measures the positions x_1 and x_2 of the two ends of the rod at the <u>same time</u> $t_1 = t_2$.

Using <u>direct</u> L.T.,

$$\begin{aligned} x_2' &= \gamma x_2 - \gamma\beta ct_2 \\ x_1' &= \gamma x_1 - \gamma\beta ct_1 \end{aligned} \qquad \text{but } t_2 = t_1$$

$$\Delta x' = \gamma \Delta x \qquad \boxed{\Delta x = \Delta x'/\gamma} \qquad (1.17)$$

The length of the rod as observed in a system moving with respect to it is always <u>shorter</u> than its proper length $\Delta x'$.

The analysis of {1.7} could have been done with the <u>direct</u> L.T., and that of {1.8} with the <u>inverse</u> L.T. More algebra would be required to get the same result.

1.9. EINSTEIN LAW OF VELOCITY ADDITION

The <u>identity</u>

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b} \qquad (1.18)$$

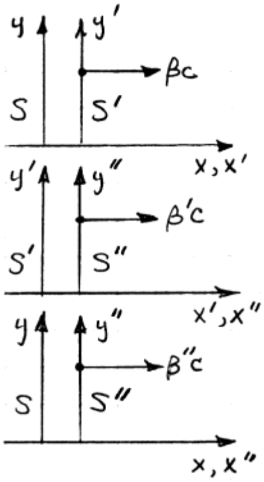
can be <u>believed</u> in analogy to the well known

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

or it can be <u>derived</u> in a few lines using

$$\tanh a \equiv \frac{e^{a} - e^{-a}}{e^{a} + e^{-a}} \qquad (1.19)$$

<u>Problem</u>: along the $\hat{x} = \hat{x}' = \hat{x}''$ direction,



S' has velocity	βc wrt	S
S''	$\beta' c$	S'
S''	$\beta'' c$	S

Given β and β', what is β''?

Since the boost parameter η is additive, we know

$$\eta'' = \eta + \eta'$$

But $\beta = \tanh \eta$ etc.

$$\therefore \quad \beta'' = \tanh \eta'' = \tanh(\eta + \eta')$$

$$= \frac{\tanh \eta + \tanh \eta'}{1 + \tanh \eta \tanh \eta'} \quad \text{using (1.18)}$$

$$\boxed{\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}} \quad \text{(Einstein law)} \qquad (1.20)$$

1.10 SPACE TRAVEL: HUMAN CONSTRAINTS

Assume that an astronaut
- must be accelerated at $\leq 1g$
- must age ≤ 40 years during the voyage.

What maximum velocity can be achieved? How far can he/she travel, and how much time will have elapsed on earth?

The voyage consists of 10 years with $a'_x = +g$, 20 years $a'_x = -g$, 10 years $a'_x = +g$. We consider only the first leg. To get the answers to the questions above, we then double the first leg distance and quadruple the first leg time.

Suppose that the astronaut at a certain moment has an $\hat{x}$ velocity equal to βc. The astronaut is accelerating and so his/her rest frame is <u>not</u> inertial. To analyze his/her motion using the Lorentz transformation we need an inertial frame, so we define a <u>comoving frame</u> S' which is instantaneously at rest with respect to the astronaut but which is <u>not</u> accelerating.

In an infinitesimal proper time interval $d\tau$ (= same in astronaut and comoving frames, since relative $\gamma_{rel} = 1$ to 2nd order in β_{rel}), the astronaut's velocity increases, relative to comoving frame, by $g\,d\tau$:

$$g\,d\tau = dv_{rel} \equiv c\,d\beta_{rel}$$

Since $\beta_{rel} = 0$ and $d\beta_{rel}$ is infinitesimally small,

$$d\eta_{rel} \cong d\beta_{rel}$$

where η is the boost parameter. Since the boost parameter is additive, as seen on the earth (frame S)

$$\eta(\tau + d\tau) = \eta(\tau) + d\eta_{rel}$$

$$d\eta/d\tau \cong d\beta_{rel}/d\tau = g/c$$

$$\eta_{max} = \int_0^{\tau_0 = 10\,yrs} \frac{d\eta}{d\tau}\,d\tau$$

$$= \int_0^{\tau_0} \frac{g}{c}\,d\tau = \frac{g\tau_0}{c} = 10.34^{*}$$

$$\beta_{max} = \tanh \eta_{max} = 1 - (2.09 \times 10^{-9}).$$

* The most boosted particles in accelerators (electrons at LEP) have $\eta \approx 12.2$.

The distance covered is

$$dx = \beta c\,dt = (\tanh\eta)c(\gamma d\tau) \quad \text{using time dilation}$$
$$= c(\tanh\eta)(\cosh\eta)\,d\tau$$
$$= c\sinh\eta\,d\tau$$
$$\Delta x = \underbrace{2}_{\text{2 legs}}c\int_0^{\tau_0}\sinh\eta\,d\tau = 2c\int_0^{\tau_0}\sinh\left(\frac{g\tau}{c}\right)d\tau$$
$$= 2\frac{c^2}{g}\left(\cosh\frac{g\tau_0}{c}-1\right) \quad \text{meters}$$
$$= 2.84\times 10^{20} \text{ meters}$$

= 29,900 light years, or $\approx 2\times10^{-7}$ the size of the universe. So only $\approx 10^{-20}$ of it can be explored by man.

The time elapsed on earth is

$$dt = \gamma\,d\tau = \cosh\eta\,d\tau$$
$$\Delta t = \underbrace{4}_{\text{4 legs}}\int_0^{\tau_0}\cosh\frac{g\tau}{c}\,d\tau = 4\frac{c}{g}\sinh\frac{g\tau_0}{c}$$
$$= 1.89\times10^{12}\text{ sec}$$
$$= 59{,}850 \text{ yrs} \qquad (\text{compare } 40 \text{ yrs.})$$

This last result is called the "twin paradox." It is not a paradox because the earthbound twin is not accelerating.

1.11 FOUR-MOMENTUM

If we wish to write Eq. (1.2) in the form

$$(r_B - r_A)^2 = c^2(t_B-t_A)^2 - (x_B-x_A)^2 - (y_B-y_A)^2 - (z_B-z_A)^2$$
$$\equiv r_B{}^2 - 2r_B\cdot r_A + r_A{}^2$$

with $r_B\cdot r_A \equiv$ inner product of 2 4-vectors in spacetime, it must be the case that

$$r_B\cdot r_A \equiv c^2 t_B t_A - x_B x_A - y_B y_A - z_B z_A$$

and that the inner product of any 2 4-vectors is independent of reference frame (invariant to Lorentz transformations).

The proper time interval $d\tau$ and the rest mass m are also Lorentz invariants.

Form $p \equiv m\dfrac{dr}{d\tau}$, that is

$$(p_0, p_x, p_y, p_z) = \left(mc\frac{dt}{d\tau}, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right).$$

Note that $dt/d\tau = \gamma$ so $dx/d\tau = \gamma\,dx/dt$. So

$$p = (\gamma mc, \gamma m v_x, \gamma m v_y, \gamma m v_z) \tag{1.21}$$
$$\equiv (E/c, \vec{p})$$

must transform like r, i.e. must also be a 4-vector. It is called the four-momentum. We can write

$$\begin{bmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{bmatrix} = \Lambda \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \quad \text{with } \Lambda \text{ as in (1.12).}$$

The length^2 of p is Lorentz invariant and we can evaluate it in a frame in which the CM of the system it describes is not moving ($\vec{p}=0$, $\gamma = 1$). Then

$$p^2 = \underbrace{E^2/c^2 - |\vec{p}|^2}_{\text{true in any frame}} = \underbrace{m^2c^2}_{\text{rest frame value}} \tag{1.22}$$

This is the fundamental equation for solving relativistic kinematics problems.

What is E? Make a Taylor series expansion

$$E = \frac{mc^2}{(1-v^2/c^2)^{1/2}} = mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}+\cdots\right)$$

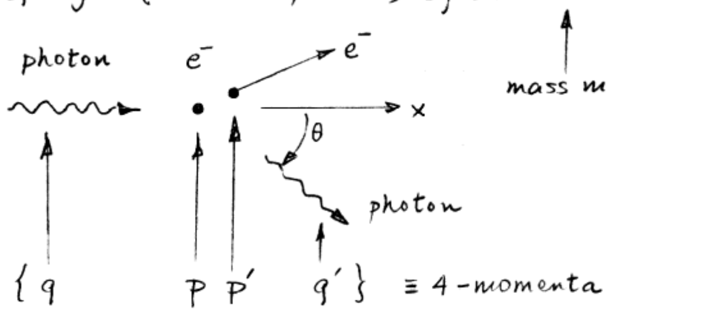
For $v \ll c$ this is $E = mc^2 + \frac{1}{2}mv^2$ where the last term is the nonrelativistic kinetic energy. We interpret the first term as the rest mass energy:

$$\underbrace{E}_{\text{total energy}} = \gamma mc^2 = \underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{T}_{\text{kinetic energy}} \tag{1.23}$$

We see the possibility of converting mass to energy (lots of energy because c^2 is large).

1.12 COMPTON (PHOTON–ELECTRON) SCATTERING

To illustrate the power of Eq. (1.22) for solving problems in relativistic kinematics, we consider the scattering of a quantum of light (massless photon) by an electron at rest.



$p = (mc, \vec{0})$ because target electron at rest.
$q = (q_0, q_x, 0, 0)$
Since photon massless, $q \cdot q = 0$ by Eq. (1.22), so $q_x = q_0$ and we can write

$$q = (q_0, q_0, 0, 0); \quad q' = (q_0', q_0'\cos\theta, q_0'\sin\theta, 0)$$

Problem: what is the relationship between the final photon energy cq_0' and its final angle θ wrt $\hat{x}$?

Use energy–momentum conservation $\Rightarrow$ 4-momentum conservation:

$$q + p = q' + p' \quad \text{(this is 4 equations!)}$$
$$q - q' + p = p'$$
$$[(q-q') + p] \cdot [(q-q') + p] = p' \cdot p'$$
$$(q-q')\cdot(q-q') + 2p\cdot(q-q') + p\cdot p = p'\cdot p'$$
$$\underbrace{q\cdot q}_{0} - 2q\cdot q' + \underbrace{q'\cdot q'}_{0} + 2p\cdot(q-q') + \underbrace{\cancel{p\cdot p}}_{\cancel{(mc)^2}} = \underbrace{\cancel{p'\cdot p'}}_{\cancel{(mc)^2}}$$
$$\cancel{2}p\cdot(q-q') = \cancel{2}q\cdot q'$$
$$(mc, 0, 0, 0)\cdot(q_0 - q_0', q_0 - q_0'\cos\theta, -q_0'\sin\theta, 0) =$$
$$= (q_0, q_0, 0, 0)\cdot(q_0', q_0'\cos\theta, q_0'\sin\theta, 0)$$
$$mc(q_0 - q_0') = q_0 q_0'(1-\cos\theta) \qquad \div q_0 q_0' mc:$$
$$\boxed{\frac{1}{q_0'} - \frac{1}{q_0} = \frac{1}{mc}(1-\cos\theta)} \qquad (1.24)$$

This is A.H. Compton's famous formula. Conventionally it is multiplied by Planck's constant h, with the photon wavelength $\lambda = h/q_0$. Then

$$\boxed{\lambda' - \lambda = \lambda_c(1-\cos\theta)} \text{ with} \qquad (1.25)$$
$$\lambda_c \equiv h/mc.$$

λ_c, the Compton wavelength of the electron, is

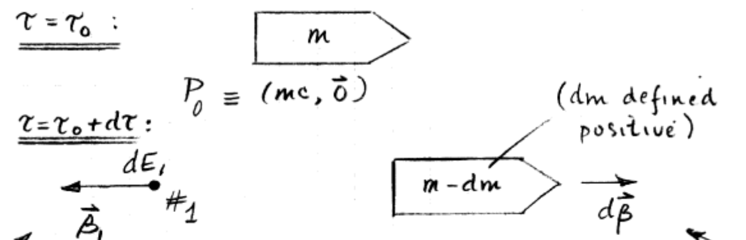
$$\lambda_c = 2\pi \times 386 \times 10^{-15}\,m$$

Planck's constant is

$$h = 2\pi \times 6.58 \times 10^{-16}\ \text{eV sec}.$$

1.13 SPACE TRAVEL: PROPULSION CONSTRAINTS

Again consider the spacecraft as viewed in the comoving frame (§1.10). In infinitesimal proper time interval $d\tau$ the rocket motor ejects particle #1 with energy dE_1 and relative velocity $\beta_1 c$.



$$p_1 \equiv \left(\frac{dE_1}{c}, \vec{\beta}_1\frac{dE_1}{c}\right) \qquad \left((m-dm)c + \frac{mc|d\vec{\beta}|^2}{2}, mc\,d\vec{\beta}\right) \equiv P'$$

In assigning the 4-momentum to particle #1, we used $p = (\gamma mc, \gamma\vec{\beta}mc)$ so that $\vec{p}/p_0 = \vec{\beta}$. In assigning the 4-momentum to the spacecraft, we used the fact that (as viewed in the comoving frame) the spacecraft is nonrelativistic, so that $E \approx mc^2 + \frac{1}{2}mv^2$.

If we assume a perfectly efficient engine, i.e. no heat energy radiated in random directions, both energy and momentum will be conserved:

$$P_0 = p_1 + P' \qquad \text{neglect, 2nd order in infinitesimals.}$$
$$\cancel{mc} = \frac{dE_1}{c} + (\cancel{m} - dm)c + \cancel{\frac{mc}{2}|d\vec{\beta}|^2} \qquad \text{(timelike part)}$$
$$\vec{0} = \vec{\beta}_1\frac{dE_1}{c} + mc\,d\vec{\beta} \qquad \text{(spacelike part)}$$

Substituting $\frac{dE_1}{c} = c\,dm$ from the timelike eq$^{\underline{n}}$, the spacelike eq$^{\underline{n}}$ becomes

$$|d\vec{\beta}| = |\vec{\beta}_1|\frac{dm}{m}.$$

Again we set $|d\vec{\beta}| \approx d\eta$, where η is the boost, since rocket is nonrelativistic in comoving frame.

As additional particles (#2, #3, etc) are ejected, the boosts $d\eta_i$ are <u>additive</u>.

$$\therefore \quad \eta_{final} - (\eta_0 \equiv 0) = \int_{m_0}^{m_{final}} \beta_1 \frac{|dm|}{m}$$

$$\boxed{\eta_{final} = \beta_1 \ln \frac{m_0}{m_{final}}} \qquad (1.26)$$

<u>Chemical rocket engines</u> achieve maximum $\beta_1 \approx 4 \times 10^3$ m/sec / c $\approx 1.33 \times 10^{-5}$. Then to achieve a boost of 10.34 (see §1.10) requires

$$\ln \frac{m_0}{m_f} = 7.8 \times 10^5$$

$$m_f = m_0 \times (\text{a number beyond calculator range}).$$

Chemical engines <u>will not suffice</u>.

<u>Relativistic engines</u> emit particles at $\beta_1 \cong 1$. If they were unit efficient,

$$\ln \frac{m_0}{m_f} = 10.34$$

$$m_f = 3.1 \times 10^4 \, m_0 .$$

<u>Manned payload</u> requires $m_f \geq 10T$ for life support; then

$$m_0 \geq 3.1 \times 10^5 \, T$$

$\Rightarrow$ a rocket heavier than an aircraft carrier. $(\leq 10^5 T)$

Note that Eq. (1.26) becomes

$$\eta_{final} = \underset{=}{\epsilon} \beta_1 \ln \frac{m_0}{m_{final}} \qquad (1.27)$$

if the efficiency ϵ of the engine is not unity.

Present relativistic engine concepts ...

- are grossly inefficient $(\epsilon \ll 1)$
- leave most of their fuel on board so that m_f / m_0 cannot be $\ll 1$.

(Example: laser powered by batteries)

1.14 OTHER FOUR-VECTORS

In addition to

$$r = (ct, \vec{r})$$

$$p = (E/c, \vec{p}) \qquad (E \equiv \gamma m c^2, \ \vec{p} \equiv \gamma m \vec{v}),$$

frequently encountered other 4-vectors are

$$\partial \equiv \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right) \qquad \left(\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

$$k \equiv \left(\frac{\omega}{c}, \vec{k}\right) \quad \text{as in } e^{i(\omega t - \vec{k}\cdot\vec{r})} \qquad (1.28)$$

"wave vector"

ω = angular freq.

$$A \equiv (\varphi, \vec{A}) \qquad (1.29)$$

"vector potential"

$$\text{where} \begin{cases} \vec{B} \equiv \vec{\nabla} \times \vec{A} \\ \vec{E} \equiv -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{c\partial t} . \end{cases}$$

Because dot products of 4-vectors are Lorentz invariant, so are

$$k \cdot r \equiv \omega t - \vec{k}\cdot\vec{r} \quad \text{"phase of a wave"}$$

$$\partial \cdot \partial \equiv \Box = \frac{\partial^2}{c^2 \partial t^2} - \nabla^2 \quad \text{"D'Alembertian"}$$

$$\partial \cdot A \equiv \frac{\partial \phi}{c \partial t} + \vec{\nabla}\cdot\vec{A}$$

(= 0 when $\vec{A}$ satisfies the "Lorentz gauge condition")

When the "de Broglie momentum" equation $|\vec{p}| = h/\lambda$ is combined with the "Planck frequency" equation $E = h\nu$, using

$$|\vec{k}| \equiv 2\pi/\lambda , \qquad (1.30)$$

both equations can be summarized by

$$(E/c, \vec{p}) \equiv \boxed{p = \frac{h}{2\pi} k} \equiv \frac{h}{2\pi}\left(\frac{\omega}{c}, \vec{k}\right) \qquad (1.31).$$

(this is 4 equations)

"generalized de Broglie eq$^{\underline{n}}$"

Another 4-vector is

$$j \equiv (c\rho, \vec{J}) \qquad \begin{matrix} \rho = \text{chg density } (\text{esu/cm}^3) \\ \vec{J} = \text{current density } (\text{esu/cm}^2\text{-sec}) \end{matrix}$$

$$\text{Lorentz invariant} \Big\} \ \partial \cdot j = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \qquad (1.32)$$

(charge conservation)

1.15 LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS

The fact that

$$\begin{bmatrix}\phi' \\ A_x' \\ A_y' \\ A_z'\end{bmatrix} = \begin{bmatrix}\gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\phi \\ A_x \\ A_y \\ A_z\end{bmatrix}$$

and

$$\left.\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A} \\ \vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{c\,\partial t}\end{aligned}\right\} \text{cgs units!}$$

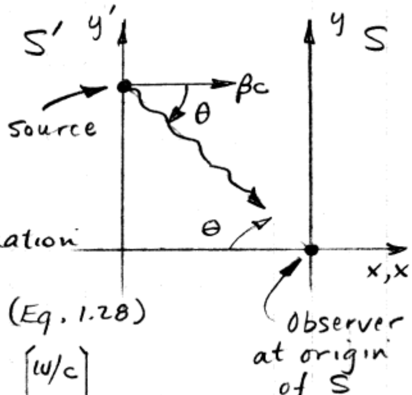
leads after some algebra to the following transformation equations for $\vec{E}$ and $\vec{B}$:

$$\begin{aligned}\vec{E}'_\perp &= \gamma(\vec{E}_\perp + \vec{\beta}\times\vec{B}) \\ \vec{B}'_\perp &= \gamma(\vec{B}_\perp - \vec{\beta}\times\vec{E}) \\ \vec{E}'_\parallel &= \vec{E}_\parallel\,,\quad \vec{B}'_\parallel = \vec{B}_\parallel\end{aligned} \qquad (1.33)$$

where "$_\perp$" means $\perp$ to $\hat{\beta}$ and "$_\parallel$" means parallel to $\hat{\beta}$.

A consequence of Eq. (1.33) is that $|\vec{E}|^2 - |\vec{B}|^2$ is a Lorentz invariant.

1.16 RELATIVISTIC DOPPLER SHIFT



Apply the direct Lorentz transformation (Eq. 1.12) to the wave 4-vector k (Eq. 1.28)

$$\begin{bmatrix}\omega'/c \\ k_x' \\ k_y' \\ k_z'\end{bmatrix} = \Lambda \begin{bmatrix}\omega/c \\ k_x \\ k_y \\ k_z\end{bmatrix}$$

$$\Rightarrow \frac{\omega'}{c} = \gamma\frac{\omega}{c} - \gamma\beta k_x \qquad (1.34)$$

Let the lab phase velocity of the wave be $\beta_s c$ ($\beta_s = 1$ for a light wave). ($T \equiv$ period)

$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{2\pi}{\beta_s c T} = \frac{2\pi\nu}{\beta_s c} = \frac{\omega}{\beta_s c}$$

so in Eq. (1.34) we may write

$$k_x = |\vec{k}|\cos\theta = \frac{\omega}{\beta_s c}\cos\theta. \quad \text{Then}$$

$$\omega' = \gamma\omega\left(1 - \frac{\beta}{\beta_s}\cos\theta\right)$$

$$\boxed{\omega = \frac{\omega'}{\gamma\left(1 - \frac{\beta}{\beta_s}\cos\theta\right)}} \quad \text{Relativistic Doppler shift} \qquad (1.35)$$

Special cases:

- light wave $\Rightarrow \beta_s = 1$

$$\boxed{\omega = \frac{\omega'}{\gamma(1 - \beta\cos\theta)}} \qquad (1.36)$$

- $\left.\begin{matrix}\text{approaching } (\theta = 0) \\ \text{receding } (\theta = \pi)\end{matrix}\right\}$ light wave:

$$\omega = \frac{\omega'}{\gamma(1 \mp \beta)} = \left(\frac{1 \pm \beta}{1 \mp \beta}\right)^{1/2}\omega'$$

- $\cos\theta = 0$ (source is at zenith, where nonrelativistically there is no Doppler shift):

$$\omega = \frac{\omega'}{\gamma}\,,\quad T = T'\gamma \quad \begin{pmatrix}\text{ordinary} \\ \text{time} \\ \text{dilation}\end{pmatrix}$$

- $\beta \ll 1$

$$\omega \cong \frac{\omega'}{1 - \frac{\beta}{\beta_s}\cos\theta} = \frac{\omega'}{1 - \frac{v_{source}}{v_{wave}}\cos\theta}$$

(= freshman physics Doppler shift. Note sonic boom at $\cos\theta = v_{wave}/v_{source}$.)

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Physics H7B Spring 1999 (*Strovink*)

PROBLEM SET 1

1. RHK problem 22.9
It is an everyday observation that hot and cold objects cool down or warm up to the temperature of their surroundings. If the temperature difference ΔT between an object and its surroundings ($\Delta T = T_{\rm obj} - T_{\rm sur}$) is not too great, the rate of cooling or warming of the object is proportional, approximately, to this difference; that is,

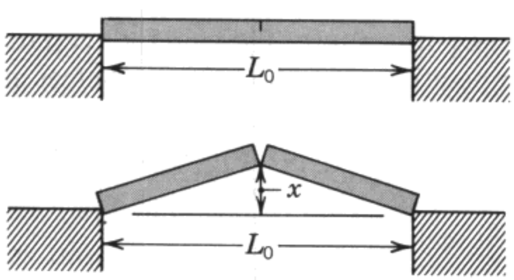
$$\frac{d\,\Delta T}{dt} = -A(\Delta T),$$

where A is a constant. Ths minus sign appears because ΔT decreases with time if ΔT is positive and increases if ΔT is negative. This is known as *Newton's law of cooling*.
(a) On what factors does A depend? What are its dimensions?
(b) If at some instant $t = 0$ the temperature difference is ΔT_0, show that it is

$$\Delta T = \Delta T_0 \exp(-At)$$

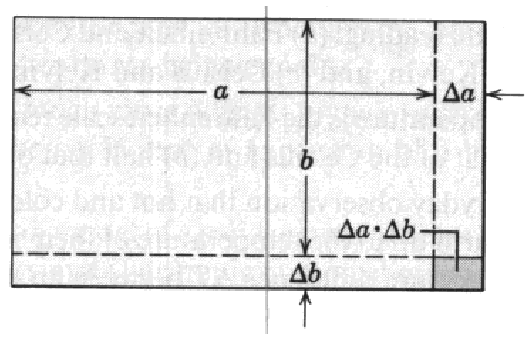
at a time t later.

2. RHK problem 22.28
As a result of a temperature rise of 32 °C, a bar with a crack at its center buckles upward, as shown in the figure. If the fixed distance $L_0 = 3.77$ m and the coefficient of linear thermal expansion is 25×10^{-6} per °C, find x, the distance to which the center rises.



3. RHK problem 22.30
The area A of a rectangular plate is ab. Its coefficient of linear thermal expansion is α. After a temperature rise ΔT, side a is longer by Δa and side b is longer by Δb. Show that if we neglect the small quantity $\Delta a\,\Delta b\,/\,ab$ (see the figure), then

$$\Delta A = 2\alpha A \Delta T\,.$$

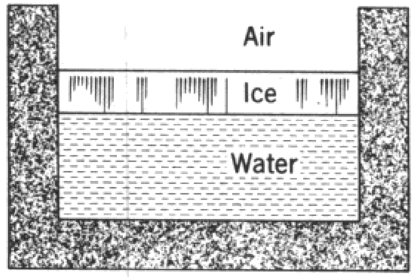


4. RHK problem 25.47
The average rate at which heat flows out through the surface of the Earth in North America is 54 mW/m^2, and the average thermal conductivity of the near surface rocks is 2.5 W/m·K. Assuming a surface temperature of 10 °C, what should be the temperature at a depth of 33 km (near the base of the crust)? Ignore the heat generated by radioactive elements in the crust; the curvature of the Earth can also be ignored.

5. RHK problem 25.50
A cylindrical silver rod of length 1.17 m and cross-sectional area 4.76 cm^2 is insulated to prevent heat loss through its surface. The ends are maintained at temperature difference of 100 °C by having one end in a water-ice mixture and the other in boiling water and steam.
(a) Find the rate (in W) at which heat is transferred along the rod.
(b) Calculate the rate (in kg/sec) at which ice melts at the cold end.
Hints: The thermal conductivity of silver is 428 W/m·K. The latent heat of fusion of water is 333 kJ/kg.

6. RHK problem 25.58
A container of water has been outdoors in cold weather until a 5.0-cm-thick slab of ice has formed on its surface (see the figure). The air above the ice is at −10 °C. Calculate the rate of

formation of ice (in centimeters per hour) on the bottom surface of the ice slab. Take the thermal conductivity and density of ice to be 1.7 W/m·K and 0.92 g/cm^3, respectively. Assume that no heat flows through the walls of the tank.



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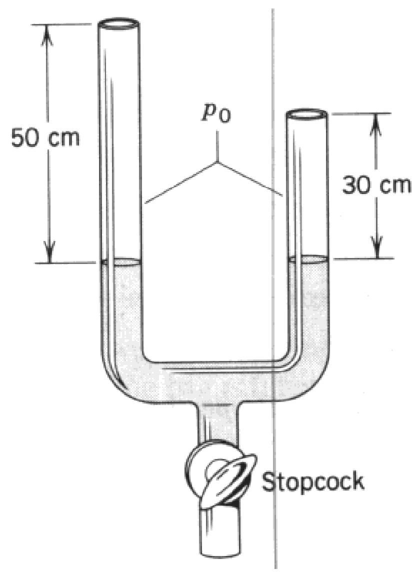
A mercury-filled manometer with two unequal-length arms of the same cross-sectional area is sealed off with the same pressure p_0 of perfect gas in the two arms (see the figure). With the temperature constant, an additional 10.0 cm^3 of mercury is admitted through the stopcock at the bottom. The level on the left increases 6.00 cm and that on the right increases 4.00 cm. Find the original pressure p_0.



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SOLUTION TO PROBLEM SET 1

1. RHK problem 22.9
It is an everyday observation that hot and cold objects cool down or warm up to the temperature of their surroundings. If the temperature difference ΔT between an object and its surroundings ($\Delta T = T_{\rm obj} - T_{\rm sur}$) is not too great, the rate of cooling or warming of the object is proportional, approximately, to this difference; that is,

$$\frac{d\,\Delta T}{dt} = -A(\Delta T),$$

where A is a constant. Ths minus sign appears because ΔT decreases with time if ΔT is positive and increases if ΔT is negative. This is known as *Newton's law of cooling*.

(a) On what factors does A depend? What are its dimensions?
Solution: The LHS (and therefore the RHS) of the above equation have dimensions C°/sec, so A must have dimension sec^{-1}. Suppose that the heat flowing between the object and its surroundings is conducted by a thermal barrier (*i.e.* a "skin" on the object that tends to insulate it from its surroundings). Then, from RHK Eq. 25.45, A should be proportional to the thermal conductivity of that barrier and to its area, and inversely proportional to the barrier's thickness.

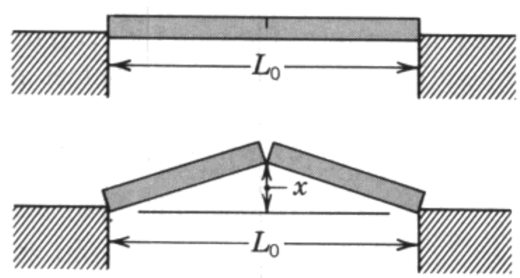
(b) If at some instant $t = 0$ the temperature difference is ΔT_0, show that it is

$$\Delta T = \Delta T_0 \exp{(-At)}$$

at a time t later.
Solution: Rearranging and solving the above equation, with dt' substituted for dt,

$$\begin{aligned}
\frac{d\,\Delta T}{\Delta T} &= -A\,dt' \\
\int_0^t \frac{d\,\Delta T}{\Delta T} &= -\int_0^t A\,dt' \\
\ln\big(\Delta T(t)\big) - \ln\big(\Delta T(0)\big) &= -At \\
\ln\left(\frac{\Delta T(t)}{\Delta T(0)}\right) &= -At \\
\frac{\Delta T(t)}{\Delta T(0)} &= \exp{(-At)} \\
\Delta T(t) &= \Delta T_0 \exp{(-At)}\ .
\end{aligned}$$

2. RHK problem 22.28
As a result of a temperature rise of 32 C°, a bar with a crack at its center buckles upward, as shown in the figure. If the fixed distance $L_0 = 3.77$ m and the coefficient of linear thermal expansion is 25×10^{-6} per C°, find x, the distance to which the center rises.



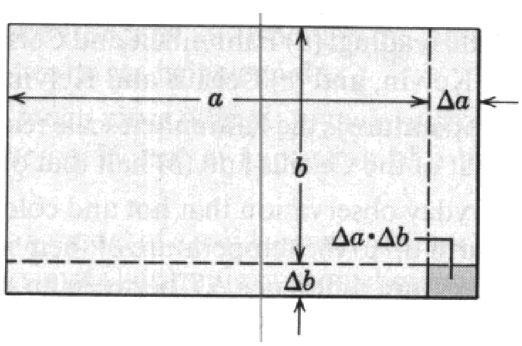
Solution: In Physics H7B, all problems involving numbers should be solved completely in terms of algebraic symbols before any numbers are plugged in (otherwise it is much more difficult to give part credit). Let
L_0 = fixed distance = 3.77 m
x = distance to which the center rises
L = thermally expanded total length of the buckled bar (twice the hypotenuse of the right triangle whose legs are x and $L_0/2$)
α = coefficient of linear thermal expansion = 25×10^{-6} per C°
ΔT = temperature rise = 32 C°

Then

$$
\begin{aligned}
L &= L_0 + \alpha\,\Delta T \\
x^2 &= (L/2)^2 - (L_0/2)^2 \\
x &= \frac{L_0}{2}\sqrt{(1+\alpha\,\Delta T)^2 - 1} \\
&= 0.0754 \text{ m} .
\end{aligned}
$$

3. RHK problem 22.30
The area A of a rectangular plate is ab. Its coefficient of linear thermal expansion is α. After a temperature rise ΔT, side a is longer by Δa and side b is longer by Δb. Show that if we neglect the small quantity $\Delta a\,\Delta b\,/\,ab$ (see the figure), then

$$\Delta A = 2\alpha A \Delta T .$$



Solution: Let
A = original area of rectangular plate
a = original width of plate
b = original height of plate
$A + \Delta A$ = thermally expanded area of plate
$a + \Delta a$ = thermally expanded width of plate
$b + \Delta b$ = thermally expanded height of plate
α = coefficient of linear thermal expansion

Then

$$
\begin{aligned}
A + \Delta A &= (a + \Delta a)(b + \Delta b) \\
&= ab + a\,\Delta b + b\,\Delta a + \Delta a\,\Delta b \\
A &= ab \\
A + \Delta A - A &= ab + a\,\Delta b + b\,\Delta a + \Delta a\,\Delta b - ab \\
\Delta A &= a\,\Delta b + b\,\Delta a + \Delta a\,\Delta b \\
&= ab\Big(\frac{\Delta b}{b} + \frac{\Delta a}{a} + \frac{\Delta a\,\Delta b}{ab}\Big) \\
\Delta A &\approx ab\Big(\frac{\Delta b}{b} + \frac{\Delta a}{a}\Big) \\
\frac{\Delta a}{a} &= \frac{\Delta b}{b} = \alpha\,\Delta T \\
\Delta A &\approx A(\alpha\,\Delta T + \alpha\,\Delta T) \\
\frac{\Delta A}{A} &\approx 2\alpha\,\Delta T .
\end{aligned}
$$

4. RHK problem 25.47
The average rate at which heat flows out through the surface of the Earth in North America is 54 mW/m^2, and the average thermal conductivity of the near surface rocks is 2.5 W/m·K. Assuming a surface temperature of 10 °C, what should be the temperature at a depth of 33 km (near the base of the crust)? Ignore the heat generated by radioactive elements in the crust; the curvature of the Earth can also be ignored.
Solution: Let
H/A = heat flow per unit area through Earth's surface = 54×10^{-3} W/m^2
k = thermal conductivity of near surface rock = 2.5 W/m·K
T_0 = temperature at earth's surface = 10 °C
D = depth at which we wish to know the temperature = 33×10^3 m
T = temperature at depth D
Then, using RHK Eq. 25.45,

$$
\begin{aligned}
\frac{H}{A} &= k\frac{\Delta T}{\Delta x} \\
&= k\frac{T - T_0}{D} \\
\frac{H}{A}\frac{D}{k} &= T - T_0 \\
T_0 + \frac{H}{A}\frac{D}{k} &= T \\
723\ ^\circ\text{C} &= T .
\end{aligned}
$$

5. RHK problem 25.50
A cylindrical silver rod of length 1.17 m and cross-sectional area 4.76 cm^2 is insulated to prevent heat loss through its surface. The ends are maintained at temperature difference of 100 C° by having one end in a water-ice mixture and the other in boiling water and steam.

(a) Find the rate (in W) at which heat is transferred along the rod.
Solution: Let
L = length of cylindrical silver rod = 1.17 m
A = area of rod = 4.76×10^{-4} m^2
k = thermal conductivity of silver = 428 W/m·K
ΔT = temperature difference between ends of rod = 100 C°.
$H = dQ/dt$ = rate at which heat is transferred along the rod.
Then, using RHK Eq. 25.45

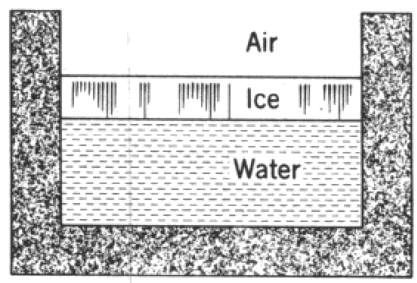
$$\begin{aligned} H &= kA\frac{\Delta T}{x} \\ &= kA\frac{\Delta T}{L} \\ &= 17.4 \text{ W} . \end{aligned}$$

(b) Calculate the rate (in kg/sec) at which ice melts at the cold end.
Solution: Let
L_{f} = latent heat of fusion of water = 333×10^3 J/kg
dm/dt = rate in kg/sec at which ice melts at the cold end
Then, using RHK Eq. 25.7,

$$\begin{aligned} Q &= L_{\mathrm{f}}\, m \\ \frac{dQ}{dt} &= L_{\mathrm{f}}\frac{dm}{dt} \\ H &= L_{\mathrm{f}}\frac{dm}{dt} \\ \frac{H}{L_{\mathrm{f}}} &= \frac{dm}{dt} \\ 5.23 \times 10^{-5} \text{ kg/sec} &= \frac{dm}{dt} . \end{aligned}$$

Hints: The thermal conductivity of silver is 428 W/m·K. The latent heat of fusion of water is 333 kJ/kg.

6. RHK problem 25.58
A container of water has been outdoors in cold weather until a 5.0-cm-thick slab of ice has formed on its surface (see the figure). The air above the ice is at −10 °C. Calculate the rate of formation of ice (in centimeters per hour) on the bottom surface of the ice slab. Take the thermal conductivity and density of ice to be 1.7 W/m·K and 0.92 g/cm^3, respectively. Assume that no heat flows through the walls of the tank.



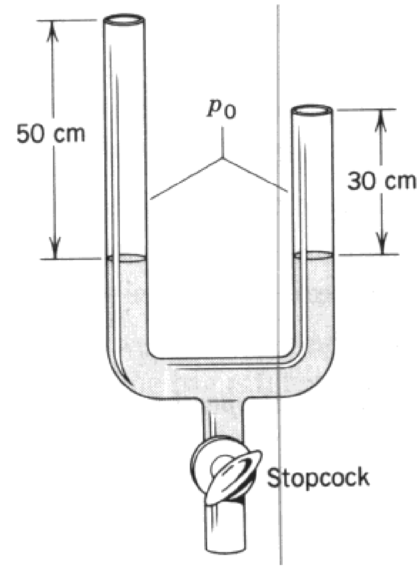
Solution: Let
A = area of slab of ice on water's surface
h = present thickness of slab = 0.05 m
T = temperature of air above ice = −10 °C
T_0 = temperature at which water freezes = 0 °C
k = thermal conductivity of ice = 1.7 W/m·K
ρ = density of ice = 0.92×10^3 kg/m^3
L_{f} = latent heat of fusion of water = 333×10^3 J/kg
$H = dQ/dt$ = heat flow (in W) through the ice
dm/dt = rate of formation of ice (in kg/sec) on the bottom surface of the slab
dh/dt = rate of change of ice thickness (in m/sec).

Then, using RHK Eqs. 25.45 and 25.7,

$$
\begin{aligned}
H &= kA\frac{\Delta T}{\Delta x} \\
&= kA\frac{T_0 - T}{h} \\
Q &= L_{\mathrm{f}}\, m \\
\frac{dQ}{dt} &= L_{\mathrm{f}}\frac{dm}{dt} \\
H &= L_{\mathrm{f}}\frac{dm}{dt} \\
kA\frac{T_0 - T}{h} &= L_{\mathrm{f}}\frac{dm}{dt} \\
\rho h A &= m \\
\frac{dh}{dt} &= \frac{1}{\rho A}\frac{dm}{dt} \\
&= \frac{1}{\rho A L_{\mathrm{f}}} L_{\mathrm{f}}\frac{dm}{dt} \\
&= \frac{kA}{\rho A L_{\mathrm{f}}}\frac{T_0 - T}{h} \\
&= \frac{k}{\rho L_{\mathrm{f}}}\frac{T_0 - T}{h} \\
&= 1.11 \times 10^{-6}\ \mathrm{m/sec} \\
&= 0.400\ \mathrm{cm/hr}\ .
\end{aligned}
$$

Note that the inverse dependence of dh/dt upon h requires h to increase only as the square root of the time t. Our numerical result for the rate of growth of the ice thickness is valid only when the ice has a particular thickness (5 cm).

7. RHK problem 23.16
A mercury-filled manometer with two unequal-length arms of the same cross-sectional area is sealed off with the same pressure p_0 of perfect gas in the two arms (see the figure). With the temperature held constant, an additional 10.0 cm^3 of mercury is admitted through the stopcock at the bottom. The level on the left increases 6.00 cm and that on the right increases 4.00 cm. Find the original pressure p_0.



Solution: Let
ρ = density of Hg = 13.6 × 10^3 kg/m^3
g = acceleration of gravity at earth's surface = 9.81 m/sec^2
L_0 = initial height of gas in left arm of manometer = 0.50 m
R_0 = initial height of gas in right arm of manometer = 0.30 m
L = final height of gas in left arm of manometer = 0.44 m
R = final height of gas in right arm of manometer = 0.26 m
A = cross-sectional area of each manometer arm
p_0 = initial pressure in both arms of manometer
p_L = final pressure in left arm of manometer
p_R = final pressure in right arm of manometer
N_L = no. of gas molecules in left arm of manometer
N_R = no. of gas molecules in right arm of manometer
k_{B} = Boltzmann's constant
T = (constant) temperature

Applying the perfect gas law,

$$\begin{aligned}
p_0 A L_0 &= N_L k_{\mathrm{B}} T \\
p_0 A R_0 &= N_R k_{\mathrm{B}} T \\
p_L A L &= N_L k_{\mathrm{B}} T \\
p_R A R &= N_R k_{\mathrm{B}} T \\
p_0 L_0 &= p_L L \\
p_0 R_0 &= p_R L \\
p_0 \frac{L_0}{L} &= p_L \\
p_0 \frac{R_0}{R} &= p_R \\
\text{(I)}\;\; p_R - p_L &= p_0 \Big(\frac{R_0}{R} - \frac{L_0}{L} \Big) \ .
\end{aligned}$$

Using Archimedes' principle (first equation on RHK page 387), the difference $(L_0 - L) - (R_0 - R)$ in final height of Hg between the two arms is proportional to the final pressure difference:

$$\begin{aligned}
A(p_R - p_L) &= \rho g A \big((L_0 - L) - (R_0 - R) \big) \\
\text{(II)}\;\; p_R - p_L &= \rho g \big((L_0 - L) - (R_0 - R) \big) \ .
\end{aligned}$$

Combining equations (I) and (II),

$$\begin{aligned}
p_0 \Big(\frac{R_0}{R} - \frac{L_0}{L} \Big) &= \rho g \big((L_0 - L) - (R_0 - R) \big) \\
p_0 &= \rho g \frac{(L_0 - L) - (R_0 - R)}{R_0/R - L_0/L} \\
&= 1.526 \times 10^5 \text{ Pa} \\
&= 1.506 \text{ atm} \ .
\end{aligned}$$

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PROBLEM SET 2

1. RHK problem 24.18

2. RHK problem 24.21

3. RHK problem 24.25

4. RHK problem 23.17

5. RHK problem 23.33

6. RHK problem 23.37

7. RHK problem 25.16

8. RHK problem 25.21

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SOLUTION TO PROBLEM SET 2

1. RHK problem 24.18
Solution: For ease of notation, here we denote the mean of any function $f(v)$ of the speed v of a gas molecule by

$$\langle f(v)\rangle \equiv \frac{\int_0^\infty f(v')n(v')dv'}{\int_0^\infty n(v')dv'}$$

where $n(v)$ (called dN/dv in lecture) is the distribution of v. If this formula is used, $n(v)$ *does not need to be normalized*. With this notation, for example, $\bar{v} \equiv \langle v\rangle$. Proceeding with the problem,

$$\begin{aligned}
v_{\rm rms} &\equiv \sqrt{\langle v^2\rangle} \text{ (RHK Eq. 23.15)}\\
0 &\le \langle (v-\bar{v})^2\rangle\\
\langle (v-\bar{v})^2\rangle &= \langle v^2\rangle - \langle 2v\bar{v}\rangle + \langle \bar{v}^2\rangle\\
&= \langle v^2\rangle - \bar{v}\langle 2v\rangle + \bar{v}^2\\
&= \langle v^2\rangle - 2\bar{v}^2 + \bar{v}^2\\
&= \langle v^2\rangle - \bar{v}^2\\
0 &\le \langle v^2\rangle - \bar{v}^2\\
\bar{v}^2 &\le \langle v^2\rangle\\
\bar{v} &\le \sqrt{\langle v^2\rangle}\\
\bar{v} &\le v_{\rm rms}\ .
\end{aligned}$$

The equality occurs only when $\langle (v-\bar{v})^2\rangle = 0$, *i.e.* all the molecules have the average speed $\bar{v}$.

2. RHK problem 24.21
Solution: Using the notation introduced above,
(b)

$$\begin{aligned}
\langle v\rangle &= \frac{\int_0^{v_0} v'Cv'^2dv'}{\int_0^{v_0} Cv'^2dv'}\\
&= \frac{\frac{1}{4}v_0^4}{\frac{1}{3}v_0^3}\\
&= \frac{3}{4}v_0\ .
\end{aligned}$$

(c)

$$\begin{aligned}
\langle v^2\rangle &= \frac{\int_0^{v_0} v'^2Cv'^2dv'}{\int_0^{v_0} Cv'^2dv'}\\
&= \frac{\frac{1}{5}v_0^5}{\frac{1}{3}v_0^3}\\
&= \frac{3}{5}v_0^2\\
v_{\rm rms} &\equiv \sqrt{\langle v^2\rangle}\\
v_{\rm rms} &= \sqrt{\frac{3}{5}}v_0\ .
\end{aligned}$$

(a)

$$\begin{aligned}
N &\equiv \int_0^{v_0} Cv'^2dv'\\
&= \frac{1}{3}Cv_0^3\\
\frac{3N}{v_0^3} &= C\ .
\end{aligned}$$

3. RHK problem 24.25
Solution:

$$\begin{aligned}
n(E) &\propto E^{1/2}\exp(-E/kT) \text{ (RHK Eq. 24.27)}\\
E_{\rm rms} &\equiv \sqrt{\langle E^2\rangle}\\
\langle E^2\rangle &= \frac{\int_0^\infty E'^2E'^{1/2}\exp(-E'/kT)dE'}{\int_0^\infty E'^{1/2}\exp(-E'/kT)dE'}\\
\beta &\equiv 1/kT\\
\langle E^2\rangle &= \frac{\int_0^\infty E'^{5/2}\exp(-\beta E')dE'}{\int_0^\infty E'^{1/2}\exp(-\beta E')dE'}\\
&= \frac{\left(d^2/d\beta^2\right)\left(\int_0^\infty E'^{1/2}\exp(-\beta E')dE'\right)}{\int_0^\infty E'^{1/2}\exp(-\beta E')dE'}\\
Z &\equiv \int_0^\infty E'^{1/2}\exp(-\beta E')dE'\\
\langle E^2\rangle &= \frac{d^2Z/d\beta^2}{Z}\ .
\end{aligned}$$

The remaining definite integral Z has dimension $(\text{energy})^{3/2}$. Since the limits of the integral are not finite, the only available quantity with which a dimensionful scale may be set is β, which has dimension 1/energy. Therefore the integral must be equal to $\beta^{-3/2}$ multiplied by some constant C:

$$\begin{aligned}
\langle E^2 \rangle &= \frac{\left(d^2/d\beta^2\right)\left(C\beta^{-3/2}\right)}{C\beta^{-3/2}} \\
&= \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(C\beta^{-7/2}\right)}{C\beta^{-3/2}} \\
&= \frac{15}{4}\beta^{-2} \\
E_{\rm rms} &= \sqrt{\frac{15}{4}}\beta^{-1} \\
&= \sqrt{\frac{15}{4}}kT \, .
\end{aligned}$$

4. RHK problem 23.17
Solution: In Physics H7B, all problems involving numbers should be solved completely in terms of algebraic symbols before any numbers are plugged in (otherwise it is much more difficult to give part credit). Let
T = temperature of interstellar space = 2.7 °K
M = molar mass of H_2 = 0.0020 kg/mole (RHK Table 23.1)
$N_{\rm A}$ = Avogadro constant
= 6.022×10^{23} molecules/mole
m = mass of H_2 molecule = $M/N_{\rm A}$
$k_{\rm B}$ = Boltzmann constant = 1.38×10^{-23} J/K
Then from RHK Eq. 23.20,

$$\begin{aligned}
\frac{1}{2}m\langle v^2 \rangle &= \frac{3}{2}k_{\rm B}T \\
\langle v^2 \rangle &= \frac{3k_{\rm B}T}{m} \\
v_{\rm rms} &\equiv \sqrt{\langle v^2 \rangle} \\
&= \sqrt{\frac{3k_{\rm B}T}{m}} \\
&= \sqrt{\frac{3k_{\rm B}N_{\rm A}T}{M}} \\
&= 183.5 \text{ m/sec} \, .
\end{aligned}$$

5. RHK problem 23.33
Solution: Let
$R_{\rm e}$ = radius of earth = 6.37×10^6 m
$R_{\rm m}$ = radius of moon = 1.74×10^6 m
$GM_{\rm e}/R_{\rm e}^2 = g$ = gravitational acceleration at earth's surface = 9.81 m/sec^2
$g_{\rm m}$ = gravitational acceleration at moon's surface = $0.16g$
$v_{\rm esc}$ = escape velocity at earth's surface
m = generic molecular mass
$N_{\rm A}$ = Avogadro constant
= 6.022×10^{23} molecules/mole
$M_{\rm Hyd}$ = molar mass of H_2 = 0.0020 kg/mole (RHK Table 23.1)
$M_{\rm Oxy}$ = molar mass of O_2 = 0.0320 kg/mole (RHK Table 23.1)
$k_{\rm B}$ = Boltzmann constant = 1.38×10^{-23} J/K
$T_{\rm esc}^{\rm Hyd}$(earth) = temperature (°K) at which rms H_2 velocity is equal to escape velocity at earth's surface
$T_{\rm esc}^{\rm Oxy}$(earth) = temperature ((°K) at which rms O_2 velocity is equal to escape velocity at earth's surface
$T_{\rm esc}^{\rm Hyd}$(m) = temperature ((°K) at which rms H_2 velocity is equal to escape velocity at moon's surface
$T_{\rm esc}^{\rm Oxy}$(m) = temperature ((°K) at which rms O_2 velocity is equal to escape velocity at moon's surface
Then

$$\begin{aligned}
\frac{1}{2}mv_{\rm esc}^2 &= \frac{GM_{\rm e}m}{R_{\rm e}} \\
v_{\rm esc}^2 &= \frac{2GM_{\rm e}}{R_{\rm e}} \\
&= 2gR_{\rm e} \\
\frac{1}{2}mv_{rms}^2 &= \frac{3}{2}k_{\rm B}T_{\rm esc} \\
v_{\rm rms} &= v_{\rm esc} \quad \text{(stated by problem)} \\
\frac{1}{2}mv_{esc}^2 &= \frac{3}{2}k_{\rm B}T_{\rm esc} \\
\frac{1}{2}m\,2gR_{\rm e} &= \frac{3}{2}k_{\rm B}T_{\rm esc} \\
\frac{2mgR_{\rm e}}{3k_{\rm B}} &= T_{\rm esc} \, .
\end{aligned}$$

We use this general result to evaluate each of the four cases posed:

$$
\begin{aligned}
m &= \frac{M_{Hyd}}{N_{\mathrm{A}}} \\
T_{\mathrm{esc}}^{\mathrm{Hyd}}(\mathrm{earth}) &= \frac{2M_{\mathrm{Hyd}} g R_{\mathrm{e}}}{3k_{\mathrm{B}} N_{\mathrm{A}}} \\
&= 1.003 \times 10^4 \ {}^\circ\mathrm{K} \\
T_{\mathrm{esc}}^{\mathrm{Oxy}}(\mathrm{earth}) &= \frac{2M_{\mathrm{Oxy}} g R_{\mathrm{e}}}{3k_{\mathrm{B}} N_{\mathrm{A}}} \\
&= 1.604 \times 10^5 \ {}^\circ\mathrm{K} \\
T_{\mathrm{esc}}^{\mathrm{Hyd}}(\mathrm{moon}) &= \frac{2M_{\mathrm{Hyd}} g_{\mathrm{m}} R_{\mathrm{m}}}{3k_{\mathrm{B}} N_{\mathrm{A}}} \\
&= 438 \ {}^\circ\mathrm{K} \\
T_{\mathrm{esc}}^{\mathrm{Oxy}}(\mathrm{moon}) &= \frac{2M_{\mathrm{Oxy}} g_{\mathrm{m}} R_{\mathrm{m}}}{3k_{\mathrm{B}} N_{\mathrm{A}}} \\
&= 7011 \ {}^\circ\mathrm{K} \ .
\end{aligned}
$$

At an altitude in the Earth's atmosphere where the temperature is ≈ 1000 K, the preceding results imply that the rms velocity would be only a factor $\sqrt{T_{\mathrm{esc}}/T} \approx \sqrt{10}$ below the escape velocity; because of leakage out of the tail of the velocity distribution, little hydrogen would be expected to remain. For oxygen, the rms velocity would be a factor $\approx \sqrt{160}$ below the escape velocity, allowing that molecule to survive as an atmospheric component.

6. RHK problem 23.37
Solution: For path 1, the work W done on the gas is

$$
\begin{aligned}
W &= -\int_{\mathrm{path}} p\,dV \quad (\mathrm{RHK}\ 23.24) \\
&= -\int_2^8 p\,dV - \int_8^8 p\,dV - \int_8^2 p\,dV \\
&= -(12.5\ \mathrm{kPa})(6\ \mathrm{m}^3) - 0 + (20\ \mathrm{kPa})(6\ \mathrm{m}^3) \\
&= 45\ \mathrm{kJ}\ ,
\end{aligned}
$$

where we have evaluated each straight-line segment by reading $\langle p\rangle$ off the graph, multiplying it by the difference in V to compute the area under the line. Similarly, for path 2,

$$
\begin{aligned}
W &= -\int_{\mathrm{path}} p\,dV \\
&= -\int_2^8 p\,dV - \int_8^2 p\,dV - \int_2^2 p\,dV \\
&= -(12.5\ \mathrm{kPa})(6\ \mathrm{m}^3) + (5\ \mathrm{kPa})(6\ \mathrm{m}^3) - 0 \\
&= -45\ \mathrm{kJ}\ .
\end{aligned}
$$

7. RHK problem 25.16
Solution: Let
m_v = (unknown) mass of vaporized material (ice), in kg
m_f = mass of fused material (ice) = 0.15 kg
L_v = latent heat of vaporization of water = 2256×10^3 J/kg
L_f = latent heat of fusion of water = 333×10^3 J/kg
c = specific heat capacity of water = 4190 J/kg·C°
T_v = temperature of steam = 100 °C
T_f = temperature of ice = 0 °C
T = final temperature of steam-ice mixture = 50 °C
The fact that the container is thermally insulated means that the total heat Q transferred out of the steam molecules is transferred into the ice molecules:

$$
\begin{aligned}
Q(\text{lost by steam}) &= Q(\text{gained by ice}) \\
m_v(L_v + c(T_v - T)) &= m_f(L_f + c(T - T_f)) \\
m_v &= m_f \frac{L_f + c(T - T_f)}{L_v + c(T_v - T)} \\
&= 0.033\ \mathrm{kg}\ .
\end{aligned}
$$

8. RHK problem 25.21
Solution: Let
Q = (unknown) heat transferred into sample
T_i = initial temperature = 6.6 K
T_f = final temperature = 15 K
m = mass of Al = 0.0012 kg
C = heat capacity per mole of Al
η = coefficient of T^3 in expression for C = 3.16×10^{-5} J/mole·K^4
M_{Al} = molar mass of Al = 0.0270 kg/mole (RHK Appendix D)
c = heat capacity per kg of Al = C/M_{Al}

With these definitions,

$$\begin{aligned}
Q &= m\int_{T_i}^{T_f} c(T)\,dT \quad \text{(RHK Eq. 25.4)}\\
&= \frac{m}{M_{\text{Al}}}\int_{T_i}^{T_f} C(T)\,dT\\
C(T) &= \eta T^3\\
Q &= \frac{m}{M_{\text{Al}}}\eta\int_{T_i}^{T_f} T^3 dT\\
&= \frac{m\eta}{4M_{\text{Al}}}\left(T_f^4 - T_i^4\right)\\
&= 0.0171 \text{ J}\ .
\end{aligned}$$

University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

PROBLEM SET 3

1. RHK problem 25.27

2. RHK problem 25.34

3. RHK problem 25.37

4. RHK problem 25.43

5. RHK problem 26.16

6. RHK problem 26.19

7. RHK problem 26.23

8. RHK problem 26.27

University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 3

1. RHK problem 25.27
Solution: In Physics H7B, all problems involving numbers should be solved completely in terms of algebraic symbols before any numbers are plugged in (otherwise it is much more difficult to give part credit). Let
Q = (unknown) heat added to gas (J)
n = no. of moles of gas = 4.34
$C_{\rm p}$ = molar specific heat of gas at constant pressure (J/mole·K)
ΔT = change in temperature of gas = 62.4 K
R = universal gas constant = 8.314 J/mole·K
$E_{\rm int}$ = internal energy of gas (J)
M = molecular weight of gas (kg/mole)
$\langle v^2\rangle$ = mean square velocity of gas molecules (m^2/sec^2)
(a)

$$\begin{aligned} Q &= nC_{\rm p}\Delta T \quad \text{(RHK Eq. 25.17)} \\ C_{\rm p} &= \frac{7}{2}R \quad \text{(RHK Eq. 25.21)} \\ Q &= \frac{7}{2}nR\Delta T \\ &= 7880 \text{ J} . \end{aligned}$$

(b)

$$\begin{aligned} E_{\rm int} &= \frac{5}{2}nRT \quad \text{(RHK Eq. 23.36)} \\ \Delta E_{\rm int} &= \frac{5}{2}nR\Delta T \\ &= 5629 \text{ J} . \end{aligned}$$

(c)

$$\begin{aligned} n\Big(\frac{1}{2}M\langle v^2\rangle\Big) &= \frac{3}{2}nRT \quad \text{(RHK Eq. 23.31)} \\ n\Big(\frac{1}{2}M\Delta\langle v^2\rangle\Big) &= \frac{3}{2}nR\Delta T \\ &= 3377 \text{ J} . \end{aligned}$$

2. RHK problem 25.34
Solution: Plunging blindly ahead, we could start by assuming that "*quickly*" means quickly enough so that a negligible amount of heat is transferred between the gas and the ice water, but slowly enough to allow the pressure nevertheless to be defined (as it is in RHK Fig. 25b); and that "*slowly*" means slowly enough that the gas and the ice water always have the same temperature. If so, the "quick" compression of the gas would occur along an adiabat, while the "slow" expansion would occur along an isotherm. Then

$$W = -\int_{V_1}^{V_2} p\,dV - \int_{V_2}^{V_2} p\,dV - \int_{V_2}^{V_1} p\,dV \ .$$

Further assuming that the gas is ideal,

$$\begin{aligned} pV^\gamma &= p_1V_1^\gamma \text{ (adiabat)} \\ pV &= p_1V_1 \text{ (isotherm)} \\ W &= -\int_{V_1}^{V_2} \frac{p_1V_1^\gamma}{V^\gamma}\,dV - 0 - \int_{V_2}^{V_1} \frac{p_1V_1}{V}\,dV \\ &= -p_1V_1^\gamma\frac{-1}{\gamma-1}\Big(\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}}\Big) - p_1V_1\ln\frac{V_1}{V_2} \\ &= \frac{p_1V_1}{\gamma-1}\Big(\frac{V_1^{\gamma-1}}{V_2^{\gamma-1}} - 1\Big) - p_1V_1\ln\frac{V_1}{V_2} \ . \end{aligned}$$

The above is correct, given the assumptions, but it does not solve the problem; we are supplied neither the initial volume nor the number of moles of gas. Instead we are told that m = 0.122 kg of ice in the surrounding ice water are melted in one cycle. The heat $-Q = L_fm$ required to melt this ice, where L_f = 333 kJ/kg is the latent heat of fusion of water, must be transferred *from* the gas (we call it $-Q$ because $+Q$ is defined to be the heat transferred *to* the gas). Around one cycle, the final temperature of gas is the same as the initial; its internal energy, which depends only on the temperature, can undergo no net change. Therefore, around the cycle, the work W done *on* the gas is given without any assumptions by

$$\begin{aligned} \Delta E_{\rm int} &= 0 \\ \Delta E_{\rm int} &= Q + W \quad (1^{\rm st} \text{ Law}) \\ W &= -Q \\ &= L_fm \\ &= 40626 \text{ J} . \end{aligned}$$

3. RHK problem 25.37
Solution:
(a)

$$
\begin{aligned}
W &= -\int_{V_1}^{V_2} p\,dV \\
\frac{V}{V_1} &= \frac{p}{p_1} \text{ (from problem)} \\
W &= -\int_{V_1}^{V_2} \frac{p_1}{V_1} V\,dV \\
&= -\frac{1}{2}\frac{p_1}{V_1}\big(V_2^2 - V_1^2\big) \\
&= -\frac{1}{2} p_1 V_1 \Big(\big(\frac{V_2}{V_1}\big)^2 - 1\Big) \\
p_1 V_1 &= nRT_1 \\
W &= -\frac{1}{2} nRT_1 \Big(\big(\frac{V_2}{V_1}\big)^2 - 1\Big) \\
V_2 &= 2V_1 \\
W &= -\frac{1}{2} nRT_1 (4-1) \\
W &= -\frac{3}{2} nRT_1 \ .
\end{aligned}
$$

(b)

$$
\begin{aligned}
E_{\text{int}} &= \frac{3}{2} nRT \text{ (ideal monatomic gas)} \\
\Delta E_{\text{int}} &= \frac{3}{2} nR(T_2 - T_1) \\
nRT_2 &= p_2 V_2 \\
&= (2p_1)(2V_1) \\
&= 4p_1 V_1 \\
&= 4nRT_1 \\
T_2 - T_1 &= 3T_1 \\
\Delta E_{\text{int}} &= \frac{3}{2} nR(3T_1) \\
&= \frac{9}{2} nRT_1 \ .
\end{aligned}
$$

(c)

$$
\begin{aligned}
\Delta E_{\text{int}} &= Q + W \quad (1^{\text{st}} \text{ Law}) \\
Q &= \Delta E_{\text{int}} - W \\
&= \frac{9}{2} nRT_1 - \big(-\frac{3}{2} nRT_1\big) \\
&= 6nRT_1 \ .
\end{aligned}
$$

(d)

$$
\begin{aligned}
C &\equiv \frac{Q}{n\Delta T} \quad \text{(RHK Eq. 25.8)} \\
&= \frac{6nRT_1}{n(4T_1 - T_1)} \\
&= 2R \ .
\end{aligned}
$$

4. RHK problem 25.43
Solution: This problem is "overconstrained": that is, too many pieces of information are provided. For example, T_C need not have been supplied; it is uniquely determined by the facts that process BC is adiabatic; that $V_B = V_A$; that $p_C = p_A$; and that the gas is ideal monatomic. This is illustrated by the following calculation (not required as part of the solution):

$$
\begin{aligned}
p_B V_B^\gamma &= p_C V_C^\gamma \quad \text{(adiabatic)} \\
p_B V_A^\gamma &= p_A V_C^\gamma \\
pV &= nRT \\
p_B &= p_A \frac{T_B}{T_A} \\
V_A^\gamma &= \Big(\frac{nRT_A}{p_A}\Big)^\gamma \\
V_C^\gamma &= \Big(\frac{nRT_C}{p_A}\Big)^\gamma \\
p_A \frac{T_B}{T_A}\Big(\frac{nRT_A}{p_A}\Big)^\gamma &= p_A \Big(\frac{nRT_C}{p_A}\Big)^\gamma \\
\frac{T_B}{T_A} T_A^\gamma &= T_C^\gamma \\
T_C &= T_A \Big(\frac{T_B}{T_A}\Big)^{1/\gamma} \\
&= 454.71497 \text{ K} \ .
\end{aligned}
$$

If we needed to get exact answers, we would need to plug in this exact value of T_C, rather than the approximate value of 455 K supplied in the problem. To proceed further, we choose not to use one known piece of information – not to use (as we did above) the specific relationship between p, V, T, and γ for an adiabatic transition. Because this choice is subjective and not unique, when our solutions are expressed in algebraic symbols we expect them also not to be unique. However, as long as the exact value of T_C is plugged in, we expect any valid solution to yield the same numerical results.
Let

T_A = temperature at point A = 300 K
T_B = temperature at point B = 600 K
T_C = temperature at point C = 454.71497 K (see above discussion)
n = no. of moles of monatomic ideal gas = 1.00
R = universal gas constant = 8.314 J/mole·K
$p_A = 1.013 \times 10^5$ Pa.
Then
(a)
Process AB:

$$\begin{aligned}\Delta E_{\rm int} &= \frac{3}{2}nR(T_B - T_A)\\ &= 3741 \text{ J} .\\ W &= -\int_{V_A}^{V_B} p\,dV\\ &= -\int_{V_A}^{V_A} p\,dV\\ &= 0 .\\ Q &= \Delta E_{\rm int} - W\\ &= \frac{3}{2}nR(T_B - T_A) \ \ (= C_{\rm V}\Delta T)\\ &= 3741 \text{ J} .\end{aligned}$$

Process BC:

$$\begin{aligned}\Delta E_{\rm int} &= \frac{3}{2}nR(T_C - T_B)\\ &= -\frac{3}{2}nR(T_B - T_C)\\ &= -1812 \text{ J} .\\ Q &= 0 \ \ \text{(adiabatic)} .\\ W &= \Delta E_{\rm int} - Q\\ &= -\frac{3}{2}nR(T_B - T_C)\\ &= -1812 \text{ J} .\end{aligned}$$

Process CA:

$$\begin{aligned}\Delta E_{\rm int} &= \frac{3}{2}nR(T_A - T_C)\\ &= -\frac{3}{2}nR(T_C - T_A)\\ &= -1929.45 \text{ J} .\\ W &= -\int_{V_C}^{V_A} p\,dV\\ &= -pV_A + pV_C \ \ (p = p_A = p_C)\\ pV &= nRT\\ W &= -nRT_A + nRT_C\\ &= nR(T_C - T_A)\\ &= 1286.30 \text{ J} .\\ Q &= \Delta E_{\rm int} - W\\ &= -\frac{3}{2}nR(T_C - T_A) - nR(T_C - T_A)\\ &= -\frac{5}{2}nR(T_C - T_A) \ \ (= C_{\rm p}\Delta T)\\ &= -3215.75 \text{ J} .\end{aligned}$$

Complete cycle:

$$\begin{aligned}\Delta E_{\rm int} &\equiv 0 \text{ (state variable)} .\\ W &= -\frac{3}{2}nR(T_B - T_C) + nR(T_C - T_A)\\ &= -nRT_A - \frac{3}{2}nRT_B + \frac{5}{2}nRT_C)\\ &= -525.55 \text{ J} .\\ Q &= \frac{3}{2}nR(T_B - T_A) - \frac{5}{2}nR(T_C - T_A)\\ &= nRT_A + \frac{3}{2}nRT_B - \frac{5}{2}nRT_C\\ &= 525.55 \text{ J} .\end{aligned}$$

(b)

$$\frac{p_B}{p_A} = \frac{T_B}{T_A} \quad (V \text{ fixed})$$

$$p_B = p_A \frac{T_B}{T_A}$$
$$= 2.026 \times 10^5 \text{ Pa} .$$
$$p_C = p_A$$
$$= 1.013 \times 10^5 \text{ Pa} .$$
$$V_A = \frac{nRT_A}{p_A}$$
$$V_B = V_A$$
$$= \frac{nRT_A}{p_A}$$
$$= 0.0246 \text{ m}^3 .$$
$$\frac{V_C}{V_A} = \frac{T_C}{T_A} \quad (p \text{ fixed})$$
$$V_C = V_A \frac{T_C}{T_A}$$
$$= 0.0373 \text{ m}^3 .$$

5. RHK problem 26.16
Solution: Consider a Carnot engine operating in reverse (as a refrigerator) between a cold reservoir at temperature $T_L = 276$ K and a hot reservoir at $T_H = 308$ K. Like all Carnot engines it is characterized by the equality

$$\frac{|Q_H|}{|Q_L|} = \frac{T_H}{T_L} \quad \text{(RHK Eq. 26.9)} .$$

For operation as a refrigerator, the heat Q_H added to the gas by the hot reservoir is negative. Conversely, Q_L is positive. The net heat $Q = Q_H + Q_L$ added to the gas over one complete cycle is negative. Since the internal energy E_{int} is a state function, over a complete cycle it must be conserved. Therefore, in one complete cycle, $-Q$ must be balanced by the mechanical work W done on the gas. A figure of merit $\mathcal{F}$ for a heat pump, the ratio of $-Q_H$ to W, is

$$\mathcal{F} = \frac{-Q_H}{W}$$
$$= \frac{-Q_H}{-Q_H - Q_L}$$
$$= \frac{T_H}{T_H - T_L}$$
$$= 9.625 .$$

The inventor claims to have achieved a figure of merit equal to

$$\mathcal{F} = \frac{-Q_H}{W}$$
$$= \frac{20 \text{ kW}}{1.9 \text{ kW}}$$
$$= 10.526 .$$

This is slightly larger than the Carnot figure of merit. Any reversible heat pump will have the same figure of merit as a Carnot engine. The only other possibility would be that the inventor's heat pump is irreversible. For example, friction in the refrigerator could convert a certain additional amount W' of work directly to heat in each cycle. In the best case, all of the heat from W' would be dumped into the hot rather than the cold reservoir. Then W' would be added both to the numerator and to the denominator of $\mathcal{F}$, reducing its physical value further below the value reported by the inventor. Therefore we are forced to reject the inventor's claim. (Nevertheless, many patents indeed have been granted for processes that violate elementary physical laws.)

6. RHK problem 26.19
Solution: Again a Carnot engine is operated in reverse between a hot reservoir at T_H and a cold reservoir at T_L. Again Q_H is negative and Q_L is positive, and, since the refrigerator is reversible,

$$\frac{|Q_H|}{|Q_L|} = \frac{T_H}{T_L} .$$

Again ΔE_{int} must be zero over a complete cycle, so that $W = -Q$ over the cycle.

(a)

$$W = -Q$$
$$= -Q_H - Q_L$$
$$= Q_L \Big(\frac{-Q_H}{Q_L} - 1 \Big)$$
$$= Q_L \Big(\frac{T_H}{T_L} - 1 \Big)$$
$$= Q_L \frac{T_H - T_L}{T_L} .$$

(b)

$$\begin{aligned}\mathcal{K} &\equiv \frac{Q_L}{W} \\ &= \frac{Q_L}{-Q_H - Q_L} \\ &= \frac{1}{\frac{-Q_H}{Q_L} - 1} \\ &= \frac{1}{\frac{T_H}{T_L} - 1} \\ &= \frac{T_L}{T_H - T_L} \; .\end{aligned}$$

(c)

$$\begin{aligned}T_L &= 260 \text{ K} \\ T_H &= 298 \text{ K} \\ \mathcal{K} &= \frac{T_L}{T_H - T_L} \\ &= 6.842 \; .\end{aligned}$$

7. RHK problem 26.23
Solution: Let
Q_1 = heat transferred to gas in engine from (hot) reservoir 1 (> 0)
Q_2 = heat transferred to gas in engine from (cold) reservoir 1 (< 0)
Q_1 = heat transferred to gas in refrigerator from (hot) reservoir 3 (< 0)
Q_1 = heat transferred to gas in engine from (cold) reservoir 4 (> 0)
W_E = mechanical work done on gas in engine (< 0)
W_R = mechanical work done on gas in refrigerator (> 0)
Then

$$\begin{aligned}-W_E &= Q_1 + Q_2 \\ &= Q_1\left(1 - \frac{-Q_2}{Q_1}\right) \\ &= Q_1\left(1 - \frac{T_2}{T_1}\right) \\ W_R &= -Q_3 - Q_4 \\ &= -Q_3\left(1 - \frac{Q_4}{-Q_3}\right) \\ &= -Q_3\left(1 - \frac{T_4}{T_3}\right) \\ 1 &= \frac{-W_R}{W_E} \\ &= \frac{-Q_3\left(1 - \frac{T_4}{T_3}\right)}{Q_1\left(1 - \frac{T_2}{T_1}\right)} \\ \frac{-Q_3}{Q_1} &= \frac{1 - \frac{T_2}{T_1}}{1 - \frac{T_4}{T_3}} \\ \frac{|Q_3|}{|Q_1|} &= \frac{1 - \frac{T_2}{T_1}}{1 - \frac{T_4}{T_3}} \; .\end{aligned}$$

8. RHK problem 26.27
Solution: Let
W_{ab} = work done on gas in stroke *ab*, etc.
W = work done *on* gas in cycle
$W_{\text{by eng}}$ = work done *by* engine in cycle
p_0 = smaller pressure = 1.01×10^5 Pa
p_1 = larger pressure = $2p_0$
V_0 = smaller volume = 0.0225 m^3
V_1 = larger volume = $2V_0$
Q_{abc} = heat added to gas during pair of expansion strokes
e = efficiency of engine
e_{Carnot} = efficiency of Carnot engine operating between two temperatures with ratio p_1V_1/p_0V_0
Then
(a)

$$\begin{aligned}W &= W_{ab} + W_{bc} + W_{cd} + W_{da} \\ &= 0 - p_1(V_1 - V_0) + 0 + p_0(V_1 - V_0) \\ &= -(p_1 - p_0)(V_1 - V_0) \\ W_{\text{by eng}} &= -W \\ &= (p_1 - p_0)(V_1 - V_0) \\ &= p_0V_0 \\ &= 22725 \text{ J} \; .\end{aligned}$$

(b)

$$
\begin{aligned}
Q_{abc} &= Q_{ab} + Q_{bc} \\
&= E_{\text{int}}(c) - E_{\text{int}}(a) - W_{ab} - W_{bc} \\
&= \frac{3}{2} nR(T_c - T_a) - 0 + p_1(V_1 - V_0) \\
&= \frac{3}{2} p_1 V_1 - \frac{3}{2} p_0 V_0 + p_1 V_1 - p_1 V_0 \\
&= \frac{5}{2} p_1 V_1 - p_1 V_0 - \frac{3}{2} p_0 V_0 \\
&= \frac{13}{2} p_0 V_0 \\
&= 147713 \text{ J} .
\end{aligned}
$$

(c)

$$
\begin{aligned}
e &\equiv \frac{W_{\text{by eng}}}{Q_{abc}} \\
&= \frac{(p_1 - p_0)(V_1 - V_0)}{\frac{5}{2} p_1 V_1 - p_1 V_0 - \frac{3}{2} p_0 V_0} \\
&= \frac{2}{13} \\
&= 0.1538 .
\end{aligned}
$$

(d)

$$
\begin{aligned}
e_{\text{Carnot}} &= \frac{T_c - T_a}{T_c} \\
&= \frac{p_1 V_1 - p_0 V_0}{p_1 V_1} \\
&= \frac{3}{4} .
\end{aligned}
$$

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PROBLEM SET 4

1. RHK problem 26.36

2. RHK problem 26.40

3. RHK problem 26.43

4. Purcell problem 1.5

5. Purcell problem 1.8

6. Purcell problem 1.14

7. Purcell problem 1.26

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SOLUTION TO PROBLEM SET 4

1. RHK problem 26.36
Solution: Let
n = no. of moles of ideal monatomic gas = 1.00
R = universal gas constant = 8.314 J/mole·K
Then
(a)

$$\begin{aligned} W_{abc} &= W_{ab} + W_{bc} \\ &= -\int_a^b p\,dV + 0 \\ &= -p_0(4V_0 - V_0) \\ &= -3p_0V_0\ . \end{aligned}$$

(b)

$$\begin{aligned} \Delta E_{\text{int}}(b \to c) &= \frac{3}{2}nR(T_c - T_b) \\ &= \frac{3}{2}(p_cV_c - p_bV_b) \\ &= \frac{3}{2}(8p_0V_0 - 4p_0V_0) \\ &= 6p_0V_0\ . \\ \Delta S_{bc} &= \int_b^c \frac{\delta Q}{T} \\ &= \int_b^c \frac{dE_{\text{int}}}{T} - W_{bc} \\ &= \int_b^c \frac{dE_{\text{int}}}{T} - 0 \\ E_{\text{int}} &= \frac{3}{2}nRT \\ \Delta S_{bc} &= \int_b^c \frac{3}{2}nR\frac{dT}{T} \\ &= \frac{3}{2}nR\ln\left(\frac{T_c}{T_b}\right) \\ &= \frac{3}{2}nR\ln 2 \\ &= 8.644\ \text{J/K}\ . \end{aligned}$$

Note that the last result does not depend on p_0 or V_0, even though the problem asks us to express it in terms of p_0 and V_0.

(c)

$$\oint dE_{\text{int}} \equiv 0 \quad \text{(state function)}\ .$$

$$\oint dS \equiv 0 \quad \text{(state function)}\ .$$

2. RHK problem 26.40
Solution: Let
n = no. of moles of ideal diatomic gas = 1.00
R = universal gas constant = 8.314 J/mole·K
Then
(a)

$$\begin{aligned} pV &= \text{constant} \quad \text{(isotherm)} \\ p_2 &= p_1\frac{V_1}{V_2} \\ &= \frac{p_1}{3}\ . \\ pV^\gamma &= \text{constant} \quad \text{(adiabat)} \\ p_3 &= p_1\left(\frac{V_1}{V_3}\right)^\gamma \\ \gamma &= \frac{7}{5} \quad \text{(diatomic)} \\ p_3 &= p_1\left(\frac{1}{3}\right)^{7/5} \\ &= 0.215\,p_1\ . \\ TV^{\gamma-1} &= \text{constant} \quad \text{(adiabat)} \\ T_3 &= T_1\left(\frac{V_1}{V_3}\right)^{\gamma-1} \\ &= T_1\left(\frac{1}{3}\right)^{2/5} \\ &= 0.644\,T_1\ . \end{aligned}$$

(b)

$$E_{\text{int}} = \frac{5}{2}nRT \quad \text{(diatomic)}$$

$$\begin{aligned}
\Delta E_{\text{int}}(1 \to 2) &= \frac{5}{2}nR(T_2 - T_1) \\
&= 0\ .
\end{aligned}$$

$$\begin{aligned}
\Delta E_{\text{int}}(2 \to 3) &= \frac{5}{2}nR(T_3 - T_2) \\
&= \frac{5}{2}nRT_1\left(\left(\frac{1}{3}\right)^{2/5} - 1\right) \\
&= -\frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) \\
&= -0.889\, p_1V_1\ .
\end{aligned}$$

$$\oint dE_{\text{int}} \equiv 0 \quad \text{(state function)}$$

$$\begin{aligned}
\Delta E_{\text{int}}(3 \to 1) &= -\Delta E_{\text{int}}(1 \to 2) \\
&\quad - \Delta E_{\text{int}}(2 \to 3) \\
&= -0 + \frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) \\
&= 0.889\, p_1V_1\ .
\end{aligned}$$

$$\begin{aligned}
W_{12} &= -\int_1^2 p\, dV \\
&= -\int_1^2 nRT_1 \frac{dV}{V} \\
&= -nRT_1 \ln\frac{V_2}{V_1} \\
&= -p_1V_1 \ln 3 \\
&= -1.099\, p_1V_1\ .
\end{aligned}$$

$$\begin{aligned}
W_{23} &= -\int_2^3 p\, dV \\
&= 0\ .
\end{aligned}$$

$$\begin{aligned}
W_{31} &= \Delta E_{\text{int}}(3 \to 1) - Q_{31} \\
&= \frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) - 0 \\
&= 0.889\, p_1V_1\ .
\end{aligned}$$

$$\begin{aligned}
Q_{12} &= \Delta E_{\text{int}}(1 \to 2) - W_{12} \\
&= 0 + p_1V_1 \ln 3 \\
&= 1.099\, p_1V_1\ .
\end{aligned}$$

$$\begin{aligned}
Q_{23} &= \Delta E_{\text{int}}(2 \to 3) - W_{23} \\
&= -\frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) - 0 \\
&= -0.889\, p_1V_1\ .
\end{aligned}$$

$$Q_{31} \equiv 0 \quad \text{(adiabat)}\ .$$

$$\Delta S_{12} = \int_1^2 \frac{\delta Q}{T}$$

$$T = T_1 \quad \text{(isotherm)}$$

$$\begin{aligned}
\Delta S_{12} &= \frac{Q_{12}}{T_1} \\
&= \frac{p_1V_1}{T_1} \ln 3 \\
&= nR \ln 3 \\
&= 9.134\ \text{J/K}\ .
\end{aligned}$$

$$\Delta S_{31} \equiv 0 \quad \text{(adiabat)}\ .$$

$$\oint dS \equiv 0 \quad \text{(state function)}$$

$$\begin{aligned}
\Delta S_{23} &= -\Delta S_{12} - \Delta S_{31} \\
&= -nR \ln 3 - 0 \\
&= -9.134\ \text{J/K}\ .
\end{aligned}$$

3. RHK problem 26.43

Solution: In Physics H7B, all problems involving numbers should be solved completely in terms of algebraic symbols before any numbers are plugged in (otherwise it is much more difficult to give part credit). Let

m_1 = initial amount of water = 1.780 kg
(initial amount of ice is 0.262 kg)
m_2 = final amount of water = $(1.780 + 0.262)/2$ = 1.021 kg
L_f = latent heat of fusion of water = 333000 J/kg
T_0 = temperature of melting ice = 273 K
Then

(a)

$$
\begin{aligned}
Q(m_1 \to m_2) &= L_f(m_2 - m_1) \\
\Delta S(m_1 \to m_2) &= \int_1^2 \frac{\delta Q}{T} \\
T &\equiv T_0 \\
\Delta S(m_1 \to m_2) &= \frac{Q(m_1 \to m_2)}{T_0} \\
&= \frac{L_f(m_2 - m_1)}{T_0} \\
&= -\frac{L_f(m_1 - m_2)}{T_0} \\
&= -925.8 \text{ J/K} .
\end{aligned}
$$

(b)

$$
\begin{aligned}
\oint dS &\equiv 0 \ \ \text{(state function)} \\
\Delta S(m_2 \to m_1) &= -\Delta S(m_1 \to m_2) \\
&= \frac{L_f(m_1 - m_2)}{T_0} \\
&= 925.8 \text{ J/K} .
\end{aligned}
$$

(c) Here the change of entropy of the environment in this cycle is calculated assuming that the heat to melt the ice is supplied at a temperature $T_{>0}$ which is greater than T_0, for example by a Bunsen burner. Nevertheless, using the fact that the entropy of the environment is a state variable, we calculate its change by making use of a hypothetical reversible process, $\Delta S = Q/T$:

$$
\begin{aligned}
\oint dS_{\text{icewater}} &\equiv 0 \\
\oint dS_{\text{environ}} &= -\frac{Q(m_1 \to m_2)}{T_0} - \frac{Q(m_2 \to m_1)}{T_{>0}} \\
&= -\Big(-\frac{L_f(m_1 - m_2)}{T_0}\Big) \\
&\quad - \frac{L_f(m_1 - m_2)}{T_{>0}} \\
&= L_f(m_1 - m_2)\Big(\frac{1}{T_0} - \frac{1}{T_{>0}}\Big) \\
&> 0 \\
\oint dS_{\text{universe}} &> 0 .
\end{aligned}
$$

4. Purcell problem 1.5
Solution: Consider an element of charge $dQ = \lambda R\, d\phi$, where $d\phi$ is an element of azimuth around the semicircle ($0 < \phi < \pi$), and $\lambda = Q/\pi R$ is the charge per unit length (in esu/cm) around the semicircle.

Construct a Cartesian coordinate system with its origin at the center of the semicircle; choose $x = R\cos\phi$ and $y = R\sin\phi$. Then the symmetry about $x = 0$ and $z = 0$ requires the electric field at the origin from the full semicircle not to have any component in the x or z directions. So the net electric field must be parallel to the y axis; it points toward $-y$ if the charge Q is positive.

At the origin, Coulomb's law requires the above mentioned charge element dQ to create an element of electric field $d\mathbf{E}$ which has a magnitude equal to dQ/R^2. However, only a fraction $\sin\phi$ of that field magnitude points in the $-y$ direction. Therefore

$$
\begin{aligned}
dE_y &= -\frac{dQ}{R^2}\sin\phi \\
&= -\frac{\lambda R\, d\phi}{R^2}\sin\phi \\
&= -\frac{\frac{Q}{\pi R} R\, d\phi}{R^2}\sin\phi \\
&= -\frac{Q}{\pi R^2}\sin\phi\, d\phi \\
E_y &= -\frac{Q}{\pi R^2}\int_0^\pi \sin\phi\, d\phi \\
&= -\frac{2}{\pi}\frac{Q}{R^2} \\
\mathbf{E} &= \Big(0, -\frac{2}{\pi}\frac{Q}{R^2}, 0\Big) .
\end{aligned}
$$

5. Purcell problem 1.8
Solution: Let a be the ionic spacing of the one-dimensional crystal. Place the first positive ion at $x = 0$, two negative ions at $x = \pm a$, two more positive ions at $x = \pm 2a$, etc. Consider Purcell's Eq. 1.9:

$$
U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{q_j q_k}{r_{jk}} .
$$

This is a double sum. As the number N of ions approaches ∞, the sum of the terms of the double sum which involve any particular ion will be the same as the sum of the terms involving any other particular ion (see the argument at the bottom of Purcell's page 14). Thus the double sum reduces to a single sum:

$$U = \frac{1}{2} N \sum_{k=2}^{N} \frac{q_1 q_k}{r_{1k}} ,$$

where we have chosen to sum only the terms involving ion 1. Furthermore, since the string of ions is symmetric about $x = 0$, we may consider in the single sum only the ions with $x > 0$, at the expense of multiplying the result by an extra factor of 2:

$$U = \frac{1}{2} 2N \sum_{k=2;\ x>0}^{N} \frac{q_1 q_k}{r_{1k}} .$$

Here we evaluate $r_{1k} = a(k-1)$, and we use the fact that the sign of $q_1 q_k$ is equal to $(-1)^{k-1}$:

$$\begin{aligned} U &= \frac{1}{2} 2N \sum_{k=2}^{N} \frac{q_1 q_k}{a(k-1)} \\ &= \frac{Ne^2}{a} \sum_{k=2}^{N} \frac{(-1)^{k-1}}{(k-1)} \\ &= \frac{Ne^2}{a} \sum_{j=1}^{N-1} \frac{(-1)^j}{j} . \end{aligned}$$

Taking $N \to \infty$ in the limit of the sum,

$$\begin{aligned} U &= \frac{Ne^2}{a} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \\ &= -\frac{Ne^2}{a} \ln(1+1) \\ \frac{U}{N} &= -\frac{e^2}{a} \ln 2 , \end{aligned}$$

where, following the hint, we have evaluated the sum by using the Taylor series expansion

$$\ln(1+b) = \sum_{j=1}^{\infty} \frac{(-b)^{j-1}}{j} .$$

6. Purcell problem 1.14

Solution: This is similar to Purcell's problem 1.5, discussed above, and we will use similar notation. Consider an element of charge $dQ = \lambda b\, d\phi$, where $d\phi$ is an element of azimuth around the circle ($0 < \phi < 2\pi$), and $\lambda = Q/2\pi b$ is the charge per unit length (in esu/cm) around the circle.

Construct a Cartesian coordinate system with its origin at the center of the circle; choose z as the coordinate along the axis normal to plane of the circle. Consider a line drawn from dQ to a point $(0,0,z)$ on this axis. Define ψ to be the angle that this line makes with the plane of the circle. With these definitions, $\tan\psi = z/b$ and $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$; the distance from dQ to $(0,0,z)$ is $b \sec\psi$. Because the configuration is symmetric about $x = 0$ and $y = 0$, on the z axis the electric field must point in the z direction, away from the plane of the ring if its charge Q is positive.

At the point $(0,0,z)$, Coulomb's law requires the above mentioned charge element dQ to create an element of electric field $d\mathbf{E}$ which has a magnitude equal to $dQ/(b\sec\psi)^2$. However, only a fraction $\sin\psi$ of that field magnitude points in the z direction. Therefore

$$\begin{aligned} dE_z &= \frac{dQ}{b^2 \sec^2\psi} \sin\psi \\ &= \frac{\lambda b\, d\phi}{b^2 \sec^2\psi} \sin\psi \\ &= \frac{\frac{Q}{2\pi b} b\, d\phi}{b^2 \sec^2\psi} \sin\psi \\ &= \frac{Q\cos^2\psi \sin\psi}{2\pi b^2} d\phi \\ E_z &= \frac{Q\cos^2\psi \sin\psi}{2\pi b^2} \int_0^{2\pi} d\phi \\ &= \frac{Q\cos^2\psi \sin\psi}{b^2} . \end{aligned}$$

The problem thus reduces to finding the value of

ψ which maximizes the product $\cos^2\psi\sin\psi$:

$$\begin{aligned}
u &\equiv \sin\psi \\
0 &= \frac{d}{du}\big(u(1-u^2)\big) \\
&= 1-3u^2 \\
u &= \sqrt{\frac{1}{3}} \\
\psi &= \arcsin\sqrt{\frac{1}{3}} \\
z &= b\tan\left(\arcsin\sqrt{\frac{1}{3}}\right) \\
&= b\sqrt{\frac{1}{2}}\ .
\end{aligned}$$

7. Purcell problem 1.26

Solution: Place the origin of a Cartesian coordinate system at the center of the semicircle, with both parallel rods lying in the xy plane. Orient the y coordinate so that the rods extend to $y=-\infty$.

At point C, the origin of this coordinate system, any electric field can point only in along the $\pm y$ direction, owing to the symmetry of the problem about $x=0$ and $z=0$. Purcell's figure refers us to two elements of charge. The element at point A has a value $dQ=\lambda b\,d\theta$ and generates an electric field at the origin of magnitude $\lambda b\,d\theta/b^2$. Only a fraction $\sin\theta$ of this field points in the $-y$ direction; thus

$$\begin{aligned}
dE_{A,y} &= -\frac{\lambda b\,d\theta}{b^2}\sin\theta \\
&= -\frac{\lambda}{b}\sin\theta\,d\theta\ .
\end{aligned}$$

The field from the element of charge at point B is slightly more complicated. This charge element has value $dQ=\lambda\,d|y|$, where $d|y|$ is an element of length along the straight rod, and $|y|=b\tan\theta$. Therefore $dQ=\lambda b\,d\tan\theta=\lambda b\sec^2\theta\,d\theta$. This element of charge lies a distance $b\sec\theta$ away from the origin. Again, only a fraction $\sin\theta$ of the field generated by this charge element points in the $+y$ direction. Putting it all together,

$$\begin{aligned}
dE_{B,y} &= +\frac{\lambda b\sec^2\theta\,d\theta}{b^2\sec^2\theta}\sin\theta \\
&= +\frac{\lambda}{b}\sin\theta\,d\theta\ .
\end{aligned}$$

We have seen that $dE_{A,y}$ exactly cancels $dE_{B,y}$ for any choice of θ; therefore $\mathbf{E}_C$ vanishes.

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PROBLEM SET 5

1. Purcell problem 1.16

2. Purcell problem 1.19

3. Purcell problem 1.29

4. Purcell problem 1.33

5. Purcell problem 2.1

6. Purcell problem 2.8

7. Purcell problem 2.19

8. Purcell problem 2.20

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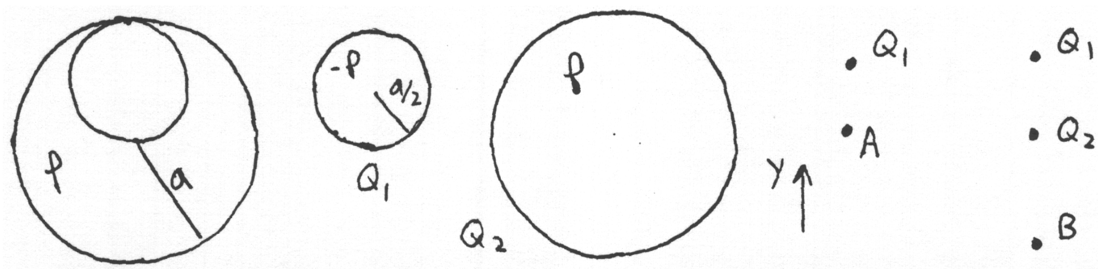
SOLUTION TO PROBLEM SET 5

Solutions by P. Pebler

1 *Purcell 1.16* *A sphere of radius a was filled with positive charge of uniform density ρ. Then a smaller sphere of radius $a/2$ was carved out, as shown, and left empty. What are the direction and magnitude of the electric field at points A and B?*



The key is to consider the given distribution as a superposition of the two distributions at right. The electric field will be the sum of the contributions from these two spheres, which are easy to evaluate. For points outside these spheres, we may treat them as point charges lying at their centers. The charges are

$$Q_1 = \frac{4}{3}\pi\left(\frac{a}{2}\right)^3(-\rho) = -\frac{\pi a^3\rho}{6},$$

$$Q_2 = \frac{4}{3}\pi a^3\rho.$$

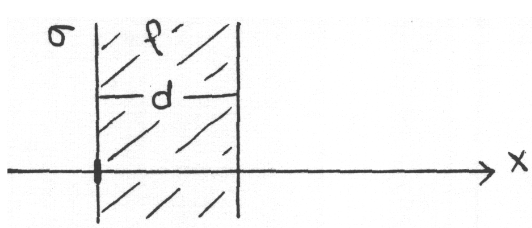
Consider the point A. The contribution from sphere 2 is zero since there is no field at the center of a spherical distribution. The point A lies outside the sphere 1.

$$\mathbf{E}_A = \frac{Q_1}{(a/2)^2}(-\hat{\mathbf{y}}) = \frac{2\pi}{3}a\rho\,\hat{\mathbf{y}}$$

Point B lies outside both spheres.

$$\mathbf{E}_B = \frac{Q_2}{a^2}(-\hat{\mathbf{y}}) + \frac{Q_1}{(3a/2)^2}(-\hat{\mathbf{y}}) = -\frac{4\pi a^3\rho}{3a^2}\,\hat{\mathbf{y}} + \frac{2\pi a^3\rho}{27a^2}\,\hat{\mathbf{y}} = -\frac{34}{27}\pi a\rho\,\hat{\mathbf{y}}$$

2 *Purcell 1.19* *An infinite plane has a uniform surface charge distribution σ on its surface. Adjacent to it is an infinite parallel layer of charge of thickness d and uniform volume charge density ρ. All charges are fixed. Find the electric field everywhere.*



The contribution due to the surface charge has magnitude $2\pi|\sigma|$ and points away from or towards the surface depending on the sign of σ. To deal with the volume charge, we can treat it as a stack of very thin layers of charge and treat these layers as surface charges. We could add up all the contributions from these infinitesimal layers by integrating. However, since the field from an infinite plane of charge does not depend on how far away you are, the contribution from each layer will be the same. So we will get the same answer by assuming the finite volume charge layer to be a surface with surface charge $\rho\, t$ where t is the thickness of the layer in question. For $x < 0$, everything pushes to the left.

$$\mathbf{E} = 2\pi\sigma(-\hat{\mathbf{x}}) + 2\pi\rho\, d(-\hat{\mathbf{x}}) = -2\pi(\sigma + \rho\, d)\,\hat{\mathbf{x}} \qquad x < 0$$

Likewise,

$$\mathbf{E} = 2\pi(\sigma + \rho\, d)\,\hat{\mathbf{x}} \qquad x \geq d.$$

For the region $0 < x < d$, the volume layer is split into two. We can thing of the right side as a single surface with surface charge $\rho(d - x)$ pushing to the left, and the left side as a surface charge ρx pushing to the right.

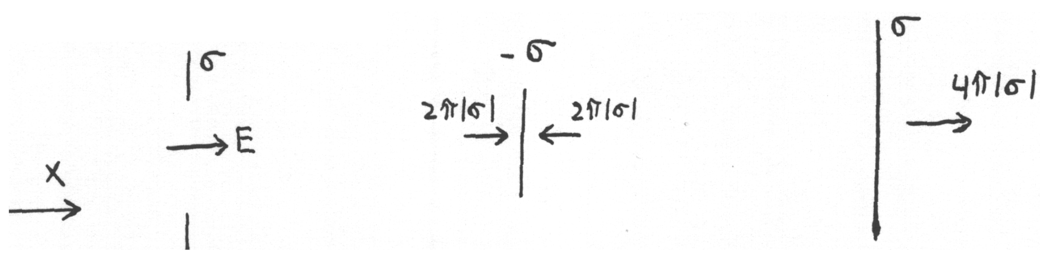
$$\mathbf{E} = 2\pi\rho\,\hat{\mathbf{x}} + 2\pi\rho\, x\,\hat{\mathbf{x}} - 2\pi\rho(d - x)\,\hat{\mathbf{x}} = (2\pi\sigma + 4\pi\rho\, x - 2\pi\rho\, d)\,\hat{\mathbf{x}} \qquad 0 < x < d$$

If we wished to consider the plane $x = 0$, we could say that the surface charge σ contributes nothing.

$$\mathbf{E} = -2\pi\rho\, d\,\hat{\mathbf{x}} \qquad x = 0$$

Notice that there is a discontinuity of $4\pi\sigma$ as we pass through zero. This is always the case for idealized surface charges. There is no discontinuity at $x = d$ however.

3 *Purcell 1.29* *A spherical shell of charge of radius a and surface charge density σ is missing a small, approximately circular, piece of "radius" $b \ll a$. What is the direction and magnitude of the field at the midpoint of the aperture?*



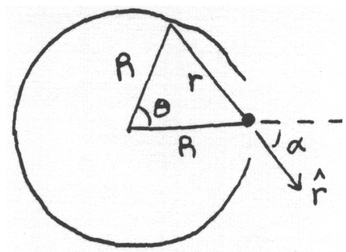
The picture above assumes for simplicity that $\sigma > 0$.

As a zero order approximation, we may consider the missing circle as infinitesimal. The field at left can be viewed as a superposition of the two distributions at right. We temporarily ignore points exactly at the surface. By considering the field to the right or to the left we find

$$\mathbf{E} = 4\pi\sigma\,\hat{\mathbf{x}} + 2\pi(-\sigma)\,\hat{\mathbf{x}} = \mathbf{0} + 2\pi(-\sigma)(-\hat{\mathbf{x}}) = 2\pi\sigma\,\hat{\mathbf{x}},$$

for the field everywhere except at the surface. But for the distribution with the circle missing, there can be no discontinuity when passing through the hole, so the field directly at the surface is also $2\pi\sigma\,\hat{\mathbf{x}}$.

This should be a good approximation when $b \ll a$. But for a finite missing piece, this will not be the exact answer even at the center. To find the contributions of higher order in b/a we can integrate. This is actually not too bad.



From symmetry considerations, we know the field is radial in the center of the aperture.

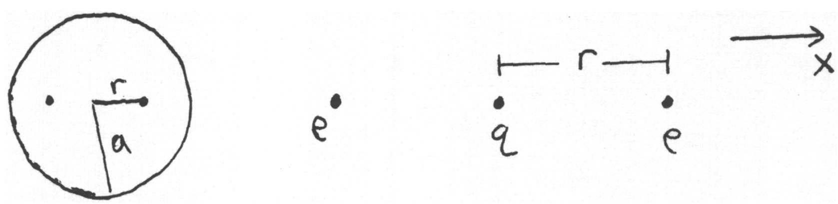
$$
\begin{aligned}
dE_r &= \frac{\sigma da}{r^2}\cos\alpha = \frac{\sigma R^2 2\pi\, d\theta}{2R^2(1-\cos\theta)}\cos(90-\theta/2) \\
&= \pi\sigma\frac{\sin\theta\sin\frac{\theta}{2}(1+\cos\theta)}{\sin^2\theta}\,d\theta = \pi\sigma\frac{(1+\cos\theta)}{2\cos\frac{\theta}{2}}\,d\theta = \pi\sigma\cos\frac{\theta}{2}\,d\theta
\end{aligned}
$$

$$
E_r = \pi\sigma\int_{\theta_o}^{\pi}\cos\frac{\theta}{2}\,d\theta = 2\pi\sigma\left(1-\sin\frac{\theta_o}{2}\right)
$$

We can take the initial angle θ_o to be b/R.

$$
E_r = 2\pi\sigma\left[1-\left(\frac{b}{2R}-\frac{1}{3!}\left(\frac{b}{2R}\right)^3+\ldots\right)\right]
$$

4 *Purcell 1.33* *Imagine a sphere of radius a filled with negative charge −2e of uniform density. Imbed in this jelly of negative charge two protons and assume that in spite of their presence the negative charge remains uniform. Where must the protons be located so that the force on each of them is zero?*



The forces on the protons from each other will be equal and opposite. Therefore, the forces on them from the negative charge distribution must be equal and opposite also. This requires that they lie on a line through the center and are equidistant from the center. The force on each proton at radius r from the negative charge will be proportional to the amount of negative charge lying inside a sphere of radius r. For purposes of finding the electric field, we may treat all of this charge as if it were a point charge sitting in the center. We ignore all negative charge outside the radius of the proton positions. The negative charge inside the radius r is

$$
q = -\frac{r^3}{a^3}2e.
$$

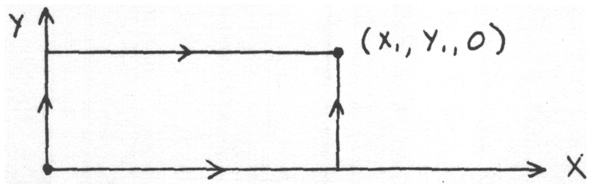
The force on the right proton must be zero.

$$
\mathbf{F} = \frac{e^2}{(2r)^2}\,\hat{\mathbf{x}} + \frac{e(-2er^3/a^3)}{r^2}\,\hat{\mathbf{x}} = \frac{e^2}{(2r)^2}\left(1-8\frac{r^3}{a^3}\right)\hat{\mathbf{x}} = \mathbf{0} \qquad r = \frac{a}{2}
$$

5 *Purcell 2.1* *The vector function*

$$
E_x = 6xy \qquad E_y = 3x^2-3y^2 \qquad E_z = 0
$$

represents a possible electrostatic field. Calculate the line integral of $\mathbf{E}$ *from the point* $(0,0,0)$ *to the point* $(x_1, y_1, 0)$ *along the path which runs straight from* $(0,0,0)$ *to* $(x_1, 0, 0)$ *and thence to* $(x_1, y_1, 0)$*. Make a similar calculation for the path which runs along the other two sides of the rectangle, via the point* $(0, y_1, 0)$*. Now you have the a potential function* $\phi(x, y, z)$*. Take the gradient of this function and see that you get back the components of the given field.*



We take the first path in two parts. While moving along the x axis we have $d\mathbf{s} = dx\,\hat{\mathbf{x}}$ so that $\mathbf{E} \cdot d\mathbf{s} = E_x\,dx$ and while moving up parallel to the y axis we have $d\mathbf{s} = dy\,\hat{\mathbf{y}}$ and $\mathbf{E} \cdot d\mathbf{s} = E_y$.

$$\int \mathbf{E} \cdot d\mathbf{s} = \int_0^{x_1} E_x\,dx + \int_0^{y_1} E_y\,dy = \int_0^{x_1} 6xy\,dx + \int_0^{y_1} (3x^2 - 3y^2)\,dy$$

When integrating along the x axis, y has the constant value $y = 0$ which we plug in to the first integral. Along the second part of the path, $x = x_1$.

$$\int \mathbf{E} \cdot d\mathbf{s} = 0 + \int_0^{y_1} (3x_1^2 - 3y^2)\,dy = 3x_1^2 y_1 - y_1^3$$

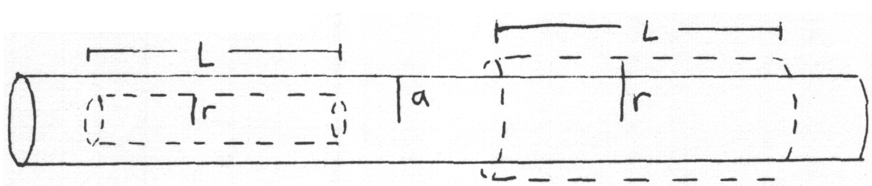
We do the same thing along the second path.

$$\int \mathbf{E} \cdot d\mathbf{s} = \int_0^{y_1} (3(0)^2 - 3y^2)\,dy + \int_0^{x_1} 6xy_1\,dx = -y_1^3 + 3x_1^2 y_1$$

$$\phi(x, y, z) = 3x^2 y - y^3$$

$$\nabla\phi(x, y, z) = \frac{\partial}{\partial x}(3x^2y - y^3)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(3x^2y - y^3)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(3x^2y - y^3)\,\hat{\mathbf{z}} = 6xy\,\hat{\mathbf{x}} + (3x^2 - 3y^2)\,\hat{\mathbf{y}}$$

6 *Purcell 2.8* *Consider an infinitely long cylinder of radius a and uniform charge density* ρ*. Use Gauss's law to find the electric field. Find the potential* ϕ *as a function of* r*, both inside and outside the cylinder, taking* $\phi = 0$ *at* $r = 0$*.*



Our Gaussian surface both inside and outside the cylinder will be a cylinder of length L. Inside we have

$$\int \mathbf{E} \cdot d\mathbf{a} = E_r 2\pi r L = 4\pi Q_{enc} = 4\pi\pi r^2 L\rho,$$

$$\mathbf{E} = 2\pi\rho\, r\,\hat{\mathbf{r}} \qquad r < a.$$

Outside,

$$\int \mathbf{E} \cdot d\mathbf{a} = E_r 2\pi r L = 4\pi Q_{enc} = 4\pi\pi a^2 L\rho,$$

$$\mathbf{E} = \frac{2\pi\rho a^2}{r}\,\hat{\mathbf{r}} \qquad\qquad r \geq a.$$

To find the potential, we integrate radially outward from the center so that $d\mathbf{s} = dr\,\hat{\mathbf{r}}$.

$$\phi(r) = \phi(0) - \int \mathbf{E}\cdot d\mathbf{s} = 0 - \int_0^r 2\pi\rho\, r'\, dr' = -\pi r^2 \rho \qquad\qquad r \leq a$$

For points outside the cylinder,

$$\phi(r) = \phi(a) - \int_a^r \frac{2\pi\rho a^2}{r'}\, dr' = -\pi a^2 \rho - 2\pi\rho\, a^2 \ln\frac{r}{a} \qquad\qquad r > a.$$

7 *Purcell 2.19 Two metal spheres of radii R_1 and R_2 are quite far apart compared with these radii. Given a total amount of charge Q, how should it be divided so as to make the potential energy of the resulting charge distribution as small as possible? Assume that any charge put on one of the spheres distributes itself uniformly over the sphere. Show that with that division the potential difference between the spheres is zero.*

Because the spheres are far apart, the energy will be essentially due to the energy of each sphere. We may assume that the charge on each sphere is uniformly distributed if the other sphere is very far away. To find this energy we can use the standard formula adapted to surface charge,

$$U = \frac{1}{2}\int \sigma\phi\, da.$$

The potential ϕ just outside a uniformly charged sphere is q/r and because the potential is continuous, this is also the potential at the surface. Then,

$$U = \frac{1}{2}\int \frac{q}{4\pi r^2}\frac{q}{r} r^2 \sin\theta\, d\phi d\theta = \frac{1}{2}\frac{q^2}{r}.$$

Breaking up the charge into q and $Q - q$,

$$U = \frac{q^2}{2R_1} + \frac{(Q-q)^2}{2R_2}.$$

If the minimum energy is obtained with $q = q_o$,

$$\frac{dU}{dq}(q_o) = \frac{q_o}{R_1} - \frac{(Q - q_o)}{R_2} = 0,$$

$$\frac{q_o}{R_1} = \frac{Q - q_o}{R_2}.$$

But these are just the potentials at both spheres.

8 *Purcell 2.20 As a distribution of electric charge, the gold nucleus can be described as a sphere of radius 6×10^{-13} cm with a charge $Q = 79e$ distributed fairly uniformly through its interior. What is the potential ϕ_o at the center of the nucleus, expressed in megavolts?*

For a uniformly charged sphere of radius a,

$$\mathbf{E} = \frac{Q}{r^2}\,\hat{\mathbf{r}} \qquad\qquad r > a,$$

$$\mathbf{E} = \frac{Qr}{a^3}\,\hat{\mathbf{r}} \qquad\qquad r \leq a.$$

Since the potential is zero at infinity, the potential at any point P is

$$\phi(P) = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{s}.$$

We can make the path of integration come radial straight in. If the point P has $r < a$,

$$\phi(r) = -\int_{\infty}^{r} E_{r'}\,dr' = -\int_{\infty}^{a} \frac{Q}{r'^2}\,dr' - \int_{a}^{r} \frac{Qr'}{a^2}\,dr' = \frac{Q}{a} - \frac{Qr^2}{2a^3} + \frac{Q}{2a}$$

In SI units,

$$\phi(0) = \frac{1}{4\pi\epsilon_o}\frac{3Q}{2a} = \frac{3 \cdot 79(1.6 \times 10^{-19}\ C)}{4\pi(8.85 \times 10^{-12}\ C^2/Nm^2)2(6 \times 19^{-15}\ m)} = 28.4\ megavolts.$$

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PROBLEM SET 6

1. Purcell problem 2.27

2. Purcell problem 2.29

3. Purcell problem 2.30

4. Purcell problem 3.1

5. Purcell problem 3.9

6. Purcell problem 3.17

7. Purcell problem 3.23

8. Purcell problem 3.24

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SOLUTION TO PROBLEM SET 6

Solutions by P. Pebler

1 *Purcell 2.27 The electrostatic potential at a point on the edge of a disc of radius r and uniform charge density σ is $\phi = 4\sigma r$. Calculate the energy stored in the electric field of a charged disk of radius a.*

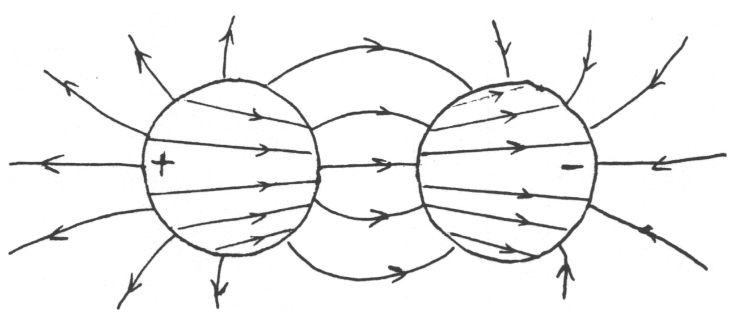
We calculate the total energy by bringing in each infinitesimal ring of charge from infinity and adding up the energy for each ring. We assume that we have already built up the disc to radius r. We now bring in a ring of width dr and stick it on the edge. Recall that the energy necessary to bring in a test charge from infinity to some point is just the potential at that point times the charge. (This is more or less the definition of the potential.) The potential just outside the disc where we are packing on the next ring is $4\sigma r$. The energy necessary is then

$$dU = \phi(r)dq = (4\sigma r)(2\pi r dr\sigma) = 8\pi\sigma^2 r^2\, dr.$$

To add up all the rings integrate from 0 to a.

$$U = 8\pi\sigma^2 \int_0^a r^2\, dr = \frac{8}{3}\pi a^3\sigma^2 = \frac{8}{3}\pi a^3 \left(\frac{Q}{\pi a^2}\right)^2 = \frac{8Q^2}{3\pi a}$$

2 *Purcell 2.29 Two nonconducting spherical shells of radius a carry charges of Q and $-Q$ uniformly distributed over their surfaces. The spheres are brought together until they touch. What does the electric field look like, both outside and inside the shells? How much work is needed to move them far apart?*



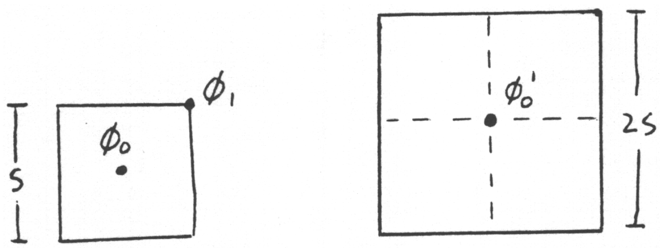
The field of a uniformly charged shell is zero inside the shell and that of a point charge outside. Outside both shells, we have the field of two point charges. Inside either shell, the field is that of a single point charge at the center of the other shell.

To find the energy we use the following argument. Consider instead a uniform shell of charge $-Q$ and a point charge Q a distance r from the center of the shell (but outside it). We know that outside the shell, the potential due to the shell is just $-Q/r$, so the energy needed to bring in the point charge is $-Q^2/r$ and the energy needed to move it out is Q^2/r. However, this must be the same energy as that required to move out the shell while keeping the point charge fixed. So we find that the energy needed to move a shell out to infinity in the field of a point charge is Q^2/r. But since the other shell creates the field of a point charge outside of it, this is also the energy needed to separate our two shells.

$$E = \frac{Q^2}{2a}$$

If you don't like this argument, you can integrate a shell distribution times the potential of a point charge which isn't too hard and find the same answer.

3 *Purcell 2.30* *Consider a cube with sides of length b and constant charge density ρ. Denote by ϕ_o the potential at the center of the cube and ϕ_1 the potential at a corner, with zero potential at inifinity. Determine the ratio ϕ_o/ϕ_1.*



We imagine another cube with the same charge density but with twice the side length. Let the potential at the center of this cube be ϕ_o'. The point at the center of this new cube lies at the corner of each of eight cubes of the original size. Because the potential is additive, we have

$$\phi_o' = 8\phi_1.$$

We can also use dimensional arguments to find ϕ_o'. We can write

$$\phi_o = f(Q, s),$$

where Q is the total charge, s is the side length and the functional form of f depends on the shape and nature of the distribution. We can now ask for what's called a scaling law which tells us what happens if we multiply the variables Q and s by numerical factors while keeping all other details of the distribution the same. Whatever the functional form of f is, we know it has units of charge per length, the units of the potential. Fortunately, the only parameters carrying units which enter into f are Q and s. The only way then to get the right units is if

$$f(Q, s) \propto \frac{Q}{s}.$$

The function f then satisfies the simple scaling

$$f(\alpha Q, \beta s) = \frac{\alpha}{\beta} f(Q, s).$$

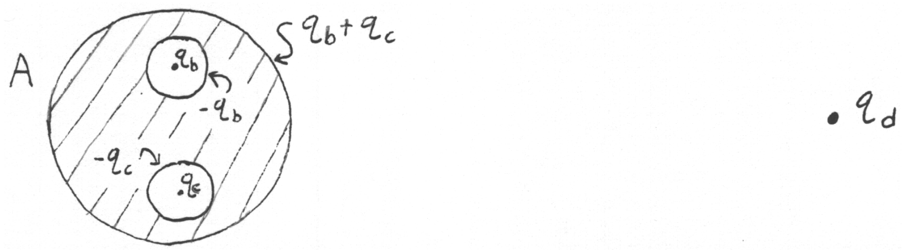
In our case $s' = 2s$ and because we are keeping the charge density constant, $Q' = \rho s'^3 = \rho(2s)^3 = 8Q$. Then

$$\phi_o' = f(8Q, 2s) = \frac{8}{2} f(Q, s) = 4\phi_o,$$

$$4\phi_o = 8\phi_1,$$

$$\frac{\phi_o}{\phi_1} = 2.$$

4 *Purcell 3.1* *A spherical conductor A contains two spherical cavities. The total charge on the conductor is zero. There are point charges q_b and q_c at the center of each cavity. A considerable distance r away is another charge q_d. What force acts on each of the four objects A, q_b, q_c, q_d? Which answers, if any, are only approximate, and depend on r being relatively large?*

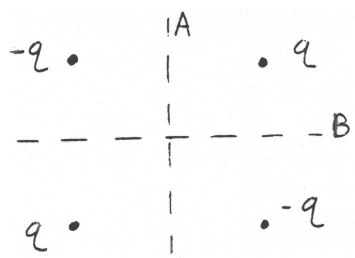


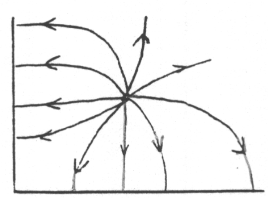
The force on q_b and q_c is zero. The field inside the spherical cavity is quite independent of anything outside. A charge $-q_b$ is uniformly distributed over the conducting surface to cancel the field from the point charge. The same happens with q_c. This leave an excess charge of $q_c + q_b$ on the outside surface of the conductor. If q_d were absent, the field outside A would be the symmetrical, radial field $E = |q_b + q_c|/r^2$, the same as a point charge because the excess charge would uniformly distribute itself over the spherical outer surface. The influence of q_d will slightly alter the distribution of the charge on A, but without affecting the total amount. Hence for large r, the force on q_d will be approximately

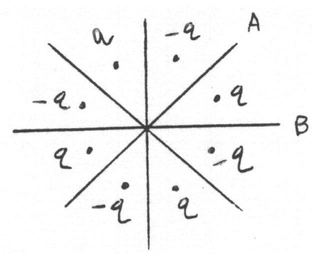
$$\mathbf{F}_d = \frac{q_d(q_b + q_c)}{r^2}\,\hat{\mathbf{r}}.$$

The force on A must be precisely equal and opposite to the force on q_d.

5 *Purcell 3.9* *Two charges q and two charges $-q$ lie at the corners of a square with like charges opposite one another. Show that there are two equipotential surfaces that are planes. Obtain and sketch qualitatively the field of a single point charge located symmetrically in the inside corner formed by bending a metal sheet through a right angle. Which configurations of conducting planes and point charges can be solved this way and which can't?*







The potential on each of the two lines A and B shown is zero because the contribution at each point on either line from any charge is cancelled by the opposite charge directly across from it. Therefore, the field of a point charge in the corner of a bent conductor is the same as the field from these four point charges. You should be able to see by looking at the first few cases that this strategy will work any time we divide the space into an even number of wedges. This allows the contributions to the potential to cancel pairwise. For example, in the picture at right the potential is zero on lines A and B because all the charges come in equal and opposite pairs. The applicable angles are $\theta_n = 2\pi/(2n) = \pi/n$, where n is an integer. This would not work for an angle of 120^o.

6 *Purcell 3.17 A spherical vacuum capacitor has radius a for the outer sphere. What radius b should be chosen for the inner spherical conductor to store the greatest amount of electrical energy subject to the constraint that the electric field strength at the surface of the inner sphere may not exceed* E_o*? How much energy can be stored?*

We first need the capacitance of this capacitor. Assuming there is a charge Q on the inner shell and a charge $-Q$ on the outer shell, the field between the shells is

$$\mathbf{E} = \frac{Q}{r^2}\,\hat{\mathbf{r}}.$$

The potential difference is

$$V = -\int_a^b \frac{Q}{r^2}\,dr = Q\left(\frac{a-b}{ab}\right),$$

and the capacitance

$$C = Q/V = \frac{ab}{a-b}.$$

The energy stored by this capacitor is

$$U = \frac{1}{2C}Q^2 = \frac{1}{2}\frac{a-b}{ab}Q^2.$$

The energy in the capacitor will depend on how much charge is on it. If we were allowed to put arbitrary amounts on, the energy would have no maximum. However, for a given b, the maximum field near the inner sphere gives us the maximum allowed charge. This gives us the maximum stored energy for a given capacitor.

$$E_o = \frac{Q_{max}}{b^2}$$

$$U_{max} = \frac{1}{2}\frac{a-b}{ab}E_o^2 b^4 = \frac{1}{2}\frac{ab^3-b^4}{a}E_o^2$$

Now we want to choose a b to make this as large as possible.

$$\frac{\partial U_{max}}{\partial b}(b_{max}) = \frac{1}{2}\frac{3ab^2-4b^3}{a}E_o^2 = 0$$

$$3a - 4b_{max} = 0$$

$$b_{max} = \frac{3}{4}a$$

The energy is then

$$U_{max} = \frac{1}{2}\frac{a-\left(\frac{3}{4}a\right)}{a\left(\frac{3}{4}a\right)}E_o^2\left(\frac{3}{4}a\right)^4 = \frac{27}{512}E_o^2 a^3.$$

7 *Purcell 3.23 Find the capacitance of a capacitor that consists of two coaxial cylinder of radii a and b and length L. Assume* $L \gg b-a$ *so that end corrections may be neglected. Check your result in the limmit* $b-a \ll a$ *with the formula for the parallel-plate capacitor.*

A cylinder of 2.00 *in outer diameter hangs with its axis vertical from one arm of a beam balance. The lower portion of the hanging cylinder is surrounded by a stationary cylinder with inner diameter* 3.00 *in. Calculate the magnitude of the force down when the potential difference between the two cylinders is* 5 *kV.*

The field between charged cylinders is

$$\mathbf{E} = \frac{2\lambda}{r}\,\hat{\mathbf{r}} = \frac{2Q}{rL}\,\hat{\mathbf{r}},$$

assuming we have Q on the inside and $-Q$ on the outside. The potential difference is

$$V = \int_a^b \frac{2Q}{L}\frac{dr}{r} = \frac{2Q}{L}\ln\frac{b}{a}.$$

Just arrange your signs so that the capacitance comes out positive.

$$C = \frac{L}{2\ln(b/a)}$$

Let us now consider the general case where the potential difference is being held constant by a battery while the capacitance is changing. Initially we have charge and energy

$$Q = CV \qquad U = \frac{1}{2}CV^2.$$

After a change in capacitance ΔC,

$$Q' = (C + \Delta C)V = Q + V\Delta C \qquad U' = \frac{1}{2}(C + \Delta C)V^2.$$

The battery has done work on this system by moving this extra charge across the potential difference.

$$W_b = (\Delta C)V^2$$

If the change in capacitance is caused by movement of the components, the electric field does work on the plates or plate.

$$W = F(\Delta L)$$

From conservation of energy we have

$$U + W_b = U' + W,$$

$$(\Delta C)V^2 = \frac{1}{2}(\Delta C)V^2 + W,$$

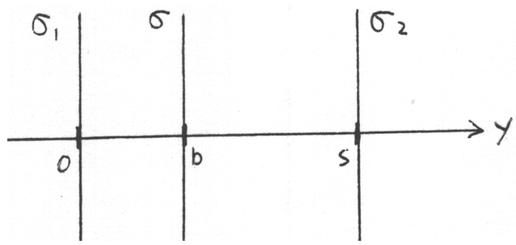
$$W = F(\Delta L) = \frac{1}{2}(\Delta C)V^2,$$

$$F = \frac{1}{2}V^2\frac{\partial C}{\partial L}.$$

In our case we have

$$F = \frac{1}{2}V^2\frac{1}{2\ln(b/a)} = \frac{1}{2}\frac{(16.7\ statvolts)^2}{2\ln(3/2)} = 172\ dynes.$$

8 *Purcell 3.24 Two parallel plates are connected by a wire. Let one plate coincide with the xz plane and the other with the plane $y = s$. The distance s is much smaller than the lateral dimensions of the plates. A point charge Q is located between the plates at $y = b$. What is the magnitude of the total surface charge on the inner surface of each plate?*



The total induced charge is $-Q$. We need to find the fraction of induced charge on either conductor. For this we may notice that the fraction of induced charge on both planes will be the same for any distribution located at $y = b$ because we may view it as the superposition of many little point charges. So we want to consider the simplest possible case which is a uniformly charged plane. (Once again, we are ignoring edge effects.) Using a Gaussian pillbox with its left face inside the left plate and its right face at y, where $0 < y < b$, the field in the left region is

$$E_l = 4\pi\sigma_1\hat{\mathbf{y}}.$$

Similarly, the field in the right region is

$$E_r = -4\pi\sigma_2\hat{\mathbf{y}}.$$

Since the two conductors are connected by a wire, they are at the same potential so the line integral from the middle to the left and right should be the same.

$$4\pi\sigma_1(-b) = -4\pi\sigma_2(s-b)$$

$$\frac{\sigma_2}{\sigma_1} = \frac{b}{s-b}$$

Now switch back to the original problem.

$$\frac{Q_2}{Q_1} = \frac{b}{s-b} \qquad Q_1 + Q_2 = -Q$$

$$Q_1 = -\frac{s-b}{s}Q \qquad Q_2 = -\frac{b}{s}Q$$

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PROBLEM SET 7

1. Purcell problem 4.8

2. Purcell problem 4.20

3. Purcell problem 4.25

4. Purcell problem 4.26

5. Purcell problem 4.30

6. Purcell problem 4.31

7. Purcell problem 4.32

8. (Taylor & Wheeler problem 19)
(a.)
Two events P and Q have a spacelike separation. Show that an inertial frame can be found in which the two events occur at the *same time*. In this frame, find the distance between the two events (this is called the *proper distance*). (*Hint*: one method of proof is to assume that such an inertial frame exists and then use the Lorentz transformation equations to show that the velocity βc of this inertial frame, relative to the frame in which the events were initially described, is such that $\beta < 1$, thus justifying the assumption made.)
(b.)
Two events P and R have a timelike separation. Show that an inertial frame can be found in which the two events occur at the *same place*. In this frame, find the time interval between the two events (this is called the *proper time*).

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SOLUTION TO PROBLEM SET 7

Solutions to Purcell problems by P. Pebler

1 *Purcell 4.8* *A copper wire* 1 *km long is connected across a* 6 *volt battery. The resistivity of copper is* 1.7×10^{-6} *ohm cm, and the number of conduction electrons per cubic centimeter is* 8×10^{22}. *What is the drift velocity of the conduction electrons under these circumstances? How long does it take an electron to drift once around the circuit?*

The current will be

$$I = \frac{V}{R} = \frac{VA}{\rho L}$$

and the current density

$$J = \frac{I}{A} = \frac{V}{\rho L}.$$

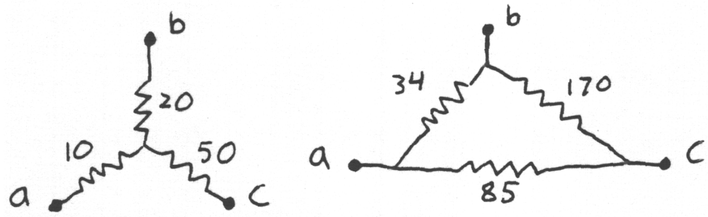
The current density is the charge density times the drift speed and the charge density is ne where n is the number density of electrons.

$$v = \frac{J}{ne} = \frac{V}{\rho L n e}$$

$$v = \frac{6\ V}{(1.7 \times 10^{-6}\ ohm\,cm)(10^5\ cm)(8 \times 10^{22}\ cm^{-3})(1.6 \times 10^{-19}\ C)} = 2.8 \times 10^{-3}\ cm/s$$

$$t = \frac{10^5\ cm}{2.8 \times 10^{-3}\ cm/s} = 3.6 \times 10^7\ s \simeq 1\ year$$

2 *Purcell 4.20* *A black box with three terminals a, b, and c contains nothing by three resistors and connecting wire. Measuring the resistance between pairs of terminals, we find* $R_{ab} = 30$ *ohm,* $R_{ac} = 60$ *ohm, and* $R_{bc} = 70$ *ohm. Show that the contents cound be either of the following.*



Is there any other possibility? Are the two boxes completely equivalent, or is there an external measurement that would distinguish between them?

For the first box, the resistance between any two terminals involves two of the resistors in series with the third resistor extraneous. For example, $R_{ab} = 10\ ohm + 20\ ohm = 30\ ohm$.

For the second box, the resistance between any two terminals involves one resistor in parallel with the other two in series. For example,

$$R_{ab} = \left(\frac{1}{34\ ohm} + \frac{1}{85\ ohm + 170\ ohm}\right)^{-1} = 30\ ohm.$$

The other two are easily verified.

These are the only two ways to make these three resistances with only three resistors.

For the two arrangements to be electrically identical, they must both draw the same currents given the same input voltages V_a, V_b, and V_c. The details are somewhat messy. I'll leave it to you to verify that given the input voltages V_a, V_b, and V_c, both arrangements draw the currents

$$I_a = \frac{1}{170}(7V_a - 5V_b - 2V_c),$$

$$I_b = \frac{1}{170}(6V_b - V_c - 5V_a),$$

$$I_c = \frac{1}{170}(3V_c - V_b - 2V_a).$$

Note that $I_a + I_b + I_c = 0$ as it must.

3 *Purcell 4.25* *A charged capacitor C discharges through a resistor R. Show that the total energy dissipated in the resistor agrees with the energy originally stored in the capacitor. Suppose someone objects that the capacitor is never really discharged because Q only becomes zero for $t = \infty$. How would you counter this objection?*

Assume that the capacitor initially has charge Q on it. The current as a function of time is

$$i(t) = i_o e^{-t/\tau}$$

where $\tau = RC$ and $i_o = V_o/R = Q/CR$. The power dissipated in a resistor is $P = i^2 R$, so the total energy dissipated is

$$E = \int_0^\infty P\,dt = \int_0^\infty R\,i_o^2 \exp\left(-2t/\tau\right) dt = R\,i_o^2 \frac{\tau}{2} = R\frac{Q^2}{C^2R^2}\frac{RC}{2} = \frac{1}{2C}Q^2.$$

This was the initial energy stored in the capacitor.

We can find the time it takes for the charge left on the capacitor to be one electron. The charge as a function of time is

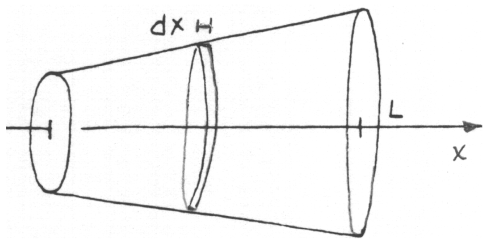
$$q = Q\exp\left(-t/\tau\right).$$

The time would be

$$t = \tau \ln\frac{Q}{e}.$$

Because of the ln, even for macroscopic initial charges, the time wouldn't be that large.

4 *Purcell 4.26 Two graphite rods are of equal length. One is a cylinder of radius a. The other is conical, tapering linearly from radius a at one end to radius b at the other. Show that the end-to-end resistance of the conical rod is a/b times that of the cylindrical rod.*



We consider the conical rod to be the series combination of little cylindrical rods of length dx. The radii of these little cylinders are

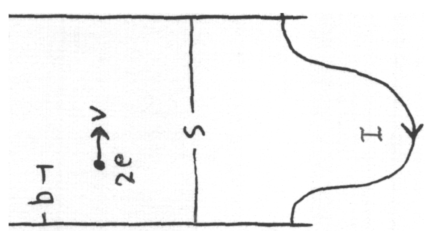
$$r(x) = a + \frac{b-a}{L}x.$$

We sum up the little resistances.

$$R = \int dR = \int_0^L \rho \frac{dx}{A} = \rho \int_0^L \frac{dx}{\pi(a+(b-a)x/L)^2} = \rho \frac{L}{\pi ab} = \frac{a}{b}\left(\rho \frac{L}{\pi a^2}\right)$$

$\rho L/\pi a^2$ is the resistance of the cylinder of radius a.

5 *Purcell 4.30 Consider two electrodes 2 mm apart in vacuum connected by a short wire. An alpha particle of charge 2e is emmitted by the left plate and travels directly towards the right plate with constant speed 10^8 cm/s and stops in this plate. Make a quantitative graph of the current in the connecting wire, plotting current against time. Do the same for an alpha particle that crosses the gap moving with the same speed but at an angle of 45^o. Suppose we had a cylindrical arrangement of electrodes. Would the current pulse have the same shape?*



The result of problem 3.24 tells us the induced charge on the electrodes for any given position of the alpha particle. The current is just the time derivative of this charge.

$$q_1 = -2e\left(\frac{s-b}{s}\right)$$

$$I = \frac{dq_1}{dt} = \frac{2e}{s}\frac{db}{dt} = \frac{2ev}{s} = 0.48 \ esu/s$$

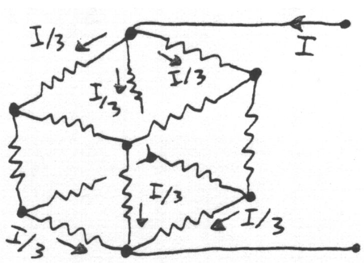
This pulse lasts for $(0.2\ cm)/(10^8\ cm/s) = 2\ ns$. If the alpha particle travels at 45^o, the normal speed and the current just decrease by $\sqrt{2}$.

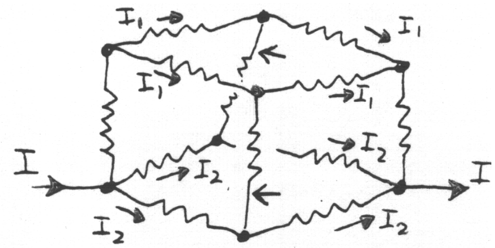
For concentric cylinders, the current pulse would have a different shape. If you work out the details,

$$I \propto \frac{1}{a+vt}$$

during the time that the alpha particle is in motion.

6 *Purcell 4.31 Suppose a cube has a resistor of resistance R_o along each edge. At each corner the leads from three resistors are soldered together. Find the equivalent resistance between two nodes that represent diagonally opposite corners of the cube. Now find the equivalent resistance between two nodes that correspond to diagonally opposite corners of one face of the cube.*





A total current I enters one node. It then has a choice of three directions to go. Because of the symmetry, each choice is identical to the others so the current must split up evenly so that $I/3$ goes through each resistor. Likewise, the current reaching the other node comes through three resistors each having current $I/3$. This leaves 6 resistors in the middle to share the current. Because each one is identical due to the symmetry, they must each have current $I/6$. To find the voltage drop between the two nodes, follow a straight path from one to the other.

$$V = \frac{I}{3}R_o + \frac{I}{6}R_o + \frac{I}{3}R_o = \frac{5}{6}RI$$

$$R_{eq} = \frac{5}{6}R_o$$

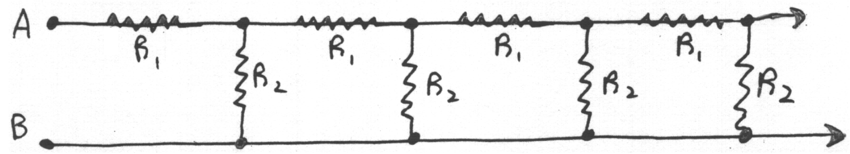
In the second situation, because of the symmetry, we notice that all the resistors on the top square carry the same magnitude of current while all the resistors on the bottom square carry the same magnitude also. This tells us that there is no current through the two resistors indicated by arrows. We can therefore ignore them, because the circuit would behave the same without them. It is then easy to combine the remaining resistors. The top and bottom squares are parallel combinations of resistors $2R_o$.

$$R = \left(\frac{1}{2R_o} + \frac{1}{2R_o}\right)^{-1} = R_o$$

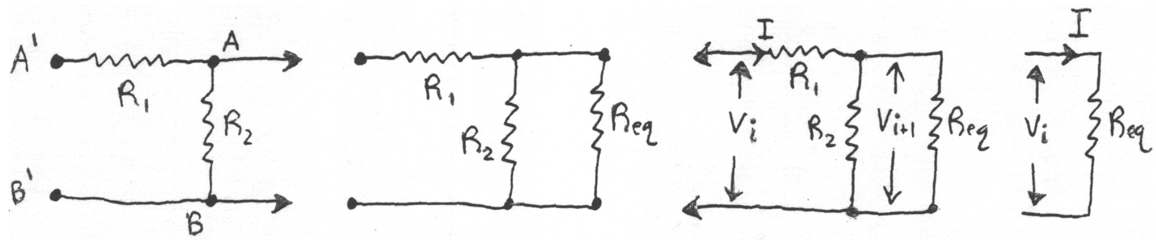
This leaves us with one resistor R_o in parallel with three resistors R_o.

$$R_{eq} = \left(\frac{1}{R_o} + \frac{1}{3R_o}\right)^{-1} = \frac{3}{4}R_o$$

7 *Purcell 4.32 Find the input resistance (between terminals A and B) of the following infinite series.*



Show that, if voltage V_o is applied at the input to such a chain, the voltage at successive nodes decreases in a geometric series. What ratio is required for the resistors to make the ladder an attenuator that halves the voltage at every step? Can you suggest a way to terminate the ladder after a few sections without introducing any error in its attenuation?



If we put another "link" on the left of this infinite chain, we get exactly the same configuration. If this infinite chain has equivalent resistance R_{eq}, the new chain with the extra link can be described by the middle circuit. We can calculate the equivalent resistance of this circuit by considering R_{eq} and R_2 in parallel, in series with R_1. But since this circuit is the same as the original, this equivalent resistance is again R_{eq}.

$$R_{eq} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_{eq}}\right)^{-1}$$

This leads to the equation

$$R_{eq}^2 - R_1 R_{eq} - R_1 R_2 = 0,$$

with positive solution

$$R_{eq} = \frac{R_1 + \sqrt{R_1^2 + 4R_1 R_2}}{2}.$$

Now consider an arbitrary link with voltage difference V_i between top and bottom. To find the voltage V_{i+1}, we can replace the rest of the series with the equivalent resistor. We could get V_{i+1} if we knew the current through R_1. For this purpose we can replace the ith link also with R_{eq}. The current through this equivalent circuit will also be the current through R_1. This current is just V_i/R_{eq}.

$$V_{i+1} = V_i - R_1 \frac{V_i}{R_{eq}} = V_i \frac{R_{eq} - R_1}{R_{eq}}$$

If we wish to halve the voltage each step,

$$\frac{R_1}{R_{eq}} = \frac{1}{2},$$

$$4R_1 = R_1 + \sqrt{R_1^2 + 4R_1 R_2},$$

$$R_2 = 2R_1.$$

If we wish to terminate the ladder without changing this property, we just replace the rest of the chain at any point with a resistor with resistance R_{eq}.

8 *Taylor & Wheeler 19 (a.) Two events P and Q have a spacelike separation. Show that an inertial frame can be found in which the two events occur at the same time. In this frame, find the distance between the two events (this is called the* proper distance*). (Hint: one method of proof is to assume that such an inertial frame exists and then use the Lorentz transformation equations to show that the velocity βc of this inertial frame, relative to the frame in which the events were initially described, is such that $\beta < 1$, thus justifying the assumption made.) (b.) Two events P and R have a timelike separation. Show that an inertial frame can be found in which the two events occur at the* same place*. In this frame, find the time interval between the two events (this is called the* proper time*).*

Denote the frame in which P and Q were initially described by $\mathcal{S}$, and the frame in which we wish them to be simultaneous by $\mathcal{S}'$. As usual, the origins of these frames coincide at $t = t' = 0$. In $\mathcal{S}$, if P and Q have a spacelike separation, their spatial separation must be nonzero. Orient the x and x' axes along the direction of this separation, so that $y_Q = y_P$ and $z_Q = z_P$ but $x_Q \neq x_P$. Applying the Lorentz transformation between $\mathcal{S}'$ and $\mathcal{S}$,

$$ct'_P = \gamma ct_P - \gamma\beta x_P$$

$$ct'_Q = \gamma ct_Q - \gamma\beta x_Q$$

$$ct'_Q - ct'_P = \gamma(ct_Q - ct_P) - \gamma\beta(x_Q - x_P)$$

We wish the left-hand side to be zero. If it is, then

$$\gamma(ct_Q - ct_P) = \gamma\beta(x_Q - x_P)$$

$$\frac{(ct_Q - ct_P)}{(x_Q - x_P)} = \beta$$

Now, we are told that the separation between events Q and P is spacelike:

$$c^2(t_Q - t_P)^2 - (x_Q - x_P)^2 < 0$$

This guarantees that $|\beta| < 1$. It is straightforward to calculate the spatial separation of the two events in $\mathcal{S}'$ by using the inverse Lorentz transformation:

$$x_P = \gamma x'_P + \gamma\beta ct'_P$$

$$x_Q = \gamma x'_Q + \gamma\beta ct'_Q$$

$$x_Q - x_P = \gamma(x'_Q - x'_P) + \gamma\beta(ct'_Q - ct'_P)$$

We have chosen β so that the two events are simultaneous in $\mathcal{S}'$; this forces the last term to vanish. Substituting the value that we found for β,

$$x_Q - x_P = \gamma(x'_Q - x'_P)$$

$$x'_Q - x'_P = \sqrt{1-\beta^2}\,(x_Q - x_P)$$

$$x'_Q - x'_P = \sqrt{1 - \Big(\frac{ct_Q - ct_P}{x_Q - x_P}\Big)^2}\,(x_Q - x_P)$$

$$x'_Q - x'_P = \pm\sqrt{(x_Q - x_P)^2 - (ct_Q - ct_P)^2}$$

where the sign is chosen so that $x'_Q - x'_P$ has the same sign as $x_Q - x_P$. This *proper distance* is the smallest distance between the two events that can be reached in any reference frame.

Similarly, when the separation between events P and R is timelike,

$$x'_R - x'_P = \gamma(x_R - x_P) - \gamma\beta(ct_R - ct_P)$$

$$\gamma(x_R - x_P) = \gamma\beta(ct_R - ct_P)$$

$$\frac{(x_R - x_P)}{(ct_R - ct_P)} = \beta$$

$$ct_R - ct_P = \gamma(ct'_R - ct'_P) + \gamma\beta(x'_R - x'_P)$$

$$ct'_R - ct'_P = \sqrt{1-\beta^2}\,(ct_R - ct_P)$$

$$ct'_R - ct'_P = \sqrt{1 - \Big(\frac{x_R - x_P}{ct_R - ct_P}\Big)^2}\,(ct_R - ct_P)$$

$$ct'_R - ct'_P = \pm\sqrt{(ct_R - ct_P)^2 - (x_R - x_P)^2}$$

where the sign is chosen so that $ct'_R - ct'_P$ has the same sign as $ct_R - ct_P$. This *proper time* is the smallest time interval between the two events that can be reached in any reference frame.

University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

PROBLEM SET 8

1. (Taylor and Wheeler problem 27)
The clock paradox, version 1.
On their twenty-first birthday, Peter leaves his twin Paul behind on the earth and goes off in the x direction for seven years of *his* time at 24/25 the speed of light, then reverses direction and in another seven years of *his* time returns at the same speed. [In this most elementary version of the problem, we assume that the necessary periods of acceleration are infinitesimal in duration, requiring Peter's acceleration to be infinite. Nonetheless, our plucky twin remains uninjured.]
(a.)
Make a spacetime diagram (ct vs. x) showing Peter's motion. Indicate on it the x and ct coordinates of the turn-around point and the point of reunion. For simplicity idealize the earth as an inertial frame, adopt this inertial frame in the construction of the diagram, and take the origin to be the event of departure.
(b.)
How old is Paul at the moment of reunion?

2. Prove that

$$\tanh(\eta_1 + \eta_2) = \frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1 \tanh\eta_2} \ .$$

Using this relation, deduce Einstein's law for the addition of velocities.

3. The thermonuclear "deuterium-tritium" reactions are:

$$\begin{aligned} {}^2\mathrm{H} + {}^2\mathrm{H} &\to {}^3\mathrm{He} + n \\ {}^2\mathrm{H} + {}^2\mathrm{H} &\to {}^3\mathrm{H} + p \\ {}^2\mathrm{H} + {}^3\mathrm{H} &\to {}^4\mathrm{He} + n \ . \end{aligned}$$

These sum to

$$5({}^2\mathrm{H}) \to {}^3\mathrm{He} + {}^4\mathrm{He} + p + 2n.$$

Using the following masses in AMU,

(proton) p	1.007825
(neutron) n	1.008665
(deuteron) ^{2}H	2.014102
(helium 3) ^{3}He	3.016030
(triton) ^{3}H	3.016050
(alpha particle) ^{4}He	4.002603 ,

calculate (to 5%) the kinetic energy released when one liter of heavy water $(^2\mathrm{H})_2\mathrm{O}$ undergoes deuterium-tritium fusion in an H-bomb. Express your answer in terms of tons of TNT (1 ton of TNT = 4.2×10^9 J of explosive energy).

4. The universe is filled with old cold photons that are remnants of the big bang. Typically their energy is $\approx 6.6 \times 10^{-4}$ eV.

A cosmonaut who is accelerated at 1 g for 10 years in her own rest frame attains a boost ($= \operatorname{arctanh}\beta$) of 10.34. As seen by her, what is the typical energy of these photons?

5. Prove that an isolated photon (zero mass) cannot split into two photons which do not both continue in the original direction.

6. The now retired Bevatron at Berkeley Lab is famous for having produced the first observed antiprotons (you may have glimpsed white-maned Nobelist Owen Chamberlain, one of the first observers, being helped to his seat at Physics Department colloquia). An economical reaction for producing antiprotons is

$$p + p \to p + p + p + \bar{p} \ ,$$

where the first proton is part of a beam, the second is at rest in a target, and $\bar{p}$ is an antiproton. Because of the CPT theorem, both p and $\bar{p}$ must have the same mass ($= 0.94 \times 10^9$ eV).

At threshold, all four final state particles have essentially zero velocity *with respect to each other*. What is the beam energy in that case? (The actual Bevatron beam energy was 6×10^9 eV).

7. Using Eqs. 1.33 in the lecture notes, prove that $E^2 - B^2$, where E (B) is the magnitude of the electric (magnetic) field, is a Lorentz invariant.

8. You shine a one-watt beam of photons on a crow, who absorbs them. Calculate the force (in N) on the crow.

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SOLUTION TO PROBLEM SET 8

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(a.)
Make a spacetime diagram (ct vs. x) showing Peter's motion. Indicate on it the x and ct coordinates of the turn-around point and the point of reunion. For simplicity idealize the earth as an inertial frame, adopt this inertial frame in the construction of the diagram, and take the origin to be the event of departure.
(b.)
How old is Paul at the moment of reunion?

Solution:
On a spacetime (ct vs. x) diagram in Paul's (unprimed) frame, Peter begins at (0,0) and proceeds with slope $\beta^{-1} = \frac{25}{24}$ for a time interval

$$\begin{aligned} c\Delta t &= \gamma c\Delta t' + \gamma\beta(\Delta x' = 0) \\ &= \gamma c\Delta t' \\ &= \sqrt{\frac{1}{1-\left(\frac{24}{25}\right)^2}}\, c\Delta t' \\ &= \frac{25}{7}\, c\Delta t' \\ &= 25 \text{ lightyr} . \end{aligned}$$

At Peter's point of maximum excursion, ($ct = 25$, $x = 24$) lightyr. Peter then returns with slope $\beta^{-1} = -\frac{25}{24}$, reaching $x = 0$ at $ct = 50$ lightyr where he reunites with Paul. Peter has aged only 14 years, while Paul has aged 50 years (and has reached the age of 71).

2. Prove that

$$\tanh(\eta_1 + \eta_2) = \frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1 \tanh\eta_2} .$$

Using this relation, deduce Einstein's law for the addition of velocities.

Solution:

$$\begin{aligned} &\tanh\eta = \\ &= \frac{\exp(2\eta) - 1}{\exp(2\eta) + 1} \\ &\tanh\eta_1 + \tanh\eta_2 = \\ &= \frac{\exp(2\eta_1) - 1}{\exp(2\eta_1) + 1} + \frac{\exp(2\eta_2) - 1}{\exp(2\eta_2) + 1} \\ &= \frac{2\exp(2\eta_1 + 2\eta_2) - 2}{\exp(2\eta_1 + 2\eta_2) + 1 + \exp(2\eta_1) + \exp(2\eta_2)} \\ &\tanh\eta_1 \tanh\eta_2 = \\ &= \frac{\exp(2\eta_1 + 2\eta_2) + 1 - \exp(2\eta_1) - \exp(2\eta_2)}{\exp(2\eta_1 + 2\eta_2) + 1 + \exp(2\eta_1) + \exp(2\eta_2)} \\ &1 + \tanh\eta_1 \tanh\eta_2 = \\ &= \frac{2\exp(2\eta_1 + 2\eta_2) + 2}{\exp(2\eta_1 + 2\eta_2) + 1 + \exp(2\eta_1) + \exp(2\eta_2)} \\ &\frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1 \tanh\eta_2} = \frac{2\exp(2\eta_1 + 2\eta_2) - 2}{2\exp(2\eta_1 + 2\eta_2) + 2} \\ &= \tanh(\eta_1 + \eta_2) . \end{aligned}$$

Suppose that all velocities are in the x direction. Take the velocity of frame $\mathcal{S}_1$ with respect to frame $\mathcal{S}$ to be $\beta_1 c$; of frame $\mathcal{S}_2$ with respect to frame $\mathcal{S}_1$ to be $\beta_2 c$; and of frame $\mathcal{S}_2$ with respect to frame $\mathcal{S}$ to be $\beta_3 c$. $\beta_{1,2,3}$ correspond to boost parameters (or rapidities) $\eta_{1,2,3}$ according to the relation

$$\beta_{1,2,3} = \tanh\eta_{1,2,3} .$$

The boost parameters have the unique property that they are additive, i.e. a boost of η_1 followed by a boost of η_2 is equivalent to a boost of $\eta_1 + \eta_2$. So, with the above definitions,

$$\begin{aligned}\eta_3 &= \eta_1 + \eta_2 \\ \beta_3 &= \tanh \eta_3 \\ &= \tanh(\eta_1 + \eta_2) \\ &= \frac{\tanh \eta_1 + \tanh \eta_2}{1 + \tanh \eta_1 \tanh \eta_2} \\ &= \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \ .\end{aligned}$$

This is Einstein's law for the addition of velocities.

3. The thermonuclear "deuterium-tritium" reactions are:

$$\begin{aligned}{}^2\mathrm{H} + {}^2\mathrm{H} &\rightarrow {}^3\mathrm{He} + n \\ {}^2\mathrm{H} + {}^2\mathrm{H} &\rightarrow {}^3\mathrm{H} + p \\ {}^2\mathrm{H} + {}^3\mathrm{H} &\rightarrow {}^4\mathrm{He} + n \ .\end{aligned}$$

These sum to

$$5({}^2\mathrm{H}) \rightarrow {}^3\mathrm{He} + {}^4\mathrm{He} + p + 2n.$$

Using the following masses in AMU,

(proton) p	1.007825
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calculate (to 5%) the kinetic energy released when one liter of heavy water $({}^2\mathrm{H})_2\mathrm{O}$ undergoes deuterium-tritium fusion in an H-bomb. Express your answer in terms of tons of TNT (1 ton of TNT = 4.2×10^9 J of explosive energy).

Solution:
The energy released in the summed reaction corresponds to a mass deficit equal to Δm = 5(2.014102) − 3.016030 − 4.002603 − 1.007825 − 2(1.008665) = 0.026722 amu. Heavy water has a density about $\frac{20}{18}$ times that of ordinary water, due to the extra two neutrons. Thus one liter of heavy water weighs 1.11 kg and corresponds to $\frac{1.11}{20} \times 10^3 = 55.5$ moles. It contains $55.5 \times N_{\mathrm{Avo}} = 55.5 \times 6.023 \times 10^{23} = 3.35 \times 10^{25}$ molecules of heavy water. Each of the summed reactions requires five deuterium nuclei, or 2.5 molecules of heavy water, so $3.35 \times 10^{25}/2.5 = 1.34 \times 10^{25}$ summed reactions take place. The mass energy in one amu is equivalent to $mc^2 = 0.9315 \times 10^9$ eV, or, with 1 eV = 1.6×10^{-19} J, 1.49×10^{-10} J. Therefore the energy released is $0.026722 \times 1.34 \times 10^{25} \times 1.49 \times 10^{-10} = 5.33 \times 10^{13}$ J. This is equivalent to the energy released by the explosion of 12.7 kilotons of TNT.

A corollary is that about 80 liters – one Jeep gasoline tank – of $({}^2\mathrm{H})_2\mathrm{O}$ are needed to make a one megaton H-bomb. This sets a lower limit on the degree to which an H-bomb can be miniaturized, irrespective of any espionage.

4. The universe is filled with old cold photons that are remnants of the big bang. Typically their energy is $\approx 6.6 \times 10^{-4}$ eV.

A cosmonaut who is accelerated at 1 g for 10 years in her own rest frame attains a boost (= arctanh β) of 10.34. As seen by her, what is the typical energy of these photons?

Solution:
We know that the energy-momentum four-vector $(E/c, \mathbf{p})$ satisfies the same Lorentz transformation equations as the spacetime four-vector $(ct, \mathbf{r})$:

$$\begin{aligned}E'/c &= \gamma E/c - \gamma\beta p_x \\ p'_x &= -\gamma\beta E/c + \gamma p_x \\ p'_y &= p_y \\ p'_z &= p_z \ ,\end{aligned}$$

where the cosmonaut (in the primed frame) is assumed to be travelling with respect to the big bang's (unprimed) frame with a velocity $\beta c = \tanh 10.34$ in the x direction. [With respect to this large velocity, here we are neglecting the much smaller speed of 370 km/sec with which the solar system moves with respect to the big bang radiation; this was first measured by a Berkeley group, including Profs. Smoot and Muller, in the 1970s.]

On average, $\langle p_x \rangle = 0$ for the big bang photons in the big bang's frame; thus

$$\begin{aligned}
\langle E'/c \rangle &= \gamma \langle E/c \rangle \\
&= \sqrt{\frac{1}{1-\beta^2}} \langle E/c \rangle \\
&= \sqrt{\frac{1}{1-\tanh^2 \eta}} \langle E/c \rangle \\
&= \langle E/c \rangle \cosh \eta \\
&= 6.6 \times 10^{-4} \text{ eV/c } \times \cosh 10.34 \\
E' &= 10.2 \text{ eV} .
\end{aligned}$$

Therefore, while the cosmic background radiation is in the far infrared as seen in the solar system, on average it is boosted to the ultraviolet as seen in the frame of the cosmonaut.

5. Prove that an isolated photon (zero mass) cannot split into two photons which do not both continue in the original direction.

Solution:
Assume that a photon decays into two other photons a and b. The photons have energy-momentum four-vectors denoted by $(E/c, \mathbf{p})$, $(E_a/c, \mathbf{p}_a)$, and $(E_b/c, \mathbf{p}_b)$, repectively. Both energy and momentum must be conserved in the decay. We can express this requirement in a single four-component equation:

$$(E/c, \mathbf{p}) = (E_a/c, \mathbf{p}_a) + (E_b/c, \mathbf{p})_b) .$$

To save writing we will use the shorthand notation $p \equiv (E/c, \mathbf{p})$; similarly for p_a and p_b. Rewriting the above equation in this shorthand notation, and taking the inner product of each side with itself,

$$\begin{aligned}
p &= p_a + p_b \\
p \cdot p &= (p_a + p_b) \cdot (p_a + p_b) \\
&= p_a \cdot p_a + p_b \cdot p_b + 2 p_a \cdot p_b .
\end{aligned}$$

In the above, the symbol "·" refers to the *four-vector* inner product, i.e. $p \cdot p \equiv E^2/c^2 - \mathbf{p} \cdot \mathbf{p}$. Since the inner product of any two four-vectors has the same value in any Lorentz frame, it is easiest to evaluate $p \cdot p$ in the rest frame of the particle; there one finds that

$$p \cdot p = E^2/c^2 - |\mathbf{p}|^2 = m^2 c^2 ,$$

where m is the particle's rest mass. This is the *fundamental equation* for solving relativistic kinematics problems. The fundamental equation tells us that $p \cdot p = 0$ and $|\mathbf{p}| = E/c$ for any massless particle like the photon. Returning to the problem,

$$\begin{aligned}
p \cdot p &= p_a \cdot p_a + p_b \cdot p_b + 2 p_a \cdot p_b \\
0 &= 0 + 0 + 2 E_a E_b / c^2 - 2 \mathbf{p}_a \cdot \mathbf{p}_b \\
&= 2 E_a E_b / c^2 (1 - \cos \theta_{ab}) \\
1 &= \cos \theta_{ab} ,
\end{aligned}$$

where θ_{ab} is the opening angle between the two photons. The last equation tells us that photons a and b must be travelling in the same direction (they are "collinear"); by conservation of momentum, that must be the direction of the initial photon.

[For the case in which photons a and b do travel in the direction of the initial photon, which is allowed by the above kinematic calculation, the decay nevertheless is prevented by conservation of angular momentum. Angular momentum nonconservation in the collinear decay arises from the photon's internal angular momentum ("spin").]

[Also, we note that the (electrically neutral) photon couples to electric charge, so, to lowest order, no electromagnetic interaction occurs when three photons meet at a common vertex. This is not the case for the strong force carriers (gluons), which also are massless; gluons both carry and couple to a different kind of charge called "color".]

[How could we express the above solution in words? "If the two decay photons are not collinear, their combined invariant mass must be greater than zero. Since the initial state has invariant mass equal to zero, this violates energy-momentum conservation."]

6. The now retired Bevatron at Berkeley Lab is famous for having produced the first observed antiprotons (you may have glimpsed white-maned Nobelist Owen Chamberlain, one of the first observers, being helped to his seat at Physics Department colloquia). An economical reaction for producing antiprotons is

$$p + p \to p + p + p + \bar{p} \ ,$$

where the first proton is part of a beam, the second is at rest in a target, and $\bar{p}$ is an antiproton. Because of the CPT theorem, both p and $\bar{p}$ must have the same mass ($= 0.94 \times 10^9$ eV).

At threshold, all four final state particles have essentially zero velocity *with respect to each other*. What is the beam energy in that case? (The actual Bevatron beam energy was 6×10^9 eV).

Solution:
We shall use the notation of the previous solution. Denote by p_a and p_b the four-momenta of the incident and target protons, each of which has mass m. At threshold, we are told that the four final-state particles are at rest with respect to each other. Therefore, for kinematic purposes, they are equivalent to a single particle of mass $4m$. Denote by p_c the four-momentum of this four-particle state. Energy-momentum conservation demands

$$\begin{aligned}
p_a + p_b &= p_c \\
(p_a + p_b) \cdot (p_a + p_b) &= p_c \cdot p_c \\
p_a \cdot p_a + p_b \cdot p_b + 2 p_a \cdot p_b &= p_c \cdot p_c \\
m^2c^2 + m^2c^2 + 2p_a \cdot p_b &= (4m)^2 c^2 \\
7m^2c^2 &= p_a \cdot p_b \\
&= (E_a, \mathbf{p}_a) \cdot (m, \mathbf{0}) \\
&= E_a m \\
E_a &= 7mc^2 \\
&= 7 \times 0.94 \times 10^9 \text{ eV} \\
&= 6.58 \times 10^9 \text{ eV} \ .
\end{aligned}$$

This is $\sim 10\%$ more proton beam energy than the Bevatron ($= 6 \times 10^9$ GeV) was able to supply!

How then were Chamberlain, Segrè, Wiegand, and Ypsilantis able to discover the antiproton at the Berkeley Bevatron in 1956? They took advantage of the fact that protons confined inside the atomic nucleus have a significant ($\sim$ 200 MeV/c) rms momentum as a result of Heisenberg's uncertainty principle. This is called "Fermi momentum". When the target proton's Fermi momentum is directed *against* the incoming beam proton, the energy available for the interaction can be augmented up to $\approx 20\%$.

7. Using Eqs. 1.33 in the lecture notes, prove that $E^2 - B^2$, where E (B) is the magnitude of the electric (magnetic) field, is a Lorentz invariant.

Solution:
The equations for Lorentz transformation of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ may be derived from three facts:

$$\begin{aligned}
(\phi, \mathbf{A}) &= \text{a four vector} \\
\mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} &= \nabla \times \mathbf{A} \ ,
\end{aligned}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. The result of the derivation is Eq. 1.33 in the distributed relativity notes:

$$\begin{aligned}
\mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B}_\perp) \\
\mathbf{B}'_\perp &= \gamma(\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}_\perp) \\
E'_\parallel &= E_\parallel \\
B'_\parallel &= B_\parallel \ ,
\end{aligned}$$

where $\boldsymbol{\beta}c$ is the velocity of frame $\mathcal{S}'$ relative to $\mathcal{S}$, the subscript $_\perp$ refers to the component perpendicular to $\boldsymbol{\beta}$, and the subscript $_\parallel$ refers to the component parallel to $\boldsymbol{\beta}$. Note that, in the first two equations, the subscript $_\perp$ may be dropped from the last term, since taking the cross product with $\boldsymbol{\beta}$ automatically picks out the perpendicular part. Using the first two equations,

$$\begin{aligned}
\gamma^{-2}(E'_\perp)^2 &= E_\perp^2 + \beta^2 B_\perp^2 + 2\mathbf{E}_\perp \cdot (\boldsymbol{\beta} \times \mathbf{B}_\perp) \\
\gamma^{-2}(B'_\perp)^2 &= B_\perp^2 + \beta^2 E_\perp^2 - 2\mathbf{B}_\perp \cdot (\boldsymbol{\beta} \times \mathbf{E}_\perp) \ .
\end{aligned}$$

We rearrange the last line using the invariance under cyclic permutation of the triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \ ,$$

a relation which may be found on the inside cover of Griffiths (distributed in class), or, more physically, may be understood from the fact that the triple product describes the (invariant) volume of a parallelopiped with sides **a**, **b**, and **c**. Cyclically permuting the triple product in the last line,

$$\begin{aligned}
\gamma^{-2}(B'_\perp)^2 &= B_\perp^2 + \beta^2 E_\perp^2 - 2\mathbf{E}_\perp \cdot (\mathbf{B}_\perp \times \boldsymbol{\beta}) \\
&= B_\perp^2 + \beta^2 E_\perp^2 + 2\mathbf{E}_\perp \cdot (\boldsymbol{\beta} \times \mathbf{B}_\perp) \\
\frac{E_\perp'^2 - B_\perp'^2}{\gamma^2} &= E_\perp^2 + \beta^2 B_\perp^2 - B_\perp^2 - \beta^2 E_\perp^2 \\
&= (1-\beta^2)(E_\perp^2 - B_\perp^2) \\
E_\perp'^2 - B_\perp'^2 &= E_\perp^2 - B_\perp^2 \ .
\end{aligned}$$

This demonstrates that $E_\perp^2 - B_\perp^2$ is conserved; $E_\parallel^2 - B_\parallel^2$ is conserved automatically since $\mathbf{E}_\parallel$ and $\mathbf{B}_\parallel$ are invariant under the transformation. Finally, $E^2 = E_\perp^2 + E_\parallel^2$, etc., because the dot product in the cross term vanishes.

8. You shine a one-watt beam of photons on a crow, who absorbs them. Calculate the force (in N) on the crow.

Solution:
Suppose that a photon in the flashlight beam has an energy E. Then it must have momentum $p = E/c$. The beam power (= 1 W) is $P = N\langle E\rangle$, where N is the number of photons emitted per second and $\langle E\rangle$ is their average energy. If the photons are totally absorbed by the crow, the momentum absorbed by the crow per second is

$$F = N\langle p\rangle = N\langle E\rangle/c = P/c \ .$$

Therefore the force F on the crow is

$$F = \frac{1\ \mathrm{W}}{c} = 3.3\times 10^{-9}\ \mathrm{N} \ .$$

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PROBLEM SET 9

1. Purcell problem 5.3

2. Purcell problem 5.7

3. Purcell problem 5.10

4. Purcell problem 5.17

5. Purcell problem 6.4

6. Purcell problem 6.14

7. Purcell problem 6.18

8. Purcell problem 6.22

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SOLUTION TO PROBLEM SET 9

Solutions by P. Pebler

1 *Purcell 5.3 A beam of* 9.5 MeV *electrons* ($\gamma = 20$) *amounting as current to* 0.05 μA, *is traveling through vacuum. The transverse dimensions of the beam are less than* 1 mm, *and there are no positive charges in or near it. In the lab frame, what is approximately the electric field strength* 1 cm *away from the beam, and what is the average distance between the electrons, measured parallel to the beam? Answer the same questions for the electron rest frame.*

With $\gamma = 20$, the speed of the electrons will be essentially c. The number of electrons passing per second is

$$\nu_e = \frac{0.05 \times 10^{-6}\ A}{1.6 \times 10^{-19}\ C} = 3.1 \times 10^{11}\ 1/s \quad .$$

The mean distance between electrons is

$$d = \frac{c}{\nu_e} = 1\ mm \quad ,$$

and the charge per unit length is

$$\lambda = -\frac{4.8 \times 10^{-10}\ esu}{0.1\ cm} = -4.8 \times 10^{-9}\ esu/cm \quad .$$

If we think of this as a continuous line charge, the field strength a distance 1 cm away is

$$E = -\frac{2\lambda}{r} = 9.6 \times 10^{-9}\ statvolt/cm \quad .$$

Since the electrons are moving, the distance between them will be contracted. Therefore, in the electron rest frame, they will be more spread out so that

$$d' = \gamma d = 2\ cm \quad .$$

We might find a new field strength by

$$E' = \frac{E}{\gamma} = 4.8 \times 10^{-10}\ statvolt/cm \quad ,$$

however, this will be only the average field strength along a line parallel to the beam. Since the electrons are so far apart in this frame, there will be big variation in the field.

2 *Purcell 5.7 A moving proton has* $\gamma = 10^{10}$. *How far away from such a proton would the field rise to* 1 V/m *as it passes?*

The electric field strength of a moving point charge is (in SI units)

$$E' = \frac{Q}{4\pi\epsilon_o r'^2} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2\theta')^{3/2}} \quad .$$

The maximum field strength is directly beside the particle where $\theta' = \pi/2$. Then

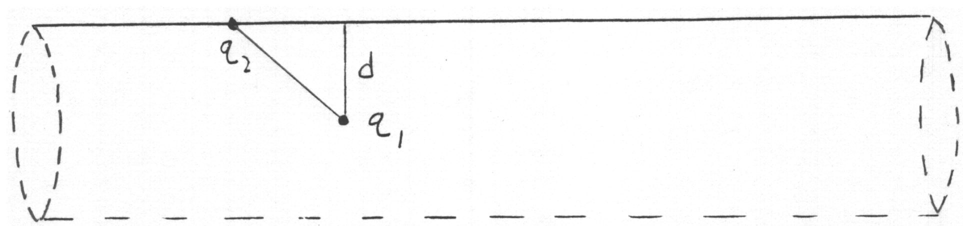
$$E' = \frac{Q}{4\pi\epsilon_o r'^2} \frac{1}{\sqrt{1-\beta^2}} = \gamma \frac{Q}{4\pi\epsilon_o r'^2} \quad .$$

We want the distance where the field is 1 V/m.

$$\frac{10^{10}(1.6\times10^{-19}\ C)}{4\pi(8.85\times10^{-12}\ C^2/Nm^2)r'^2}=1\ V/m$$

$$r'=3.8\ m$$

3 *Purcell 5.10* *In the rest frame of a particle with charge q_1 another particle with charge q_2 is approaching, moving with velocity v not small compared with c. If it continues to move in a straight line, it will pass a distance d from the position of the first particle. It is so massive that its displacement from the straight path during the encounter is small compared with d. Likewise, the first particle is so massive that its displacement from its initial position is small compared with d. Show that the increment in momentum acquired by each particle as a result of the encounter is perpendicular to* $\mathbf{v}$ *and has magnitude $2q_1q_2/vd$. Expressed in terms of other quantities, how large must the masses of the particles be to justify our assumptions?*



In the first approximation, we consider the trajectory to be a straight line. For this trajectory, the net impulse in the x direction will be zero. Using cylindrical coordinates,

$$\Delta p_r=\int F_r dt=\frac{1}{v}\int F_r\,dx=\frac{q_2}{2\pi dv}\int E_r 2\pi d\,dx\quad.$$

The purpose of transforming this integral in this way is so that it becomes the electric flux through an infinite cylinder of radius d. We can evaluate it easily by using Gauss's law.

$$\Delta p_r=\frac{q_2}{2\pi dv}\int \mathbf{E}\cdot d\mathbf{a}=\frac{q_2}{2\pi dv}4\pi q_1=\frac{2q_1q_2}{dv}$$

In the rest frame of q_2, the situation is reversed, and q_1 acquires the same y momentum in the opposite direction. This will be the same momentum in the rest frame of q_1 because the perpendicular component of the momentum is unchanged by a Lorentz transformation.

For this approximation to be good, the acquired y momentum must be much smaller than the x momentum, so that

$$\gamma mv\gg\frac{2q_1q_2}{dv}\quad,$$

where m can represent either mass.

4 *Purcell 5.17* *Two protons are moving parallel to one another a distance r apart, with the same velocity βc in the lab frame. At the instantaneous position of one of the protons the electric field strength caused by the other is $\gamma e/r^2$. But the force on the proton measured in the lab frame is not $\gamma e^2/r^2$. Verify that by finding the force in the proton rest frame and transforming that force back to the lab frame. Show that the discrepancy can be accounted for if there is a magnetic field β times as strong as the electric field, accompanying this proton as it travels through the lab frame.*

In the rest frame of the protons, the force is e^2/r^2. In the lab frame this force is

$$F_y = \frac{1}{\gamma}\frac{e^2}{r^2} = \gamma\frac{e^2}{r^2} + F_y^{(b)} \quad ,$$

where the first term is the electric force in the lab frame. The extra term is

$$F_y^{(b)} = \gamma\frac{e^2}{r^2}\left(\frac{1}{\gamma^2} - 1\right) = -\gamma\beta^2\frac{e^2}{r^2} = -\left(\beta\gamma\frac{e}{r^2}\right)e\frac{v}{c} \quad .$$

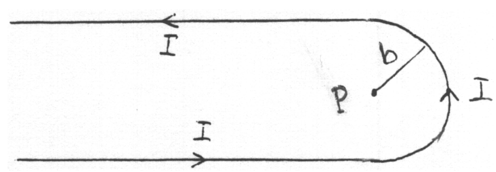
From the Lorentz force law

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v}\times\mathbf{B} \quad ,$$

we can account for this extra force with a magnetic field out of the page

$$\mathbf{B} = \beta E\,\hat{\mathbf{z}} \quad .$$

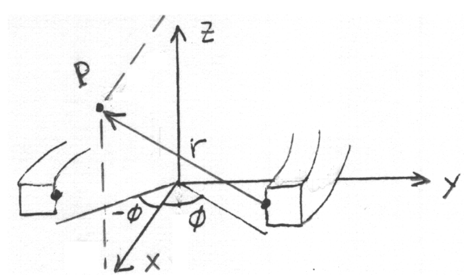
5 *Purcell 6.4* *A long wire is bent into the hairpin like shape shown. find an exact expression for the magnetic field at the point P which lies at the center of the half-circle.*



In the integral for the magnetic field of an infinite wire, each half of the wire contributes the same amount in the same direction, so for each of the half wires, we can take half of the formula for an infinite wire. Also, the half circle will contribute half of the ring formula. All contributions point out of the page so

$$B_z = \frac{1}{2}\frac{2\pi I}{cb} + \frac{1}{2}\frac{2I}{cb} + \frac{1}{2}\frac{2I}{cb} = \frac{(2+\pi)I}{bc} \quad .$$

6 *Purcell 6.14* *A coil is wound evenly on a torus of rectangular cross section. There are N turns of wire in all. Assume that the current on the surface of the torus flows exactly radially on the annular end faces, and exactly longitudinally on the inner and outer cylindrical surfaces. Show that the magnetic field everywhere would be circumferential. Second, prove that the field is zero at all points outside the torus, including the interior of the central hole. Third, find the magnitude of the field inside the torus as a function of radius.*



We set a coordinate system so the point P under consideration is in the $z-x$ plane with coordinates $(x_o, 0, z_o)$. Consider a current loop at an angle ϕ from the x axis. One small section of the loop has coordinates $(r\cos\phi, r\sin\phi, z)$ and

$$\mathbf{r} = (x_o - r\cos\phi, -r\sin\phi, z_o - z) \quad .$$

The direction of this little current is arbitrary within the plane so the current direction vector can be written

$$\mathbf{I} = I_r\,\hat{\mathbf{r}} + I_z\,\hat{\mathbf{z}} = (I_r \cos\phi, I_r \sin\phi, I_z) \quad .$$

Then the contribution to the magnetic field will be in the direction

$$\mathbf{I} \times \mathbf{r} = [\sin\phi(I_r(z_o - z) + rI_z)]\,\hat{\mathbf{x}} + [I_z(x_o - r\cos\phi) - I_r \cos\phi(z_o - z)]\,\hat{\mathbf{y}} - I_r x_o \sin\phi\,\hat{\mathbf{z}} \quad .$$

However, for each section of current, there is a similar section that is identical except that $\phi \rightarrow -\phi$. We see that the x and z components change sign. Therefore, these components cancel out and the net field at the point P is in the y direction. For our coordinate system, this in the circumferential direction.

Once we have this information, it is easy to use Ampere's law.

$$\int \mathbf{B} \cdot d\mathbf{l} = 2\pi r B = \frac{4\pi}{c} NI$$

$$B = \frac{2NI}{cr} \qquad\qquad a < r < b$$

If we are outside the solenoid, the enclosed current will be zero and so the field is zero.

7 *Purcell 6.18* *Two long coaxial aluminum cylinders are charged to a potential difference of* 50 *statvolts. The inner cylinder has an outer diameter of* 6 *cm, the outer cylinder an inner diameter of* 8 *cm. With the outer cylinder stationary the inner cylinder is rotated around its axis at a constant frequency of* 30 *Hz. Describe the magnetic field this produces and determine its intensity in gauss. What if both cylinders are rotated in the same direction at* 30 *Hz?*

The capacitance for two coaxial cylinders is

$$C = \frac{L}{2\ln(b/a)} \quad .$$

Assuming we have equal amounts of positive and negative charge on the inside and outside respectively, we can find the charge per unit length.

$$\frac{Q}{L} = \frac{CV}{L} = \frac{V}{2\ln(b/a)} = 87\ esu/cm$$

The spinning charged cylinders are essentially perfect solenoids. The field of a long solenoid is constant inside and practically zero outside. The solenoid formula is

$$B = \frac{4\pi I n}{c} \quad ,$$

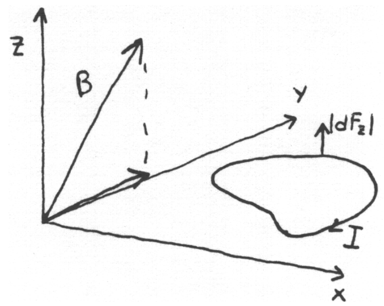
where n is the number of turns per unit length. The quantity nI can be thought of as the charge per unit time per unit length passing through a line running parallel to the cylinder. You should be able to convince yourself that this quantity in our case is $Q\nu/L$, where ν is the frequency of revolution. With just the inner cylinder rotating with $\nu = 30\ Hz$,

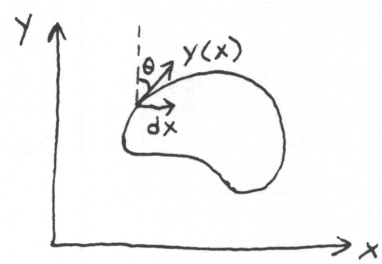
$$B = \frac{4\pi Q\nu}{cL} = \frac{4\pi(87\ esu/cm)(30\ Hz)}{3 \times 10^{10}\ cm/s} = 1.1 \times 10^{-6}\ gauss \qquad\qquad r < a \quad ,$$

and the field elsewhere is zero. With a clockwise rotation as shown, the field is into the page.

If the outer cylinder rotates in the same direction, it will produce a field with the same magnitude but opposite direction. These two fields will cancel for $r < a$, but for $a < r < b$, the field is 1.1×10^{-6} *gauss* out of the page. It is zero again for $r > b$.

8 *Purcell 6.22 A constant* $\mathbf{B}$ *field lies in the* $y-z$ *plane. An arbitrary current loop lies in the* $x-y$ *plane. Show, by calculating the torque about the* x *axis, that the torque on the current loop can be written* $\mathbf{N} = \mathbf{m} \times \mathbf{B}$*, where the magnetic moment* $\mathbf{m}$ *of the loop is defined as a vector of magnitude* IA/c *where* I *is the current in esu/s and a is the area of the loop in* cm^2*, and the direction of the vector is normal to the loop with a right-hand relation to the current. What about the net force on the loop?*





The force on a small section of loop is

$$d\mathbf{F} = \frac{1}{c} I d\mathbf{l} \times \mathbf{B} \quad .$$

Only the part of the force in the z direction will contribute to the torque about the x axis. We split the magnetic field into two parts in the z and y directions. $d\mathbf{l} \times (B_z \hat{\mathbf{z}})$ is in the $x-y$ plane, so we only need the y part of $\mathbf{B}$.

$$dF_z = \frac{1}{c} I B_y \sin\theta \, dl$$

From the figure we see that $dl \sin\theta = dx$ so that $dF_z = \frac{1}{c} I B_y \, dx$ and the torque is

$$N_x = \int y \, dF_z = \frac{1}{c} I B_y \int_{loop} y(x) \, dx \quad .$$

This integral if we go forward along the top of the loop and back along the bottom is the area of the loop and

$$N_x = \frac{1}{c} I B_y a \quad .$$

With $m = Ia/c$ and pointing in the $-z$ direction,

$$\mathbf{m} \times \mathbf{B} = \mathbf{m} \times (B_y \hat{\mathbf{y}}) = m B_y \hat{\mathbf{x}} = \frac{1}{c} I B_y a \, \hat{\mathbf{x}} \quad .$$

For a constant field there is no net force because

$$\mathbf{F} = \int d\mathbf{F} = \int \frac{1}{c} I \, d\mathbf{l} \times \mathbf{B} = -\frac{1}{c} I \mathbf{B} \times \int d\mathbf{l}$$

and $\int d\mathbf{l}$ around the loop is zero.

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PROBLEM SET 10

1. Purcell problem 6.26

2. Purcell problem 6.28

3. Purcell problem 6.32

4. Purcell problem 7.4

5. Purcell problem 7.9

6. Purcell problem 7.11

7. Purcell problem 7.14

8. Purcell problem 7.16

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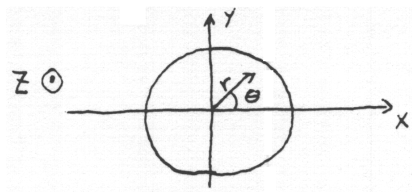
SOLUTION TO PROBLEM SET 10
Solutions by P. Pebler

1 *Purcell 6.26 A round wire of radius r_o carries a current I distributed uniformly over the cross section of the wire. Let the axis of the wire be the z axis, with $\hat{\mathbf{z}}$ the direction of the current. Show that a vector potential of the form $\mathbf{A} = A_o(x^2 + y^2)\hat{\mathbf{z}}$ will correctly give the magnetic field $\mathbf{B}$ of this current at all points inside the wire. What is the value of the constant A_o?*



The magnetic field is the curl of the vector potential.

$$\mathbf{B} = \nabla \times \mathbf{A} = 2A_o y\,\hat{\mathbf{x}} - 2A_o x\,\hat{\mathbf{y}}$$

If we use plane polar coordinates in the $x - y$ plane,

$$\mathbf{B} = 2A_o r(\sin\phi\,\hat{\mathbf{x}} - \cos\phi\,\hat{\mathbf{y}}) = -2A_o r\,\hat{\phi} \quad .$$

We know that the magnetic field circles in the counterclockwise direction for a current coming out of the page. We can find the magnitude from Ampere's law.

$$2\pi r B = \frac{4\pi}{c} I \frac{r^2}{r_o^2}$$

$$B = \frac{2I}{c r_o^2} r$$

The vector potential $\mathbf{A}$ therefore works with the constant

$$A_o = -\frac{I}{c r_o^2} \quad .$$

2 *Purcell 6.28 A proton with kinetic energy 10^{16} eV ($\gamma = 10^7$) is moving perpendicular to the interstellar magnetic field which in that region of the galaxy has a strength 3×10^{-6} gauss. What is the radius of curvature of its path and how long does it take to complete one revolution?*

Magnetic forces do no work. They can only change the direction of the momentum. Because the force is perpendicular to the velocity, we can instantaneously think about the motion as being along a circle of some radius R. Because the field and velocity are perpendicular, the magnitude of the force is

$$F = \frac{evB}{c} \quad .$$

If this were a non-relativistic problem, we could find the radius R by equating the force with mv^2/R. In the relativistic case, this formula turns out to be correct with the replacement $m \to \gamma m$, but this is something that must be proved. If we wait a time Δt, the momentum will swing through some angle $\Delta\theta$, and $|\Delta\mathbf{p}| = p\Delta\theta$. (Please note that Δp is not the same thing as $|\Delta\mathbf{p}|$.) This angle $\Delta\theta$ will also be the angle of the circle we go through in this time. Therefore, in the infinitesimal limit $\Delta \to d$, $v = \omega r$. Consequently,

$$\left|\frac{d\mathbf{p}}{dt}\right| = p\frac{d\theta}{dt} = p\omega = \frac{pv}{R} \quad .$$

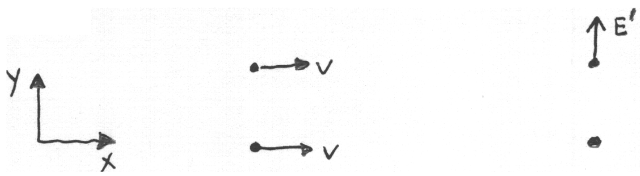
Equating this with the force,

$$R = \frac{pc}{eB} \simeq \frac{\gamma mc^2}{eB} = \frac{10^7(1.5\times10^{-3}\ ergs)}{(4.8\times10^{-10}\ esu)(3\times10^{-6}\ gauss)} = 1\times10^{19}\ cm \quad .$$

The period is

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi R}{v} \simeq \frac{2\pi R}{c} = 2.1\times10^9\ s \quad .$$

3 *Purcell 6.32 Two electrons move along parallel paths, side by side, with the same speed v. The paths are a distance r apart. Find the force acting on one of them in two ways. First, find the force in the rest frame of the electrons and tranform this force back to the lab frame. Second, calculate the force from the fields in the lab frame. What can be said about the force between them in the limit $v \to c$?*



In the particle rest frame, the field is just the Coulomb field and the force magnitude is e^2/r^2. If we use the transformation formulas (14) in Purcell, the primed frame must be the particle rest frame.

$$\mathbf{F} = \frac{1}{\gamma}\frac{e^2}{r^2}\,\hat{\mathbf{y}}$$

To find the fields in the lab frame, it is easiest to transform them back from the rest frame where

$$\mathbf{E}' = -\frac{e}{r^2}\,\hat{\mathbf{y}} \qquad \mathbf{B}' = 0 \quad .$$

Please note that most transformation formulas found in books assume that the primed frame is moving in the positive x direction of the unprimed frame. If you wish to use these formulas verbatim, you must choose your frames correctly. Here the particles are going to the right so the rest frame is the primed frame. To switch back to the lab frame, we need the inverse of the equations (6.60) in Purcell. We can accomplish this by simply switching the primes and the sign of $\boldsymbol{\beta}$. Then

$$\mathbf{E}_\parallel = \mathbf{E}'_\parallel = \mathbf{0} \quad ,$$

$$\mathbf{E}_\perp = \gamma(\mathbf{E}'_\perp - \boldsymbol{\beta}\times\mathbf{B}'_\perp) = \gamma\mathbf{E}'_\perp = -\gamma\frac{e}{r^2}\,\hat{\mathbf{y}} \quad ,$$

$$\mathbf{B}_\parallel = \mathbf{B}'_\parallel = \mathbf{0} \quad ,$$

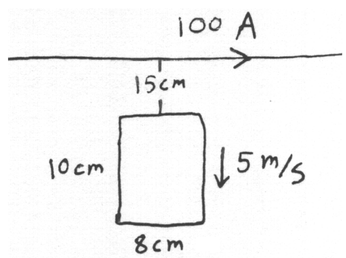
$$\mathbf{B}_\perp = \gamma(\mathbf{B}'_\perp + \boldsymbol{\beta} \times \mathbf{E}'_\perp) = -\gamma\beta \frac{e}{r^2}\,\hat{\mathbf{z}} \quad .$$

The force is then

$$\mathbf{F} = (-e)\mathbf{E} + \frac{(-e)}{c}\mathbf{v} \times \mathbf{B} = \gamma\frac{e^2}{r^2}\,\hat{\mathbf{y}} - \gamma\beta^2\frac{e^2}{r^2}\,\hat{\mathbf{y}} = \gamma\frac{e^2}{r^2}(1-\beta^2)\,\hat{\mathbf{y}} = \frac{1}{\gamma}\frac{e^2}{r^2}\,\hat{\mathbf{y}} \quad .$$

In the limit $v \to c$, we see that $\mathbf{F} \to \mathbf{0}$.

4 *Purcell 7.4* *Calculate the electromotive force in the moving loop in the figure at the instant when it is in the position there shown. Assume the resistance of the loop is so great that the effect of the current in the loop itself is negligible. Estimate very roughly how large a resistance would be safe, in this respect. Indicate the direction in which current would flow in the loop, at the instant shown.*



We first calculate the flux. We will define the positive direction to be into the page. The field is that of a wire. The current is given in SI, so we must use SI formulas.

$$\int \mathbf{B} \cdot d\mathbf{a} = \int_x^{x+L} \frac{\mu_o I}{2\pi r} w\,dr = \frac{\mu_o I w}{2\pi} \ln \frac{x+L}{x}$$

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = -\frac{\mu_o I w}{2\pi}\frac{x}{x+L}\left(\frac{v}{x} - \frac{(x+L)v}{x^2}\right) = \frac{\mu_o I w}{2\pi}\frac{Lv}{x(x+L)}$$

$$\mathcal{E} = 2.1 \times 10^{-5}\ V$$

By choosing into the page as positive for flux, we have also defined clockwise as the positive way to go around the loop. Since $\mathcal{E}$ is positive, the induced current will be clockwise.

5 *Purcell 7.9* *Derive an approximate formula for the mutual inductance of two circular rings of the same radius a, arranged like wheels on the same axle with their centers a distance b apart. Use an approximation good for $b \gg a$.*

From Purcell Eq. 6.41 (where a and b are interchanged relative to this problem) the field along the axis of a ring is

$$B_z = \frac{2\pi a^2 I}{c(a^2+z^2)^{3/2}} \quad .$$

We may use the information $b \gg a$, and approximate the z component of the field everywhere in the second loop as

$$B_z = \frac{2\pi a^2 I}{c(a^2+b^2)^{3/2}} \simeq \frac{2\pi a^2 I}{cb^3} \quad ,$$

so the flux is

$$\Phi_B = \frac{2\pi a^2 I}{cb^3}\pi a^2 = \frac{2\pi^2 a^4}{cb^3} I \quad .$$

The induced emf is then

$$\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\Phi_B = -\frac{2\pi^2 a^4}{c^2 b^3}\frac{dI}{dt} \quad ,$$

and the mutual induction is

$$M = \frac{2\pi^2 a^4}{c^2 b^3} \quad ,$$

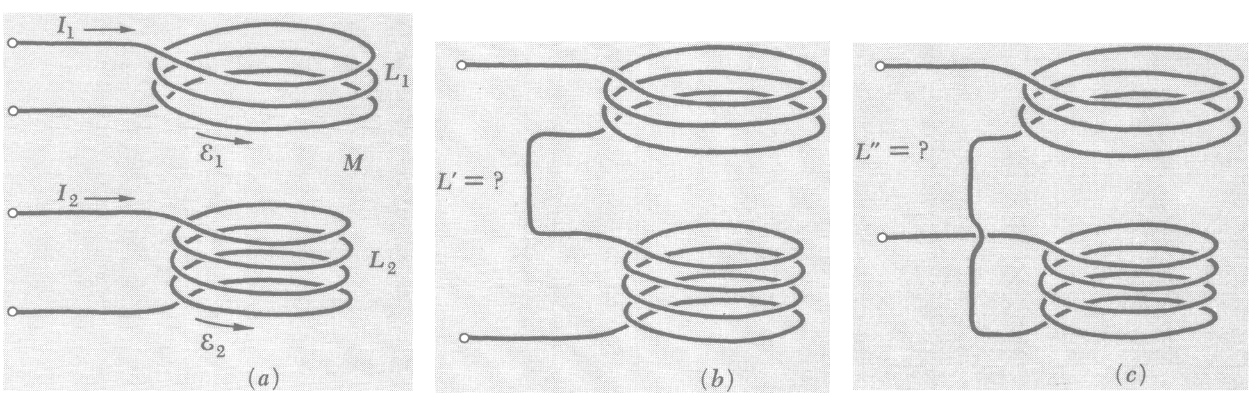
in cgs units. If you use SI formulas this becomes

$$M = \frac{\mu_o}{4\pi}\frac{2\pi^2 a^4}{b^3} \quad .$$

6 *Purcell 7.11* *Two coils with self-inductances L_1 and L_2 and mutual inductance M are shown with the positive direction for current and electromotive force indicated. The equations relating currents and emf's are*

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} \pm M\frac{dI_2}{dt} \qquad\qquad \mathcal{E}_2 = -L_2\frac{dI_2}{dt} \pm M\frac{dI_1}{dt} \quad .$$

Given that M is always to be taken as positive, how must the signs be chosen in these equations? What if we had chosen the other direction for positive current and emf in the lower coil? Now connect the two coils together as in b. What is the inductance L' of this circuit? What is the inductance L'' of the circuit formed as shown in c? Which circuit has the greater self-inductance? Considering that the self-inductance of any circuit must be a positive quantity, see if you can deduce anything concerning the relative magnitudes of L_1, L_2, and M.



Imagine first that the current I_2 is positive and increasing so that $dI_2/dt > 0$. In this case the magnetic field due to coil 2 will point up through coil 1. As the current I_2 increases, the field it creates will increase and the flux up through coil 1 will increase. By using Lenz's law, we find we need an induced current that will create a magnetic field that will oppose this *change* in the flux. In this case, the field should point down through coil 1. To do this the induced current must flow in the negative direction as it is defined for coil 1. Thus, the induced emf must be negative and we need the negative sign. The same argument will tell you to choose the negative sign in the second equation also. (You should go through it yourself however.)

If the sign convention for coil 2 had been switched, the same argument would switch the sign in both equations. (Do it yourself though.)

With the circuit in b, since both emf positive directions point in the same way, the total emf across the new circuit is

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} - L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} \quad .$$

We also have $I = I_1 = I_2$ so that

$$\mathcal{E} = -(L_1 + L_2 + 2M)\frac{dI}{dt} \quad ,$$

and the self inductance is

$$L' = L_1 + L_2 + 2M \quad .$$

With the circuit in c, the sign conventions "conflict" so that

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} + L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} \quad ,$$

but with $I = I_1 = -I_2$ so that

$$\mathcal{E} = -(L_1 + L_2 - 2M)\frac{dI}{dt} \quad ,$$

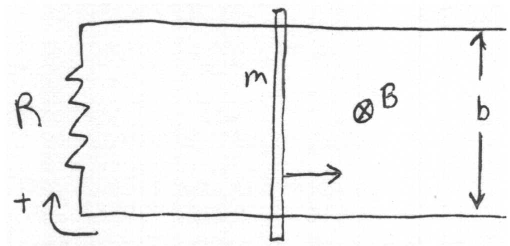
and the self inductance is

$$L'' = L_1 + L_2 - 2M \quad .$$

If the self inductance of a coil were negative, the circuit would be unstable – any change in current would result in more current the same direction which would built indefinitely. Therefore we must have $L'' > 0$ and

$$M \leq \frac{L_1 + L_2}{2} \quad .$$

7 *Purcell 7.14 A metal crossbar of mass m slides without friction on two long parallel conducting rails a distance b apart. A resistor R is connected across the rails at one end; compared with R, the resistance of bar and rails is negligible. There is a uniform field* $\mathbf{B}$ *perpendicular to the plane of the figure. At time $t = 0$ the crossbar is given a velocity v_o toward the right. What happens then? Does the rod ever stop moving? If so, when? How far does it go? How about conservation of energy?*



Let us assume the magnetic field is into the page, and let's make that positive so that clockwise is positive for the loop. The flux is then

$$\Phi_B = bxB \quad .$$

The emf (in SI) is

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = -bvB = IR \quad ,$$

so the current is counterclockwise. The bar will feel a force due to the magnetic charges moving through it. The force is

$$\mathbf{F} = |I|\mathbf{L} \times \mathbf{B} = -|I|bB\,\hat{\mathbf{x}} \quad .$$

We can solve for the motion using $F = ma$.

$$m\frac{dv}{dt} = -|I|bB = -\frac{b^2B^2}{R}v$$

$$v(t) = v_o e^{-b^2B^2t/mR}$$

The bar never stops moving. It will approach the distance

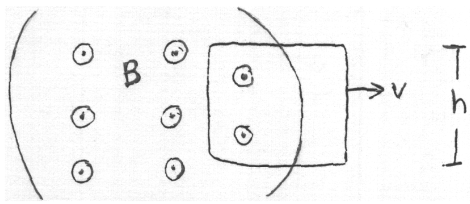
$$d = \int_0^\infty v_o e^{-b^2B^2t/mR} dt = \frac{mRv_o}{b^2B^2} \quad .$$

The lost kinetic energy is dissipated by the resistor.

$$I = \frac{b}{B}Rv$$

$$U = \int_o^\infty I^2 R dt = \int_o^\infty \frac{b^2B^2}{R} v_o^2 e^{-2b^2B^2t/mR} = \frac{b^2B^2}{R} v_o^2 \frac{mR}{2b^2B^2} = \frac{1}{2}mv_o^2$$

8 *Purcell 7.16 The shaded region represents the pole of an electromagnet where there is a strong magnetic field perpendicular to the plane of the paper. The rectangular frame is made of* 5 *mm diameter aluminum. suppose that a steady force of* 1 *N can pull the frame out in* 1 *s. If the force is doubled, how long does it take? If the frame is made of* 5 *mm brass, with about twice the resistivity, what force is needed to pull it out in* 1 *s? If the frame were* 1 *cm diameter aluminum, what force is needed to pull it out in* 1 *s? Neglect inertia of the frame and assume it moves with constant velocity.*



If we assume a constant velocity, the force necessary to pull out the loop will be equal in magnitude to the magnetic force on the loop. We will ignore signs here. The net force will be on the left wire of the frame.

$$F = |I|hB$$

The magnetic flux will be something like $\Phi = LxB$, where x is the length of loop in the field. Then the emf is

$$|\mathcal{E}| = \left| -\frac{d}{dt}\Phi \right| = hvB \quad .$$

The current is $|I| = |\mathcal{E}|/R$ and the resistance is $R = \rho L/A = \rho L/\pi r^2$, where L is the total length of the loop and r is the radius of the wire of which it is made.

$$F = hBhBv\frac{A}{\rho L} \propto \frac{vr^2}{\rho}$$

If the force is doubled, the speed doubles and it takes half the time or 0.5 s. If the resistivity doubles with the same speed, the force is halved so that $F = 1$ N. If the radius doubles with the same speed, the force is four times as great or 4 N.

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PROBLEM SET 11

1. Purcell problem 7.21. Assume that $b_2 \gg a_2$ and $b_2 \gg b_1$.

2. Purcell problem 7.22. Assume that the charge is distributed uniformly around the ring.

3. Purcell problem 7.23

4. Purcell problem 7.29

5. Purcell problem 8.5. For simplicity, if you wish, in parts (a) and (b) you may assume that the battery is connected only until just before the switch is closed.

6. Purcell problem 8.7

7. Purcell problem 8.11

8. Purcell problem 8.16. An infinite L-C ladder of the type studied in this problem is called a "lumped-element delay line". It is of great practical importance when one needs to delay an analog electrical pulse by a longer time than would be conveniently achieved by using the finite speed ($\approx c$) of propagation in a coaxial cable.

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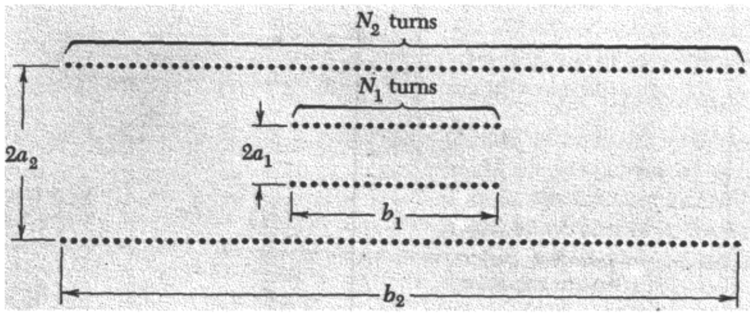
SOLUTION TO PROBLEM SET 11
Solutions by P. Pebler

1 *Purcell 7.21 A solenoid of radius a_1 and length b_1 is located inside a longer solenoid of radius a_2 and length b_2. The total number of turns is N_1 in the inner coil and N_2 on the outer. Work out a formula for the mutual inductance M.*



The mutual inductances M_{12} and M_{21} are equal, so we are free to calculate the inductance with whatever coil is more convenient. We find the flux through the inner coil. If we assume $b_2 \gg b_1$, the field through coil 1 will be fairly uniform and with the sign conventions shown,

$$\mathbf{B}_2 = \frac{\mu_o N_2 I_2}{b_2}\hat{\mathbf{x}} \quad ,$$

along the common axis of both coils. Since there are N_1 loops in this coil, the flux through all of them is

$$\phi_{12} = \pi a_1^2 N_1 \frac{\mu_o N_2 I_2}{b_2} \quad ,$$

and the induced emf is

$$\mathcal{E}_{12} = -\frac{d}{dt}\Phi_{12} = -\frac{\mu_o \pi a_1^2 N_1 N_2}{b_2}\frac{dI_2}{dt} \quad ,$$

and the mutual inductance is

$$M = \frac{\mu_o \pi a_1^2 N_1 N_2}{b_2} \quad .$$

2 *Purcell 7.22 A thin ring of radius a carries a static charge q. This ring is in a magnetic field of strength B_o, parallel to the ring's axis, and is supported so that it is free to rotate about that axis. If the field is switched off, how much angular momentum will be added to the ring? If the ring has mass m, show that it will acquire an angular velocity $\omega = qB_o/2mc$.*

We'll assume the charge is uniformly distributed around the ring, with linear density $\lambda = q/2\pi a$. Then the torque about the z axis is

$$\tau_z = \int d\tau_z = \int a\lambda \mathbf{f}\cdot d\mathbf{l} = \frac{q}{2\pi}\int \mathbf{E}\cdot d\mathbf{l} \;= \frac{q}{2\pi}\mathcal{E} \; .$$

The emf is

$$\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\Phi \quad ,$$

so that

$$\tau_z = \frac{dL_z}{dt} = -\frac{q}{2\pi c}\frac{d\Phi}{dt} \quad ,$$

and

$$L_{zf} - L_{zi} = -\frac{q}{2\pi c(\Phi_f - \Phi_i)} = \frac{q}{2\pi c}\pi a^2 B_o \quad ,$$

$$L_{zf} = \frac{qa^2 B_o}{2c} \quad .$$

The moment of inertia of the ring is $I = ma^2$, and

$$L_{zf} = I\omega = ma^2\omega = \frac{qa^2 B_o}{2c} \quad ,$$

$$\omega = \frac{qB_o}{2mc} \quad ,$$

equal to half the cyclotron frequency of a particle with mass m and charge q in a field B_0.

3 *Purcell 7.23 There is evidence that a magnetic field exists in most of the interstellar space with a strength between 10^{-6} and 10^{-5} gauss. Adopting 3×10^{-6} gauss as a typical value, find the total energy stored in the magnetic field of the galaxy. Assume the galaxy is a disk roughly 10^{23} cm in diameter and 10^{21} cm thick. Assuming stars radiate about 10^{44} ergs/s, how many years of starlight is the magnetic energy worth?*

The magnetic energy is

$$U = \frac{1}{8\pi}\int B^2\, dV = \frac{1}{8\pi}(3 \times 10^{-6}\ gauss)^2(10^{21}\ cm)\pi(10^{23}/2\ cm)^2 = 3 \times 10^{54}\ ergs \quad ,$$

and this is

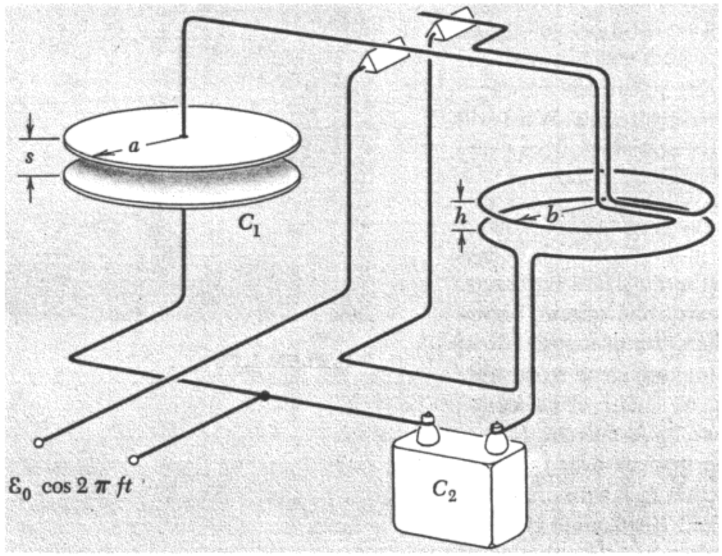
$$\frac{3 \times 10^{54}\ ergs}{10^{44}\ ergs/s} = 3 \times 10^{10}\ s = 900\ yr$$

of starlight.

4 *Purcell 7.29 Consider the arrangement shown. The force between capacitor plates is balanced against the force between parallel wires. An alternating voltage of frequency f is applied to the capacitors C_1 and C_2. The charge flowing through C_2 constitutes the current through the rings. Suppose the time-average downward force on C_2 exactly balances the time averaged force on the wire loop. Show that under these conditions the constant c is*

$$c = (2\pi)^{3/2} a \left(\frac{b}{h}\right)^{1/2} \left(\frac{C_2}{C_1}\right) f \quad .$$

Assume $h \ll b$ and ignore the self-inductance of the wire loop.



The electric field in capacitor C_1 is

$$E = \frac{1}{s}\mathcal{E}_o \cos \omega t \quad ,$$

and the charge on it is

$$Q = C_1 V = C_1 \mathcal{E}_o \cos \omega t \quad .$$

The downward force will be the charge times the electric field due to the bottom plate. This field will be half of the total field.

$$F_1 = \frac{1}{2} E Q = \frac{\mathcal{E}_o^2 C_1}{2s} \cos^2 \omega t$$

Because the capacitance is

$$C_1 = \frac{\pi a^2}{4\pi s} = \frac{a^2}{4s} \quad ,$$

and the time average of $\cos^2$ is $1/2$, we may rewrite this as

$$\bar{F_1} = \frac{\mathcal{E}_o^2 C_1^2}{a^2} \quad .$$

If we have $h \ll b$, we can use the force of two long wires. The field due to a wire is

$$B = \frac{2|I|}{cr} \quad ,$$

and the total force on the wire will be

$$F_2 = \frac{1}{c}|I| L B = \frac{1}{c} 2\pi b \frac{2I^2}{ch} = \frac{4\pi b I^2}{c^2 h} \quad .$$

The current is the time derivative of the charge on capacitor 2.

$$I = \frac{dQ_2}{dt} = \frac{d}{dt}(C_2 V) = -C_2 \mathcal{E}_o 2\pi f \sin \omega t$$

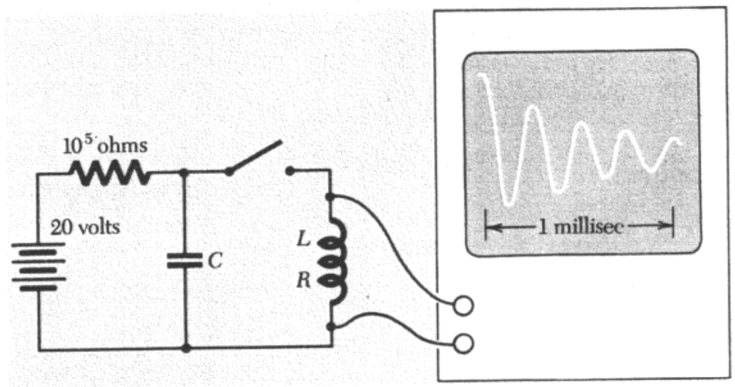
$$F_2 = \frac{4\pi b}{c^2 h} C_2^2 \mathcal{E}_o^2 (2\pi f)^2 \sin^2 \omega t$$

The time average of $\sin^2$ is also $1/2$ so

$$\bar{F}_2 = \frac{8\pi^3 b C_2^2 f^2 \mathcal{E}_o^2}{c^2 h} = \bar{F}_1 = \frac{C_1^2 \mathcal{E}_o^2}{a^2} \quad ,$$

$$c = (2\pi)^{3/2} a \left(\frac{b}{h}\right)^{1/2} \left(\frac{C_2}{C_1}\right) f \quad .$$

5 *Purcell 8.5* *The coil in the circuit shown in the diagram is known to have an inductance of* 0.01 *henry. when the switch is closed, the oscilloscope sweep is triggered. Determine the capacitance* C*. Estimate the value of the resistance* R *of the coil. What is the magnitude of the voltage across the oscilloscope input a long time, say 1 second after the switch has been closed?*



Parts (a.) and (b.) of this problem may be approximated by assuming that the battery is disconnected when the switch is closed. However, the problem actually is not too bad with the battery connected, so we will solve the original problem. The answers are the same except for the final voltage.

If you work out the equation for the charge on the capacitor C, you will find

$$L\frac{d^2Q}{dt^2} + \left(R_2 + \frac{L}{R_1 C}\right)\frac{dQ}{dt} + \left(\frac{R_1 + R_2}{R_1}\right)\frac{Q}{C} = \frac{VR_2}{R_1} \quad .$$

If we assume that the resistance R_2 of the inductor is much less that R_1, this becomes the LCR circuit equation. From the trace we see that

$$\omega = \frac{2\pi \cdot 4}{10^{-3}\ s} = 8\pi \times 10^3\ Hz \quad .$$

For low damping the capacitance is approximately

$$C = \frac{1}{L\omega^2} = 1.6 \times 10^{-7}\ F \quad .$$

Also from the trace, the amplitude falls off by a factor of e in about $0.5 \times 10^{-3}\ s$.

$$e^{-Rt/2L} = e^{-1}$$

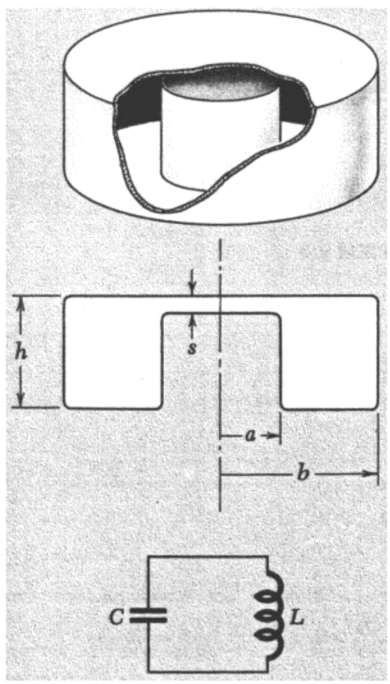
$$R = \frac{2L}{t} = 40\ ohms$$

If we wait one second, a long time, things will settle so that a steady current passes through the inductor and the voltage across it will be due to the resistance. If the current is I, the voltage is $V_2 = IR_2$ and

$$20\ V = I(10^5\ ohm + R_2) = V_2 \left(\frac{10^5 + 40}{40}\right) \quad ,$$

$$V = 8\ mV \quad .$$

6 *Purcell 8.7 A resonant cavity of the form illustrated is an essential part of many microwave oscillators. It can be regarded as a simple LC circuit. The inductance is that of a toroid with one turn. Find an expression for the resonant frequency of this circuit and show by a sketch the configuration of the magnetic and electric fields.*



The inner narrow circle will act as the capacitor while the outer ring is the solenoid. For a single turn toroid, the inductance is

$$L = \frac{2h}{c^2} \ln \frac{b}{a} \quad ,$$

and for the parallel plate capacitor,

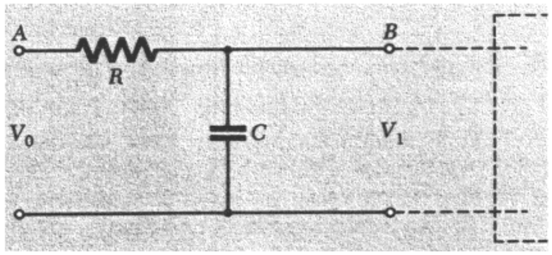
$$C = \frac{\pi a^2}{4\pi s} = \frac{a^2}{4s} \quad ,$$

so that

$$\omega = \sqrt{\frac{1}{LC}} = \frac{c}{a}\sqrt{\frac{2s}{h \ln(b/a)}} \quad .$$

The electric field is concentrated in the circular gap, where its direction is vertical; the magnetic field in the toroidal cavity is azimuthal in direction, with magnitude proportional to r^{-1}.

7 *Purcell 8.11* *An alternating voltage $V_o \cos \omega t$ is applied to the terminals at A. The terminals at B are connected to an audio amplifier of very high input impedance. Calculate the ratio $|V_1|^2/V_o^2$, where V_1 is the complex voltage at terminals B. Choose values for R and C to make $|V_1|^2/V_o^2 = 0.1$ for a 5000 hz signal. Show that for sufficiently high frequencies, the signal power is reduced by a factor 1/4 for every doubling of the frequency.*



Since the impedance on the right is very large, the impedance of the circuit is approximately

$$Z = R + \frac{1}{i\omega C} \quad ,$$

and the magnitude is

$$|Z| = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \quad .$$

This gives us the magnitude of the complex current.

$$I_o = \frac{V_o}{|Z|} = \frac{V_o}{\sqrt{R^2 + 1/\omega^2 C^2}}$$

The impedance of just the capacitor is

$$Z_C = \frac{1}{i\omega C} \qquad |Z_C| = \frac{1}{\omega C} \quad .$$

This gives us the magnitude of the voltage across C.

$$|V_1| = I_o |Z_C| = \frac{V_o}{\sqrt{\omega^2 R^2 C^2 + 1}}$$

$$\frac{|V_1|^2}{V_o^2} = \frac{1}{\omega^2 R^2 C^2 + 1}$$

We would like an R and C such that

$$\frac{1}{[2\pi(5000\ hz)]^2 R^2 C^2 + 1} = 0.1 \quad ,$$

which can be done with many values of R and C, for example

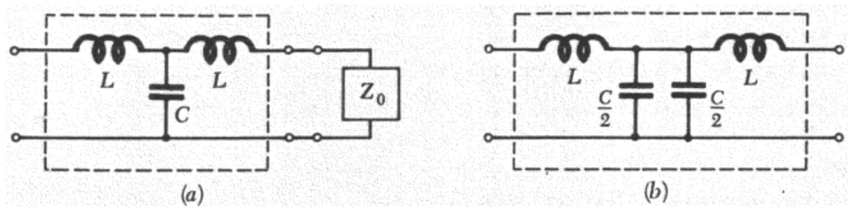
$$R = 100\ ohm \qquad C = 1\ \mu F \quad .$$

For sufficiently high frequencies we have

$$P \propto |V_1|^2 \propto \omega^{-2} \quad .$$

A reduction of signal power by a factor 8 rather than 4 per octave can be achieved by substituting an inductor L for the resistor R.

8 *Purcell 8.16* *An impedance Z_o is to be connected to the terminals on the right. For given frequency ω find the value which Z_o must have if the resulting impedance between the left terminals is Z_o. The required Z_o is a pure resistance R_o provided $\omega^2 < 2/LC$. What is Z_o in the special case $\omega = \sqrt{2/LC}$?*



We combine the impedances like resistances so that the total impedance is

$$Z = Z_L + \frac{1}{\frac{1}{Z_C} + \frac{1}{Z_L + Z_o}} \quad ,$$

with $Z_L = i\omega L$ and $Z_C = 1/i\omega C$. We set this equal to Z_o and simplify to obtain

$$Z_o = \sqrt{-\omega^2 L^2 + 2L/C} \quad .$$

This will be pure real and thus a pure resistance if

$$-\omega^2 L^2 + 2\frac{L}{C} > 0 \quad ,$$

$$\omega^2 < \frac{2}{LC} \quad .$$

In the special case $\omega = \sqrt{2/LC}$, we have $Z_o = 0$.

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PROBLEM SET 12

1. Purcell problem 9.3

2. Purcell problem 9.5

3. Purcell problem 9.9

4. Purcell problem 9.10

5. Purcell problem 10.7

6. Purcell problem 10.17

7. Purcell problem 10.19

8. Purcell problem 10.21

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SOLUTION TO PROBLEM SET 12

Solutions by P. Pebler

1 *Purcell 9.3* *A free proton was at rest at the origin before the wave*

$$\mathbf{E} = \frac{(5\ statvolt/cm)\,\hat{\mathbf{y}}}{1+[k(x+ct)]^2} \qquad\qquad \mathbf{B} = \frac{(-5\ gauss)\,\hat{\mathbf{z}}}{1+[k(x+ct)]^2}$$

came past with $k = 1\ cm^{-1}$. *Where would you expect to find the proton after* $1\ \mu s$*? The proton mass is* $1.6\times10^{-24}\ g$.

To begin, we will neglect the magnetic force and see later if this is justified. In this case, the impulse due to the electric force will be in the y direction. The pulse only has an appreciable magnitude for a few nanoseconds, so we may extend the integral to infinity.

$$\Delta\mathbf{p} = \int \mathbf{F}_e dt = e\,(5\ statvolt/cm)\hat{\mathbf{y}}\int_{-\infty}^{\infty}\frac{dt}{1+(kct)^2} = \frac{(5\ statvolt/cm)\pi e}{kc}\,\hat{\mathbf{y}}$$

$$\Delta\mathbf{p} = (2.5\times10^{-19}\ g\,cm/s)\,\hat{\mathbf{y}}$$

This corresponds to a speed of

$$v = 1.6\times10^{5}\ cm/s \quad .$$

From the Lorentz force law

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v}\times\mathbf{B} \quad ,$$

we see that because the electric and magnetic fields have the same strength, the magnetic force is smaller by

$$F_B \simeq \frac{v}{c}F_e \simeq (5\times10^{-4})F_e \quad ,$$

so our approximation is pretty good. The acceleration while the pulse is passing occurs for a very small time, so the position of the proton after one microsecond is essentially

$$y = (1.6\times10^{5}\ cm/s)(1\times10^{-6}\ s) = 0.16\ cm \quad .$$

2 *Purcell 9.5* *Consider the wave in free space*

$$E_x = 0 \qquad E_y = E_o\sin(kx-\omega t) \qquad E_z = 0$$

$$B_x = 0 \qquad B_y = 0 \qquad B_z = -E_o\sin(kx-\omega t) \quad .$$

Show that this field can satisfy Maxwell's equations if ω *and* k *are related in a certain way. Suppose* $\omega = 10^{10}\ hz$ *and* $E_o = 0.05\ statvolt/cm$. *What is the wavelength in cm? What is the energy density in* $ergs/cm^3$*, averaged over a large region? From this calculate the power density, the energy flow in* $ergs/cm^2s$.

We see immediately that $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$. It's also easy to calculate

$$\nabla \times \mathbf{E} = kE_o \cos(kx + \omega t)\,\hat{\mathbf{z}} \quad ,$$

$$\nabla \times \mathbf{B} = kE_o \cos(kx + \omega t)\,\hat{\mathbf{y}} \quad ,$$

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \frac{\omega}{c} E_o \cos(kx + \omega t)\,\hat{\mathbf{z}} \quad ,$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{\omega}{c} E_o \cos(kx - \omega t)\,\hat{\mathbf{y}} \quad .$$

The other two Maxwell's equations will be satisfied if

$$c = \frac{\omega}{k} \quad .$$

In this case,

$$k = \frac{2\pi}{\lambda} = \frac{10^{10}\ 1/s}{3 \times 10^{10}\ cm/s} = \frac{1}{3\ cm} \quad ,$$

$$\lambda = 6\pi\ cm = 18.8\ cm \quad .$$

The average energy density is

$$\frac{E_o^2}{8\pi} = \frac{(0.05\ statvolt/cm)^2}{8\pi} = 9.95 \times 10^{-5}\ erg/cm^3 \quad ,$$

and the average intensity (= power density) is

$$\frac{cE_o^2}{8\pi} = 3 \times 10^6\ erg/cm^2 s \quad .$$

3 *Purcell 9.9 The cosmic microwave background radiation apparently fills all space with an energy density of* 4×10^{-13} *erg/cm*^{3}*. Calculate the rms electric field strength in statvolt/cm and in V/m. Roughly how far away from a 1 kW radio transmitter would you find a comparable electromagnetic wave intensity?*

The average energy density is

$$\frac{E_{rms}^2}{4\pi} = 4 \times 10^{-13}\ erg/cm^3 \quad ,$$

and

$$E_{rms} = 2.2 \times 10^{-6}\ statvolt/cm = 0.067\ V/m \quad .$$

Assuming the transmitter projects in all directions, a distance r away, the intensity of the transmitter is

$$\frac{1\ kW}{4\pi r^2} = \frac{E_{rms}^2}{\sqrt{\mu_o/\epsilon_o}} = \frac{(0.067\ V/m)^2}{376.73\ ohms} \quad ,$$

$$r = 2584\ m \quad .$$

4 *Purcell 9.10* *Find the magnetic field at a point P midway between the plates of capacitor a distance r from the axis of symmetry. A current I is flowing through the capacitor.*

We assume that the magnetic field circles the capacitor axis. If the capacitor spacing is small, the electric field will be fairly uniform and

$$E = \frac{V}{s} = \frac{Q}{sC} = \frac{4\pi s Q}{s\pi b^2} = \frac{4Q}{b^2} \quad ,$$

$$\Phi_E = \pi r^2 E = \frac{4\pi r^2 Q}{b^2} \quad ,$$

and ignoring signs,

$$2\pi r B = \frac{1}{c}\frac{4\pi r^2}{b^2}\frac{dQ}{dt} \quad ,$$

$$B = \frac{2rI}{cb^2} \quad .$$

At the edge of the capacitor ($r = b$) this is the same as the magnetic field around a long wire.

5 *Purcell 10.7* *A cell membrane typically has a capacitance around 1 $\mu F/cm^2$. It is believed the membrane consists of material having a dielectric constant of about 3. Find the thickness this implies. Other electrical measurements have indicated that the resistance of 1 cm^2 of cell membrane is around 1000 ohms. Show that the time constant of such a leaky capacitor is independent of the area of the capacitor. How large is it in this case? What is the resistivity?*

The capacitance is given in Farads so we will use SI. The constant ϵ_o appears in SI formulas. To deal with a dielectric material, we make the replacement $\epsilon_o \to \epsilon$. However, in SI, ϵ is not dimensionless. For example, if $\epsilon = 3$ in cgs, the value in SI is $3\epsilon_o$. The capacitance of a parallel plate capacitor is

$$C = \epsilon\frac{A}{s} = 3\frac{1\ cm^2}{s} = 1 \times 10^{-6}\ F \quad ,$$

so

$$s = 2.66 \times 10^{-9}\ m \quad .$$

We may view the leaky capacitor as a simple RC circuit, where the resistor and the capacitor are really the same element. The time constant is

$$\tau = RC = \frac{\rho s}{A}\epsilon\frac{A}{s} = \rho\epsilon \quad ,$$

which is independent of the area of the membrane. It is also independent of its thickness..

$$\tau = (1000\ ohms)(1 \times 10^{-6}\ F) = 1 \times 10^{-3}\ s = \rho\, 3(8.85 \times 10^{-12}\ C^2/Nm^2)$$

$$\rho = 3.8 \times 10^7\ ohm\, m$$

$$\tau = sec$$

6 *Purcell 10.17*

In the first two cases, we assume the left dipole to be present and we bring in the right dipole from infinity. We would like to do this in such a way that the work required is zero. This will be the case if the path of the dipole coming in is perpendicular to the force on it. We will bring in the second dipole on a straight line from the right. The force on it is

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E} \quad ,$$

where $\mathbf{E}$ is the field created by the other dipole. We wish to find an orientation for the right dipole so that this force is perpendicular to the path. We can do this if the dipole is pointed to the right. In this case the force is

$$\mathbf{F} = p\frac{\partial \mathbf{E}}{\partial x} \quad .$$

The field from the left dipole on the line of the path is

$$\mathbf{E} = -\frac{p}{r^3}\hat{\mathbf{z}} = -\frac{p}{x^3}\hat{\mathbf{z}} \quad .$$

The force is then

$$\mathbf{F} = 3\frac{p^2}{x^4}\hat{\mathbf{z}} \quad ,$$

which is perpendicular to the path. Intuitively we can think of the dipole as two charges. The positive charge feels a force down and the negative charge feels a force up. But the positive charge is further away so the force on it is smaller. The net force is then up. But since this is perpendicular to the path, it still requires no work to bring in the dipole.

The field at the right dipole is

$$\mathbf{E} = -\frac{p}{d^3}\hat{\mathbf{z}} \quad ,$$

and the torque exerted on this is

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} = pE\sin\theta\,\hat{\mathbf{y}} \quad .$$

The work we do in rotating the dipole is minus the work done by the field. So

$$W = \int_{\pi/2}^{\pi} pE\sin\theta\, d\theta = pE = \frac{p^2}{d^3} \quad .$$

This is good since the dipoles don't like to point in the same direction in this orientation so it should take positive work to arrange it.

In the second case, we rotate the dipole the same amount in the opposite direction and do the opposite work.

$$W = -pE = -\frac{p^2}{d^3}$$

In the third situation, the field from the left dipole points to the right along the path, so we can't bring in the second dipole pointing to the right. However, if we bring it in pointing up, the force on it will be up, perpendicular to the path. Taking the derivative is a little messy in this case, but we can find the direction of the derivative intuitively. If you think of the dipole as two charges, the force on the positive charge will be up to the right, and the force on the negative charge will be up to the left. But since the field is symmetric with respect to the x axis, the x components cancel out leaving a force up.

The field at the second dipole in this case has strength $E = 2p/d^3$. In analogy with the above results, the work to rotate it to the right is

$$W = -2\frac{p^2}{d^3} \quad ,$$

because the dipole wants to be in this orientation. The work for the final situation is then

$$W = 2\frac{p^2}{d^3} \quad .$$

7 *Purcell 10.19*

If the ion is positive, the dipole will point away from it. The dipole field at the ion location will then point towards the dipole and the force will be attractive. If the ion is negative, the dipole will point towards it and the dipole field at the ion will be away from the dipole. The force again will be attractive. The polarization is $p = \alpha E = \alpha q/r^2$ where q is the ion charge. The force on the ion is

$$F = q\frac{2p}{r^3} = \frac{2q^2\alpha}{r^5} \quad .$$

To find the potential energy, we bring in the ion from infinity. The work required is

$$U = -\int_\infty^r \frac{2\alpha q^2}{r^5} dr = -\frac{\alpha q^2}{2r^4} \quad .$$

For sodium, $\alpha = 27 \times 10^{-24}\ cm^3$.

$$4 \times 10^{-14}\ erg = \frac{(4.8 \times 10^{-10}\ esu)^2 (27 \times 10^{-24}\ cm^3)}{2r^4}$$

$$r = 9.4 \times 10^{-8}\ cm$$

8 *Purcell 10.21*

The maximum field strength is

$$E_m = \frac{14 \times 10^3 \ V}{0.0000254 \ m} = 5.5 \times 10^8 \ V/m \quad .$$

The energy density is

$$\frac{1}{2}\epsilon E_m^2 = \frac{1}{2}(3.25)(8.85 \times 10^{-12} \ C^2/N\,m^2)(5.5 \times 10^8 \ V/m)^2 = 4.35 \times 10^6 \ J/m^3 \quad .$$

Remember that in SI, we need to insert the ϵ_o. The energy per mass is then

$$\frac{4.35 \times 10^6 \ J/m^3}{1400 \ kg/m^3} = 3100 \ J/kg \quad .$$

This can raise the capacitor to a height

$$mgh = (0.75)m(3107 \ J/kg) \quad ,$$

$$h = 238 \ m \quad .$$

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Physics H7B Spring 1999 (*Strovink*)

PROBLEM SET 13

1. Purcell problem 10.13

2. Purcell problem 10.16

3. Purcell problem 11.2

4. Purcell problem 11.4

5. Purcell problem 11.7

6. Purcell problem 11.9

7. Purcell problem 11.12

8. Purcell problem 11.16

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SOLUTION TO PROBLEM SET 13

Solutions by P. Pebler

1 *Purcell 10.13*

Consider a parallel plate capacitor. The energy required to charge it to a potential difference V is $E = CV^2/2$. The capacitance increases with a dielectric to $C = \epsilon C_o = \epsilon A/4\pi s$. The potential difference is Es. Then

$$E = \frac{1}{2}CV^2 = \frac{\epsilon A E^2 s^2}{8\pi s} = \frac{\epsilon}{8\pi}E^2(As) \quad ,$$

and the energy density is

$$\epsilon\frac{E^2}{8\pi} \quad .$$

For a wave in a dielectric $B = \sqrt{\epsilon}E$ and the energy density in the magnetic field is

$$\frac{B^2}{8\pi} = \epsilon\frac{E^2}{8\pi} \quad .$$

2 *Purcell 10.16*

We use Gauss's law inside the uniform spherical charge distribution.

$$4\pi r^2 E_r = 4\pi Q_{enc} = 4\pi\frac{4\pi}{3}r^3\rho$$

$$\mathbf{E} = \frac{4\pi}{3}\rho\mathbf{r}$$

Let the sphere of density ρ be centered at the origin, and the sphere of density $-\rho$ be centered at the location $\mathbf{s}$. The total field is

$$\mathbf{E} = \frac{4\pi}{3}\rho\mathbf{r} + \frac{4\pi}{3}(-\rho)(\mathbf{r}-\mathbf{s}) = \frac{4\pi}{3}\rho\mathbf{s} \quad .$$

In the middle of a long cylinder, we can find the field from Gauss's law.

$$2\pi r L E_r = 4\pi(\pi r^2 L\rho)$$

$$\mathbf{E} = 2\pi\rho r\hat{\mathbf{r}}$$

We are using cylindrical coordinates here so $\hat{\mathbf{r}}$ points away from the axis. The total field of two cylinders with their axes displaced by $\mathbf{s}$ is

$$\mathbf{E} = 2\pi\rho r\hat{\mathbf{r}} + 2\pi(-\rho)(r\hat{\mathbf{r}} - \mathbf{s}) = 2\pi\rho\mathbf{s} \quad .$$

3 *Purcell 11.2*

The magnetic field of a current loop with its axis on the z axis has only a z component with

$$B_z = \frac{2\pi b^2 I}{c(b^2+z^2)^{3/2}} = \frac{2m}{(b^2+z^2)^{3/2}} \quad .$$

The dipole field on this axis is all radial, which here is the z direction.

$$B'_z = B_r = \frac{2m}{r^3} = \frac{2m}{z^3}$$

So

$$B_z = \frac{z^3}{(b^2+z^2)^{3/2}} B'_z$$

and the loop field approaches the dipole field when $z \gg b$. There is a 1% difference when

$$\frac{z^3}{(b^2+z^2)^{3/2}} = 0.99 \quad ,$$

$$z = 12.2\, b \quad .$$

4 *Purcell 11.4*

The earth's radius is about $6 \times 10^8\ cm$ so

$$0.62\ gauss = \frac{2m}{(6 \times 10^8\ cm)^3} \quad ,$$

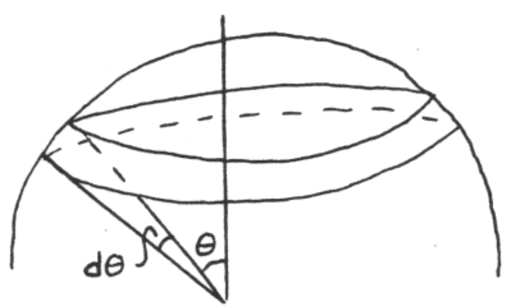
$$m = 6.7 \times 10^{25}\ erg/gauss = 6.7 \times 10^{22}\ J/T \quad .$$

If we have a current loop of radius $3 \times 10^8\ cm$, we need a current I where

$$0.62\ gauss = \frac{2\pi(3 \times 10^8\ cm)^2 I}{c[(3 \times 10^8\ cm)^2 + (6 \times 10^6\ cm)^2]^{3/2}} \quad ,$$

$$I = 9.9 \times 10^{18}\ esu/s = 3.3 \times 10^9\ A \quad .$$

5 *Purcell 11.7*



We will use polar coordinates for the integration. We divide the surface into little strips subtended by the small change in polar angle $d\theta$. The surface area of one of these strips is

$$da = 2\pi(R\sin\theta)(Rd\theta) \quad .$$

The amount of charge on this strip is

$$dq = \sigma da = \frac{Q}{4\pi R^2} 2\pi R^2 \sin\theta \, d\theta = \frac{1}{2} Q \sin\theta \, d\theta \quad .$$

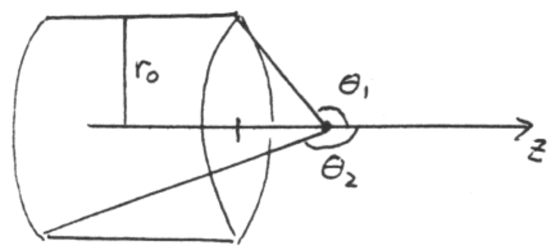
This charge revolves around with a frequency $f = \omega/2\pi$, so it represents a little current

$$dI = f \, dq = \frac{\omega Q}{4\pi} \sin\theta \, d\theta \quad .$$

Each strip contributes a moment $dm = A \, dI/c$.

$$m = \frac{1}{c} \int A \, dI = \frac{2}{c} \int_0^{\pi/2} \pi (R \sin\theta)^2 \frac{\omega Q}{4\pi} \sin\theta \, d\theta = \frac{\omega Q R^2}{2c} \int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{\omega Q R^2}{3c}$$

6 *Purcell 11.9*



From Chapter 6, the field from a finite solenoid is

$$B_z = \frac{2\pi I n}{c} (\cos\theta_1 - \cos\theta_2) \quad .$$

For a semi-infinite solenoid, $\theta_2 = \pi$ and with z measuring the distance of the point outside the top of the solenoid,

$$B_z = \frac{2\pi I n}{c} \left(1 - \frac{z}{\sqrt{z^2 + r_o^2}} \right) \quad .$$

We want to maximize $B_z (dB_z/dz)$.

$$\frac{dB_z}{dz} = -\frac{2\pi I n}{c} \left(\frac{1}{\sqrt{z^2 + r_o^2}} - \frac{z^2}{(z^2 + r_o^2)^{3/2}} \right) = -\frac{2\pi I n}{c} \left(\frac{r_o^2}{(z^2 + r_o^2)^{3/2}} \right)$$

$$B_z \frac{dB_z}{dz} \propto \frac{1}{(z^2 + r_o^2)^{3/2}} \left(1 - \frac{z}{\sqrt{z^2 + r_o^2}} \right)$$

Taking a derivative and setting to zero yields the equation

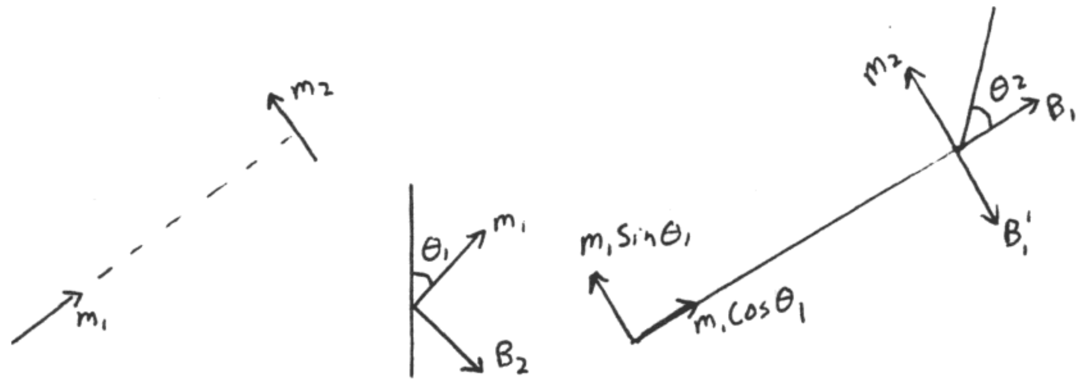
$$3z^2 - r_o^2 = 3z\sqrt{z^2 + r_o^2} \quad .$$

Squaring and solving the quadratic equation gives $z^2 = r_o^2/15$. Only the negative root solves the original equation so

$$z = -r_o \sqrt{\frac{1}{15}} \quad .$$

This is slightly inside the solenoid.

7 *Purcell 11.12*



The potential of a single dipole in a magnetic field can be chosen to be

$$U = -\mathbf{m} \cdot \mathbf{B} \quad .$$

This does not have the zero where we want our zero to be. However, for the purposes of finding work done in rotating the dipoles we may use

$$W = U_f - U_i \quad .$$

In the initial configuration, the field due to dipole 2 at m_1 is as shown above with

$$B_2 = \frac{m_2}{r^3} \quad .$$

The work required to rotate m_1 is

$$W_1 = U_f - U_i = -m_1 B_2 \cos(90 + \theta_1) - 0 = m_1 B_2 \sin\theta_1 = \frac{m_1 m_2}{r^3} \sin\theta_1 \quad .$$

To rotate m_2, we break up the field from m_1 into two parts with

$$B_1 = \frac{2m_1 \cos\theta_1}{r^3} \qquad\qquad B_1' = \frac{m_1 \sin\theta_1}{r^3} \quad .$$

The work to rotate m_2 is then

$$\begin{aligned} W_2 &= U_f - U_i = [-m_2 B_1 \cos\theta_2 - m_2 B_1' \cos(90 + \theta_2)] - [-m_2 B_1' \cos\pi] \\ &= -m_2 B_1 \cos\theta_2 + m_2 B_1' \sin\theta_2 - m_2 B_1' \\ &= -\frac{2m_1 m_2}{r^3} \cos\theta_1 \cos\theta_2 + \frac{m_1 m_2}{r^3} \sin\theta_1 \sin\theta_2 - \frac{m_1 m_2}{r^3} \sin\theta_1 \quad . \end{aligned}$$

the total work is

$$W = W_1 + W_2 = \frac{m_1 m_2}{r^3} (\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2) \quad .$$

8 *Purcell 11.16*

The exterior field of a uniformly magnetized sphere turns out to be that of a magnetic dipole with dipole moment

$$m = \frac{4\pi}{3} r^3 M \quad .$$

This is something that needs to be proved, however. One can prove this by finding the field from the bound current. The bound current density is

$$\mathbf{J}_b = c\,\nabla \times \mathbf{M} = 0 \quad ,$$

and the bound surface current is

$$\mathbf{K}_b = c\,\mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta\,\hat{\phi} \quad .$$

This is identical to the surface current of a rotating sphere with uniform surface charge. One can integrate to find the vector potential which is that of a magnetic dipole at the center. We leave this to you as an exercise.

The field at the pole is

$$B = \frac{2m}{r^3} = \frac{8\pi}{3}(750\ erg/gauss\,cm^3) = 6280\ gauss \quad .$$

At the equator

$$B = \frac{m}{r^3} = 3140\ gauss \quad .$$

To find the force, we need to know the force on a uniformly magnetized sphere in the field of a dipole. Fortunately, this is simple due to the following argument. The force on the sphere on the right must be the same if we replace the sphere on the left with a dipole at its center. This force must be equal and opposite to the force on the imaginary dipole. But the field from the sphere on the right at the dipole is that of a dipole, so the force between spheres is the same as the force between two dipoles. (This is not obvious without the argument just given.)

$$F = m_2 \left| \frac{db_{1z}}{dz} \right| = m_2 \left| \frac{d}{dz}\left(\frac{2m_1}{z^3}\right) \right| = 6\frac{m_1 m_2}{z^4} = \frac{3}{8}\frac{m_1 m_2}{r^4}$$

$$F = \frac{3}{8}\left(\frac{4\pi}{3}M\right)^2 r^2 = 3.7 \times 10^6\ dynes = 37\ N$$

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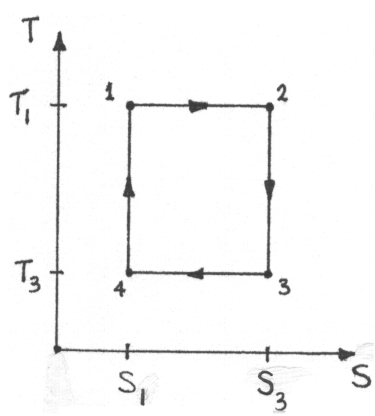
EXAMINATION 1

Directions. Do all four problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (25 points) A heat engine for which the working material is an ideal monatomic gas moves slowly enough that all parts of it are always in mutual equilibrium. It is described by a rectangular path on the T (absolute temperature) – S (entropy) plane, as in the figure. While on the path $1 \rightarrow 2$, the gas in the engine takes heat from a bath at high temperature T_1; on the path $3 \rightarrow 4$, it returns heat to bath at lower temperature T_3. On the paths $2 \rightarrow 3$ and $4 \rightarrow 1$, the entropy has constant values S_3 and S_1, respectively.



a. (5 points) Write down the net change

$$(\Delta U_{23} + \Delta U_{41})$$

in internal energy for the sum of the two paths $2 \rightarrow 3$ and $4 \rightarrow 1$.

b. (5 points) Compute the net change

$$\oint_{12341} T\, dS$$

over one complete cycle of the engine.

c. (8 points) Deduce the value of the mechanical work

$$\oint_{12341} p\, dV$$

done **by** the gas **on** the rest of the universe over one complete cycle of the engine.

d. (7 points) In one cycle, what fraction of the heat withdrawn from the hot reservoir is converted to mechanical work done by the gas on the rest of the universe?

Hint: Keep in mind that the only parameters given in this problem are T_1, T_3, S_1, and S_3; your answers, if nontrivial, must be expressed in terms of these parameters.

2. (25 points) In a hypothetical one-dimensional system, thermal motion of atoms in the y and z directions is "frozen out", so, effectively, the atoms are able to move only in the x direction. In that direction, an atom has velocity v $(-\infty < v < \infty)$. The fraction dF of atoms with velocity between v and $v + dv$ is

$$dF \equiv f_v(v)\, dv = \frac{\exp\left(-\frac{mv^2}{2kT}\right) dv}{\int_{-\infty}^{\infty} \exp\left(-\frac{mv^2}{2kT}\right) dv} \ ,$$

where $f_v(v)$ is the probability density (RHK: "relative probability") of the value v, m is the atomic mass, k is Boltzmann's constant, and T is the absolute temperature.

a. (10 points) Calculate the mean value of the square of v, *i.e.* $\langle v^2 \rangle$. If you wish, you may leave your answer in the form of a ratio of definite integrals. *Do not merely guess the answer.*

b. (15 points) Define E to be the kinetic energy $\frac{1}{2}mv^2$ of an atom. The fraction dF of atoms with kinetic energy between E and $E+dE$ is

$$dF \equiv f_E(E)\,dE\ ,$$

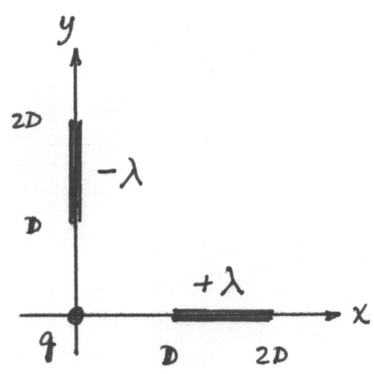
where $f_E(E)$ is the probability density of the value E. One might imagine $f_E(E)$ to take the possible forms:

$$\begin{aligned} f_E(E) &\propto E^{-1/2}\exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto \exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto E^{1/2}\exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto E\exp\left(-\frac{E}{kT}\right)\ ? \end{aligned}$$

Which one form is correct, and why?

3. (25 points)

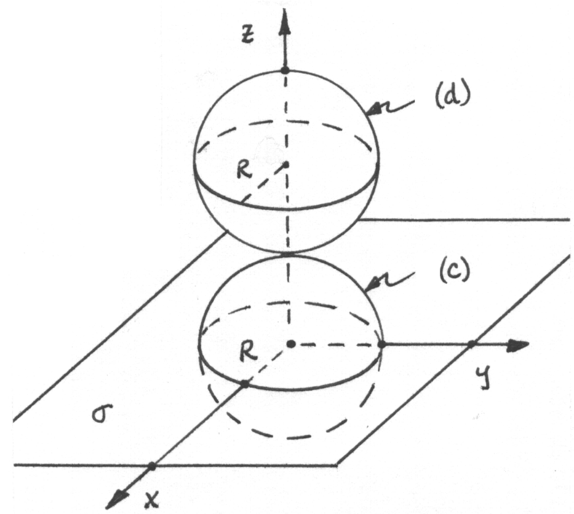
A fixed line charge of $+\lambda$ esu/cm on the x axis extends from $x = D$ to $x = 2D$, and a fixed line charge of $-\lambda$ esu/cm on the y axis extends from $y = D$ to $y = 2D$.



a. (10 points) Find the work required to bring a test point charge q from infinity to the origin. Does your answer depend on the path you chose? If so, specify the path.

b. (15 points) Calculate the mechanical force (magnitude and direction) that is required to keep the test charge at the origin.

4. (25 points) The infinite plane $z = 0$ carries a uniform surface charge density σ esu/cm^2. There are no other charges in the problem.



a. (5 points) Find the magnitude and direction of the electric field $\mathbf{E}_+$ everywhere in the region $z > 0$.

b. (5 points) Find the magnitude and direction of the electric field $\mathbf{E}_-$ everywhere in the region $z < 0$.

c. (8 points) Consider a spherical surface of radius R centered at the origin. Find the electric flux

$$\iint \mathbf{E}\cdot d\mathbf{a}$$

through the *top half* (top hemisphere) of this surface.

d. (7 points) Consider a second spherical surface, again of radius R, but now centered at the point $(0, 0, 2R)$, so that it does not enclose any charge. Find the electric flux

$$\iint \mathbf{E}\cdot d\mathbf{a}$$

through the *bottom half* (bottom hemisphere) of this surface.

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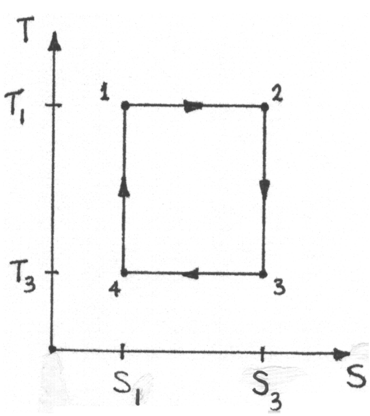
SOLUTION TO EXAMINATION 1

Directions. Do all four problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (25 points) A heat engine for which the working material is an ideal monatomic gas moves slowly enough that all parts of it are always in mutual equilibrium. It is described by a rectangular path on the T (absolute temperature) – S (entropy) plane, as in the figure. While on the path $1 \rightarrow 2$, the gas in the engine takes heat from a bath at high temperature T_1; on the path $3 \rightarrow 4$, it returns heat to bath at lower temperature T_3. On the paths $2 \rightarrow 3$ and $4 \rightarrow 1$, the entropy has constant values S_3 and S_1, respectively.



a. (5 points) Write down the net change

$$(\Delta U_{23} + \Delta U_{41})$$

in internal energy for the sum of the two paths $2 \rightarrow 3$ and $4 \rightarrow 1$.

b. (5 points) Compute the net change

$$\oint_{12341} T\, dS$$

over one complete cycle of the engine.

c. (8 points) Deduce the value of the mechanical work

$$\oint_{12341} p\, dV$$

done **by** the gas **on** the rest of the universe over one complete cycle of the engine.

d. (7 points) In one cycle, what fraction of the heat withdrawn from the hot reservoir is converted to mechanical work done by the gas on the rest of the universe?

Hint: Keep in mind that the only parameters given in this problem are T_1, T_3, S_1, and S_3; your answers, if nontrivial, must be expressed in terms of these parameters.

Solution:

(a.)

U of an ideal gas is a function only of T, so the isothermal segments $1 \rightarrow 2$ and $3 \rightarrow 4$ cause no change in U. Therefore

$$(\Delta U_{23} + \Delta U_{41}) = \oint_{12341} dU = 0$$

because U is a state function.

(b.)

$$\oint_{12341} T\, dS = (T_1 - T_3)(S_3 - S_1) \ ,$$

the area of the rectangle in the figure.

(c.)

$$\begin{aligned}\oint_{12341} p\,dV &= -\oint_{12341} \delta W \\ &= -\oint_{12341} dU + \oint_{12341} \delta Q \\ &= 0 + \oint_{12341} T\,dS \\ &= (T_1 - T_3)(S_3 - S_1)\ .\end{aligned}$$

(d.)

$$\begin{aligned}\frac{\oint_{12341} p\,dV}{Q_2} &= \frac{\oint_{12341} p\,dV}{\int_1^2 T\,dS} \\ &= \frac{(T_1 - T_3)(S_3 - S_1)}{T_1(S_3 - S_1)} \\ &= 1 - \frac{T_3}{T_1}\ .\end{aligned}$$

It is also acceptable to state that this is a Carnot engine and quote this standard result for its efficiency.

2. (25 points) In a hypothetical one-dimensional system, thermal motion of atoms in the y and z directions is "frozen out", so, effectively, the atoms are able to move only in the x direction. In that direction, an atom has velocity v ($-\infty < v < \infty$). The fraction dF of atoms with velocity between v and $v + dv$ is

$$dF \equiv f_v(v)\,dv = \frac{\exp\left(-\frac{mv^2}{2kT}\right) dv}{\int_{-\infty}^{\infty} \exp\left(-\frac{mv^2}{2kT}\right) dv}\ ,$$

where $f_v(v)$ is the probability density (RHK: "relative probability") of the value v, m is the atomic mass, k is Boltzmann's constant, and T is the absolute temperature.

a. (10 points) Calculate the mean value of the square of v, *i.e.* $\langle v^2 \rangle$. If you wish, you may leave your answer in the form of a ratio of definite integrals. *Do not merely guess the answer.*

b. (15 points) Define E to be the kinetic energy $\frac{1}{2}mv^2$ of an atom. The fraction dF of atoms with kinetic energy between E and $E+dE$ is

$$dF \equiv f_E(E)\,dE\ ,$$

where $f_E(E)$ is the probability density of the value E. One might imagine $f_E(E)$ to take the possible forms:

$$\begin{aligned} f_E(E) &\propto E^{-1/2} \exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto \exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto E^{1/2} \exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto E \exp\left(-\frac{E}{kT}\right)\ ? \end{aligned}$$

Which one form is correct, and why?

Solution:

(a.)

From the definition of it that is given, this probability density is explicitly normalized:

$$\int_{-\infty}^{\infty} f_v(v)\,dv \equiv 1\ .$$

Using the standard method for taking the average, when f_v is normalized,

$$\begin{aligned}\langle v^2 \rangle &= \int_{-\infty}^{\infty} v^2 f_v\,dv \\ &= \frac{\int_{-\infty}^{\infty} v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv}{\int_{-\infty}^{\infty} \exp\left(-\frac{mv^2}{2kT}\right) dv}\ .\end{aligned}$$

This answer is enough to earn full credit. For completeness, defining $\beta \equiv 1/kT$ and $u \equiv \frac{1}{2}mv^2$, we can rewrite this quotient as

$$\begin{aligned}\langle v^2 \rangle &= \frac{\frac{2}{m}\int_{-\infty}^{\infty} u^{1/2} \exp\left(-\beta u\right) du}{\int_{-\infty}^{\infty} u^{-1/2} \exp\left(-\beta u\right) du} \\ &= -\frac{2}{m}\frac{\partial}{\partial\beta} \ln\left(\int_{-\infty}^{\infty} u^{-1/2} \exp\left(-\beta u\right) du\right) \\ &= -\frac{2}{m}\frac{\partial}{\partial\beta} \ln\left(C\beta^{-1/2}\right) \\ &= \frac{1}{m\beta} \\ &= \frac{kT}{m}\ ,\end{aligned}$$

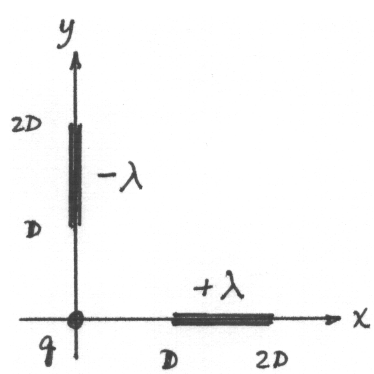
where, in the above, C is a constant whose value is immaterial here.

(b.)

$$\begin{aligned} dF &\equiv f_E(E)\, dE \\ f_E &= \frac{dF}{dE} \\ &= \frac{dF}{dv}\frac{dv}{dE} \\ &\equiv f_v(v)\frac{dv}{dE} \\ &= \frac{f_v(v)}{\frac{d}{dv}\left(\frac{1}{2}mv^2\right)} \\ &= \frac{f_v(v)}{mv} \\ &\propto \frac{\exp\left(-\frac{E}{kT}\right)}{E^{1/2}} \ . \end{aligned}$$

3. (25 points)

A fixed line charge of $+\lambda$ esu/cm on the x axis extends from $x = D$ to $x = 2D$, and a fixed line charge of $-\lambda$ esu/cm on the y axis extends from $y = D$ to $y = 2D$.



a. (10 points) Find the work required to bring a test point charge q from infinity to the origin. Does your answer depend on the path you chose? If so, specify the path.

b. (15 points) Calculate the mechanical force (magnitude and direction) that is required to keep the test charge at the origin.

Solution:

(a.)

For every positive charge element that is a certain distance from the origin, there is a corresponding negative charge element located at the same distance from the origin (but in an orthogonal direction). Therefore, by symmetry, the electrostatic potential ϕ vanishes at the origin, as does the work W required to bring the charge in from infinity:

$$W = q\big(\phi(0) - \phi(\infty)\big) = 0 \ .$$

(b.)

From the positive part of the charge distribution, the electric field at the origin is

$$\begin{aligned} \mathbf{E}_+ &= -\hat{\mathbf{x}} \int_D^{2D} \frac{\lambda}{x^2}\, dx \\ &= -\hat{\mathbf{x}}\Big(\frac{\lambda}{D} - \frac{\lambda}{2D}\Big) \\ &= -\hat{\mathbf{x}}\frac{\lambda}{2D} \ . \end{aligned}$$

Likewise, from the negative part of the charge distribution,

$$\mathbf{E}_- = +\hat{\mathbf{y}}\frac{\lambda}{2D} \ .$$

The total electric field is

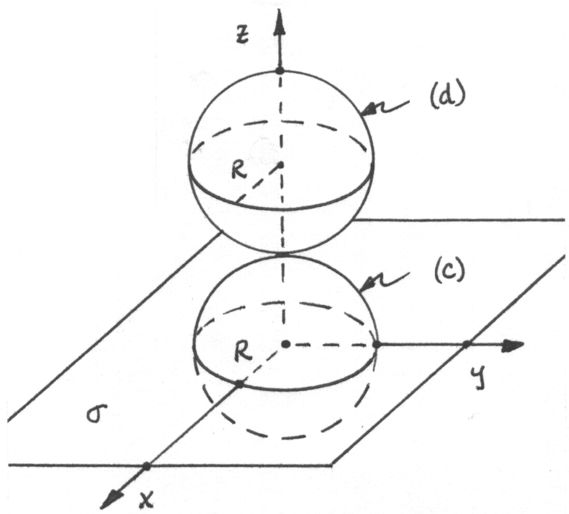
$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\hat{\mathbf{y}} - \hat{\mathbf{x}}}{\sqrt{2}}\frac{\lambda}{D\sqrt{2}} \ .$$

The mechanical force $\mathbf{F}$ required to keep the test charge at the origin must oppose $q\mathbf{E}$:

$$\mathbf{F} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}}{\sqrt{2}}\frac{q\lambda}{D\sqrt{2}} \ ,$$

where the first factor is its direction (at $-45°$ to the x axis), and the second is its magnitude.

4. (25 points) The infinite plane $z = 0$ carries a uniform surface charge density σ esu/cm^2. There are no other charges in the problem.



a. (5 points) Find the magnitude and direction of the electric field $\mathbf{E}_+$ everywhere in the region $z > 0$.

b. (5 points) Find the magnitude and direction of the electric field $\mathbf{E}_-$ everywhere in the region $z < 0$.

c. (8 points) Consider a spherical surface of radius R centered at the origin. Find the electric flux

$$\iint \mathbf{E} \cdot d\mathbf{a}$$

through the *top half* (top hemisphere) of this surface.

d. (7 points) Consider a second spherical surface, again of radius R, but now centered at the point $(0, 0, 2R)$, so that it does not enclose any charge. Find the electric flux

$$\iint \mathbf{E} \cdot d\mathbf{a}$$

through the *bottom half* (bottom hemisphere) of this surface.

Solution:
(a.) (b.)
The charge distribution is symmetric about the plane $z = 0$, so

$$\mathbf{E}_+ = -\mathbf{E}_- \ ,$$

and both fields are normal to the $z = 0$ plane. Using a Gaussian pillbox with flat surface area A parallel to the $z = 0$ plane,

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= 4\pi Q_{\mathrm{encl}} \\
((E_+)_z - (E_-)_z)A &= 4\pi\sigma A \\
((E_+)_z + (E_+)_z)A &= 4\pi\sigma A \\
(E_+)_z = -(E_-)_z &= 2\pi\sigma \\
\mathbf{E}_+ = -\mathbf{E}_- &= \hat{\mathbf{z}}\, 2\pi\sigma \ .
\end{aligned}$$

It is acceptable simply to recall that the electric field on either side of an infinite sheet of charge has this value, in the absence of other charges.
(c.)
Again because the charge distribution is symmetric about the plane $z = 0$, substituting a sphere of radius R centered at the origin for the Gaussian pillbox used in the solution of part (a.),

$$\begin{aligned}
\iint_{\mathrm{hemi}}^{\mathrm{top}} \mathbf{E} \cdot d\mathbf{a} = \iint_{\mathrm{hemi}}^{\mathrm{bot}} \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{2} 4\pi Q_{\mathrm{encl}} \\
\iint_{\mathrm{hemi}}^{\mathrm{top}} \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{2} 4\pi\pi R^2 \sigma \\
&= 2\pi^2 R^2 \sigma \ .
\end{aligned}$$

(d.)
Because $\mathbf{E}$ is constant throughout the semi-infinite region $z > 0$, the flux of $\mathbf{E}$ through the top of the hemisphere centered at $(0, 0, 2R)$ is the same as the flux in part (c.) through the top of the hemisphere centered at the origin. Since the hemisphere centered at $(0, 0, 2R)$ contains no charge, the flux of $\mathbf{E}$ through its bottom half must cancel the flux through its top half. Therefore

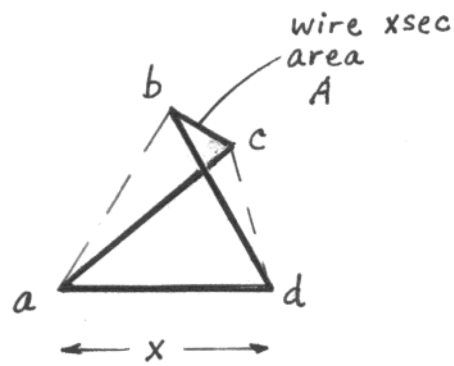
$$\iint_{\mathrm{hemi}}^{\mathrm{bot}} \mathbf{E} \cdot d\mathbf{a} = -2\pi^2 R^2 \sigma \ .$$

University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

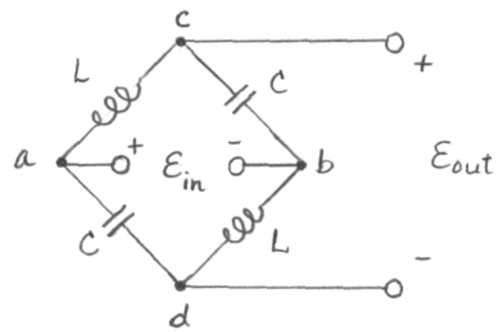
Problem 1. (35 points)
Four straight stainless steel wires of length x, cross-sectional area A, and resistivity ρ are welded together so that they lie along four of the six edges of a regular tetrahedron, as shown.



The remaining two sides ab and cd are empty. Consider a and b to be electrical input terminals, and c and d to be electrical output terminals.

a. (15 points)
When c is shorted to d, what resistance is measured between a and b?

b. (20 points)
Suppose that wires ac and bd are replaced by two identical inductors L, and wires bc and ad are replaced by two identical capacitors C. Across terminals a and b is placed a source of input EMF $\mathcal{E}_{\rm in}(t) = \mathcal{E}_0 \cos\omega t$ where $\mathcal{E}_0$ and ω are real constants. Across terminals c and d, an output EMF $\mathcal{E}_{\rm out}(t)$ is measured by an ideal voltmeter which draws no current.



Is there a value of ω for which the voltmeter will measure $\mathcal{E}_{\rm out}(t) = 0$? Explain.

Problem 2. (30 points)
Experiments using heavy electrons ("muons") have exploited the fact that, when a pi meson ("π^+") decays at rest into a muon ("μ^+") plus a neutrino ("ν"), the muon has a unique momentum with which its spin angular momentum is fully aligned. The reaction is

$$\pi^+ \rightarrow \mu^+ + \nu \ .$$

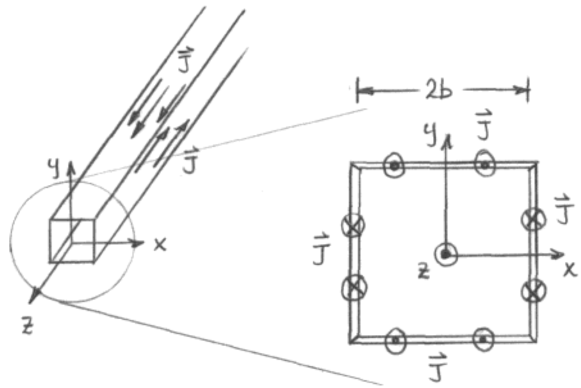
A beam of such muons is called a *surface muon beam* (because the pion is stopped near the surface of a solid target where it was produced by protons from a cyclotron). The muons in a surface beam are so well defined that, if they were allowed to impinge normally on a book, nearly all would stop in the same page.

Given that the muon mass is 3/4 of the pion mass, while the neutrino mass is negligibly small, compute the velocity of the muons in a surface beam, expressed as a fraction of the speed of light.

Problem 3. (35 points)
A *Panofsky quadrupole* magnet consists of four long thin copper bars, pointing in the $\hat{\mathbf{z}}$ direction (out of the page), arranged so that their inside surfaces form a square box of side $2b$. The bars at $y = \pm b$ carry a uniform current density in the $+\hat{\mathbf{z}}$ direction while the bars at $x = \pm b$ carry the same current in the $-\hat{\mathbf{z}}$ direction. Within the box enclosed by the bars, the magnetic vector potential is

$$\mathbf{A} = \frac{\alpha}{2}\hat{\mathbf{z}}(y^2 - x^2) ,$$

where α is a constant.



a. (10 points)
Suppose that a particle of charge $+e$ travels along $\hat{\mathbf{z}}$ at position $(x, y) = (0, y)$. Show that the particle is deflected toward (0,0) with a force that is proportional to y. (This means that, in the y projection, the Panofsky quadrupole acts as a *converging lens*. However, it acts as a *diverging lens* in the x projection. Fortunately, the combination of a converging and a diverging lens of equal strength remains slightly converging, if the two lenses are separated along their axis; this allows a pair of quadrupole magnets to focus a particle beam weakly in both the x and y projections. One of the first experiments to use this fact discovered the antiproton at the Berkeley Bevatron in 1956.)

b. (10 points)
Prove that the current density $\mathbf{J}$ within the box enclosed by the bars ($|x| < b$ and $|y| < b$) must be zero. (This allows the box to be evacuated so that a particle beam can travel unimpeded within it.)

c. (15 points)
Suppose that a different region of space has

$$\mathbf{B}(x, y) = A_0(\hat{\mathbf{x}}y - \hat{\mathbf{y}}x) ,$$

where A_0 is a constant (a "bullseye" magnetic field). Show that the current density along $\hat{\mathbf{z}}$ must be nonzero everywhere in the region; give its magnitude and any dependence that it may have on x and y. (This magnetic field acts as a converging lens in *both* the x and y projections. However, the beam particles are required to pass through the magnet's current-carrying element, which needs to be made as light as possible, e.g. of molten lithium.)

Problem 4. (35 points)
Consider a uniform region of space containing an insulating material with fixed dielectric constant ϵ and magnetic permeability μ.

a. (3 points)
Write Faraday's law in differential form.

b. (3 points)
For this material there are two differential versions of Ampere's Law, as modified by Maxwell – one version uses the free current density $\mathbf{J}_{\text{free}}$, the other uses the total current density. Write down the version that uses $\mathbf{J}_{\text{free}}$ (which is zero for this insulating material).

c. (3 points)
Expressing $\mathbf{H}$ in terms of $\mathbf{B}$ and μ, and $\mathbf{D}$ in terms of $\mathbf{E}$ and ϵ, taking advantage of the fact that ϵ and μ are constant in this material, rewrite equation (b.) in terms of $\mathbf{B}$ and $\mathbf{E}$.

d. (3 points)
Take $\frac{1}{c}\frac{\partial}{\partial t}$ of both sides of equation (c.). On the left-hand side, interchange the order of differentiation, i.e. apply $\frac{1}{c}\frac{\partial}{\partial t}$ to $\mathbf{B}$ before taking its curl.

e. (3 points)
Use equation (a.) to substitute for $\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}$. Now you should have an equation in which $\mathbf{E}$ is the only vector field that appears.

f. (3 points)
Use the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$$

to eliminate $\nabla \times (\nabla \times \mathbf{E})$ from the left-hand side of equation (e.).

g. (6 points)
Give an argument, based on the absence of free charges in this insulator, and the strict proportionality of $\mathbf{E}$ to $\mathbf{D}$, which allows you to ignore one of the terms on the left-hand side of equation (f.).

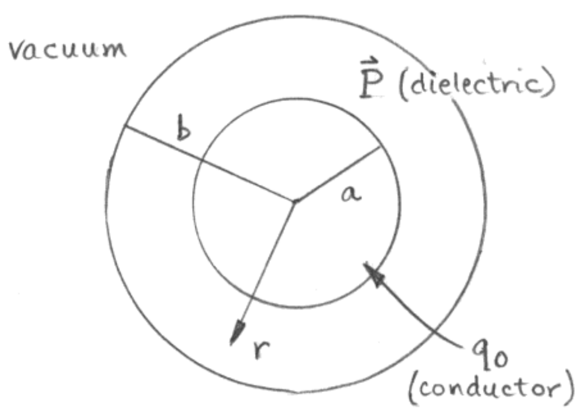
h. (6 points)
Your result should be a wave equation for $\mathbf{E}$. Show that any function of $(kx - \omega t)$, where k and ω are constants, solves this equation.

i. (5 points)
Calculate ω/k, the phase velocity of the solution (h.). Evaluate it in terms of c, ϵ, and μ.

Problem 5. (30 points)
An (insulating) hollow spherical shell of dielectric with inner (outer) radius a (b) has "frozen-in" polarization

$$\mathbf{P} = \hat{\mathbf{r}} \frac{q_0}{2\pi(a+b)r} \ ,$$

where r is the radius measured from its center and q_0 is a constant. The dielectric encloses a conducting sphere of radius a which holds total free charge q_0 (see the figure).



At what values of r does $\mathbf{E}$ vanish? [Your answer may include particular values of r (including those which are not finite), and/or ranges of r.] Note that this (nonlinear) dielectric's electric susceptibility is not defined or supplied here, and it should not appear in your answer.

Problem 6. (35 points)
Consider a hollow cubical box containing particles which make elastic collisions with its walls.

a. (10 points)
Suppose that the particles are molecules of a perfect gas. Using standard arguments about the number of molecules bouncing off an area of wall per unit time, and the momentum per collision that is imparted to the wall, prove that the pressure p (N/m^2) and the kinetic energy density u (J/m^3) of the gas are related by

$$p = \frac{2}{3} u \ .$$

b. (15 points)
Suppose that the box is filled not with molecules, but with electromagnetic radiation, which is quantized into photons. These photons can be considered to be massless particles which, like the perfect gas molecules, do not interact with each other and bounce elastically off the walls. Deduce the relationship between the pressure p and the energy density u of the electromagnetic radiation.

b. (10 points)
At sufficiently high temperature, the electromagnetic radiation pressure inside the box would be sufficient to balance the ambient pressure (10^6 dynes/cm^2) of the earth's atmosphere at sea level. If this were to occur, what would be the root mean square magnetic field (in gauss) inside the cavity?

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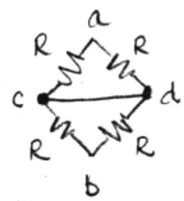
Physics H7B — SOLUTION TO FINAL EXAM — S'99 (Strovink)

1. (a) Equivalent to



Resistance between a and b is unaffected by short from c to d because all R's the same.

It is $(R \parallel R) + (R \parallel R) = R/2 + R/2 = R$

where $R = \boxed{\dfrac{\rho x}{A}}$

(b) $\underset{\sim}{\mathcal{E}}_c - \underset{\sim}{\mathcal{E}}_b = \underset{\sim}{\mathcal{E}}_{in} \dfrac{1/i\omega C}{i\omega L + 1/i\omega C}$

$\underset{\sim}{\mathcal{E}}_d - \underset{\sim}{\mathcal{E}}_b = \underset{\sim}{\mathcal{E}}_{in} \dfrac{i\omega L}{i\omega L + 1/i\omega C}$

$\underset{\sim}{\mathcal{E}}_{out} = \underset{\sim}{\mathcal{E}}_c - \underset{\sim}{\mathcal{E}}_d = \dfrac{1/i\omega C - i\omega L}{1/i\omega C + i\omega L}$

$= \dfrac{1 + \omega^2 LC}{1 - \omega^2 LC} \quad \boxed{\neq 0 \ \forall \omega}$

2. $\pi^+ \longrightarrow \mu^+ + \nu$

Assume μ^+ along $\hat{x}$:

$(m_\pi c, 0, 0, 0) \rightarrow (\sqrt{p^2 + m_\mu^2 c^2}, p, 0, 0) + (p, -p, 0, 0)$

E consv:

$m_\pi c - p = \sqrt{p^2 + m_\mu^2 c^2}$

$m_\pi^2 c^2 - 2 m_\pi c p + \cancel{p^2} = \cancel{p^2} + m_\mu^2 c^2$

$p = \dfrac{(m_\pi^2 - m_\mu^2) c^2}{2 m_\pi c} = \dfrac{m_\pi c}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)$

$\dfrac{E_\mu}{c} = \sqrt{p^2 + m_\mu^2 c^2}$

But $p = \gamma_\mu \beta_\mu m c$, $E_\mu = \gamma_\mu m c^2$

$\therefore \ \beta_\mu^2 = \dfrac{p^2}{E_\mu^2/c^2} = \dfrac{p^2}{p^2 + m_\mu^2 c^2} = \dfrac{1}{1 + \dfrac{m_\mu^2 c^2}{p^2}}$

$\beta_\mu^2 = \dfrac{1}{1 + \dfrac{m_\mu^2 c^2}{\dfrac{m_\pi^2 c^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2}}$

Define $r = m_\mu / m_\pi$. Then

$\beta_\mu^2 = \dfrac{1}{1 + \dfrac{4r^2}{(1-r^2)^2}} = \dfrac{(1-r^2)^2}{1 - 2r^2 + r^4 + 4r^2}$

$\beta_\mu^2 = \dfrac{(1-r^2)^2}{(1+r^2)^2}$, $\beta_\mu = \dfrac{1-r^2}{1+r^2}$

$\beta_\mu = \dfrac{1 - 9/16}{1 + 9/16} = \boxed{\dfrac{7}{25}}$

3. (a) $\vec{A} = \dfrac{\alpha}{2} \hat{z} (y^2 - x^2)$

$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{x} \dfrac{\partial A_z}{\partial y} - \hat{y} \dfrac{\partial A_z}{\partial x} = \alpha (\hat{x} y + \hat{y} x)$

$\vec{F} = -\dfrac{e}{c} \vec{v} \times \vec{B} = -e\alpha (0, 0, \tfrac{v}{c}) \times (y, x=0, 0)$

$\vec{F} = \boxed{-e\alpha \dfrac{v}{c} y \hat{y}}$

(b) $\dfrac{4\pi}{c} \vec{J} = \vec{\nabla} \times \vec{B} = \hat{z}\left(\underbrace{\dfrac{\partial B_y}{\partial x}}_{\alpha} - \underbrace{\dfrac{\partial B_x}{\partial y}}_{\alpha}\right)$

$+ \hat{x}\left(\underbrace{\dfrac{\partial B_z}{\partial y}}_{0} - \underbrace{\dfrac{\partial B_y}{\partial z}}_{0}\right) + \hat{y}\left(\underbrace{\dfrac{\partial B_x}{\partial z}}_{0} - \underbrace{\dfrac{\partial B_z}{\partial x}}_{0}\right)$

$\boxed{= 0}$

(c) $\dfrac{4\pi}{c} \vec{J} = A_0 \vec{\nabla} \times (\hat{x} y - \hat{y} x) = A_0 \hat{z} (-1 - 1)$

(using results of part (b))

$\vec{J} = \boxed{-\hat{z} \dfrac{c}{2\pi} A_0}$

4. (a) $\vec{\nabla} \times \vec{E} = -\dfrac{1}{c} \dfrac{\partial \vec{B}}{\partial t}$

(b) $\vec{\nabla} \times \vec{H} = \dfrac{4\pi}{c} \cancelto{0}{\vec{J}_{free}} + \dfrac{1}{c} \dfrac{\partial \vec{D}}{\partial t}$

(c) $\vec{D} = \epsilon \vec{E}$, $\vec{H} = \vec{B}/\mu$

$\vec{\nabla} \times (\vec{B}/\mu) = \dfrac{\epsilon}{c} \dfrac{\partial \vec{E}}{\partial t}$

(d) $\vec{\nabla} \times \dfrac{1}{\mu} \dfrac{\partial \vec{B}}{c\, \partial t} = \dfrac{\epsilon}{c^2} \dfrac{\partial^2 \vec{E}}{\partial t^2}$

(e) $-\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \dfrac{\mu \epsilon}{c^2} \dfrac{\partial^2 \vec{E}}{\partial t^2}$

(f) $-\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) + \nabla^2 \vec{E} = \dfrac{\mu\epsilon}{c^2} \dfrac{\partial^2 \vec{E}}{\partial t^2}$

(g) $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{D}/\epsilon = \rho_{free}/\epsilon = 0$

4. (h) $\vec{E} = \vec{E}(kx - \omega t)$

$$\nabla^2 \vec{E} = k^2 \vec{E}$$

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\epsilon\mu}{c^2}\omega^2 \vec{E}$$

cancel if $\frac{\epsilon\mu}{c^2}\omega^2 = k^2$

(i) $\frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}}$ from (h)

---

5. Spherical symmetry ⇒ use Gauss' Law (for $\vec{E}$ or $\vec{D}$) with spherical shell

Statically, $\vec{E}$ must be 0 within any conductor

$$\Rightarrow \boxed{\vec{E} = 0, \; r < a}$$

Place a shell at $r > b$

$$\oint \vec{D} \cdot d\vec{a} = 4\pi q_{free}$$

$\hat{D} = \hat{r}$ by symmetry

$$4\pi r^2 D_r = 4\pi q_0$$

$$\vec{D} = \hat{r}\frac{q_0}{r^2}$$

$$\boxed{\vec{D} = 0, \; r = \infty}$$

Place a shell at $a < r < b$

$\vec{D} = \hat{r}\frac{q_0}{r^2}$ as above

$$4\pi \vec{P} = \hat{r}\frac{2q_0}{(a+b)r}$$

$$\vec{E} = \vec{D} - 4\pi\vec{P} = \hat{r}\frac{q_0}{r^2}\left(1 - \frac{2r}{a+b}\right)$$

$$\boxed{\vec{E} = 0 \text{ when } r = \frac{a+b}{2}}$$

---

6. (a) Consider wall at constant positive x. In time Δt the # n of molecules hitting the wall is

$$n = \frac{NA}{2} v_x \Delta t \quad (N = \text{number density}, \; A = \text{area})$$

other half going wrong way

After elastic collision, wall receives momentum $2mv_x$:

$$\Delta p_x = (Nm)\frac{v_x}{\cancel{2}}\cancel{2}v_x \; A\Delta t$$

ρ = mass density

Pressure $p = \frac{F_x}{A} = \frac{\Delta p_x}{A\Delta t} = \rho\langle v_x^2\rangle$

average over molecules

KE density $u = N\left(\frac{1}{2}m\langle v^2\rangle\right)$

$$= \frac{1}{2}\rho\langle v^2\rangle$$

$$u = \frac{3}{2}\rho\langle v_x^2\rangle$$

$= \langle v_y^2\rangle = \langle v_z^2\rangle$

$$\frac{p}{u} = \frac{\rho\langle v_x^2\rangle}{\frac{3}{2}\rho\langle v_x^2\rangle} = \boxed{\frac{2}{3}}$$

(b.) Now the particles are fully relativistic, with $v = \hat{x}v_x + \hat{y}v_y + \hat{z}v_z$,

$$v_x^2 + v_y^2 + v_z^2 = c^2$$

Again $n = \frac{NA}{2} v_x \Delta t$.

But $p_x = \frac{E}{c} \cdot \frac{v_x}{c}$ $\quad E$ = photon energy

total momentum; fraction in x direction

Wall receives momentum

$$\Delta p_x = \left(\frac{NA}{\cancel{2}} v_x \Delta t\right)\left(\cancel{2}\frac{E}{c}\frac{v_x}{c}\right)$$

$$= (NE)\left(v_x^2/c^2\right) A\Delta t$$

energy density u

With $\left\langle\frac{v_x^2}{c^2}\right\rangle = \frac{1}{3}$ this becomes

$$\boxed{p = \frac{u}{3}}$$

(c.) $u = \frac{\langle E^2 + B^2\rangle}{8\pi}$

In an EM wave in vacuum $\langle E^2\rangle = \langle B^2\rangle$

so $u = \frac{\langle B^2\rangle}{4\pi}$

$$\langle B^2\rangle = 4\pi u = 12\pi p$$

given $p = 10^6$ dynes/cm²

$$\Rightarrow \langle B^2\rangle^{1/2} \approx 6000 \text{ gauss}$$

University of California, Berkeley
Physics H7C, Fall 1999 (Strovink)

General Information (27 Aug 99)

Web site for this class: First link on `http://d0lbln.lbl.gov` .

Instructors: Prof. **Mark Strovink**, 437 LeConte; (LBL) 486-7087; (home, before 10) 486-8079; (UC) 642-9685. Email: `strovink@lbl.gov` . Web: `http://d0lbln.lbl.gov` . Office hours: M 3:15-4:15, 5:30-6:30. Mr. **Derek Kimball**, (UC research lab) 221 Birge, 643-1829; (home, before 11) 548-3115. Email: `dfk@uclink4.berkeley.edu` . Office hours (to be held in 211 LeConte): Tu 2-4. You may also get help in the 7C Course Center, 262 LeConte.

Lectures: Tu-Th 11:10-12:30, 3 LeConte. Lecture attendance is essential, since not all of the course content can be found in the course text or handouts. On one or two occasions it is possible that the lecture normally held on Thursday will be given instead on Wednesday, at 4:30-5:50 PM, in 343 LeConte.

Labs: Begin in the third week, in 278 LeConte. As soon as possible, please enroll in any one of the 7C lab sections that fits your schedule.

Discussion Sections: Begin in the second week. **Section 1**: W 4-5, 343 LeConte; **Section 2**: W 5:30-6:30, also in 343 LeConte. You are welcome at either or both sections. You are especially encouraged to attend discussion section regularly. There you will learn techniques of problem solving, with particular application to the assigned exercises.

Texts (required): Fowles, **Introduction to Modern Optics**, *Second Edition* (Dover paperback, 1989). Rohlf, **Modern Physics from α to Z^0** (Wiley, 1994). **Supplementary text** (recommended): Hecht, **Schaum's Outline of Theory and Problems of Optics** (McGraw-Hill paperback, 1975). (Don't confuse this with Hecht's hardbound books on optics.)

Problem Sets: Twelve problem sets are assigned and graded. Solutions will be available. Problem sets are due on Thursday at 5 PM on weeks in which there is no exam, beginning in week 2. Deposit problem sets in the box labeled "H7C" outside 211 LeConte. You are encouraged to attempt all the problems. Students who do not do them find it almost impossible to learn the material and to succeed on the examinations. Discuss these problems with your classmates as well as with the teaching staff; however, when the time comes to write up your solutions, *work independently*. Credit for collective writeups, which are easy to identify, will be divided among the collectivists. Late papers will not be graded. Your lowest problem set score will be dropped, in lieu of due date extensions for any reason.

Syllabus: H7C has one mandatory syllabus card. It will be collected when the first midterm examination is handed back in lecture. This card pays for the 7C laboratory experiment descriptions and instructions. Copies of solutions to each problem set will also be available for separate purchase at Copy Central.

Exams: There will be two 80-minute midterm examinations and one 3-hour final examination. Before confirming your enrollment in this class, please check that its final Exam Group 9 does not conflict with the Exam Group for any other class in which you are enrolled. Please verify that you will be available for the midterm examinations (Th 7 Oct and Th 11 Nov, 11:10-12:30), and for the final examination, F 10 Dec, 5-8 PM. Except for unforeseeable emergencies, it will not be possible for the midterm or final exams to be rescheduled. Passing H7C requires passing the final exam.

Grading: 30% midterms; 25% problem sets; 40% final exam; 5% lab. Grading is not "curved" -- it does not depend on your performance relative to that of your H7C classmates. Rather it is based on comparing your work to that of a generation of earlier lower division Berkeley physics students, with due allowance for educational trends.

Course Outline

Week No.	Week of...	Reading chapter	Topic (Pu=Purcell, Fo=Fowles, Ro=Rohlf)	Problem Set No.	Due 5 PM on...	7C lab
1	23-Aug	Ro 4, Pu A, Fo App. I	Review of special relativity; relativistic transformation of EM fields			NONE
2	30-Aug	Pu 9	Review of Maxwell's equations and EM waves…			NONE
		Pu 10,11	…in vacuum and in material; boundary conditions	1	2-Sep	
3	6-Sep		LABOR DAY			
		Pu B	Radiation by an accelerated charge			reflect/
		Fo 2	Polarization	2	9-Sep	refract
4	13-Sep	Fo 2	Plane reflection/refraction			geom
		Fo 3	Interference	3	16-Sep	optics
5	20-Sep	Fo 3	Coherence			michelson
		Fo 4	Multiple beams	4	23-Sep	interferom
6	27-Sep	Fo 4	Multiple beams			diffract/
		Fo 5	Diffraction	5	30-Sep	interfer
7	4-Oct	Fo 5	Diffraction			NONE
		Fo 6	Optics of solids			
	7-Oct		MIDTERM 1 (covers PS 1-5)			
8	11-Oct	Ro 2	Optics of solids, Maxwell-Boltzmann distribution			polarI-
		Ro 3	Planck's constant	6	14-Oct	zation
9	18-Oct	Ro 3	Planck's constant			
		Ro 5	Wave properties of matter	7	21-Oct	NONE
10	25-Oct	Ro 5	Uncertainty principle			photo-
		Ro 6	Probing the structure of matter	8	28-Oct	electric
11	1-Nov	Ro 7	Schroedinger equation			NONE
		Ro 8	Hydrogen atom	9	4-Nov	
12	8-Nov	Ro 9	Periodic table			atomic
	11-Nov		MIDTERM 2 (covers PS 1-9)			spectra
13	15-Nov	Ro 12	Quantum statistics			radio
		Ro 17	Quarks and leptons	10	18-Nov	half-life
14	22-Nov	Ro 18	Unification of the forces	11	24-Nov	NONE
	25-Nov		THANKSGIVING			
15	29-Nov	Ro 19	Cosmology			makeups
			LAST LECTURE (review)	12	2-Dec	
16	6-Dec					
	8-Dec		Final exams begin			
	10-Dec	5-8 PM	H7C FINAL EXAM (Group 9) (covers PS 1-12)			

updated 12/31/99

NOTES ON H7C TEXTS:

RELATIVITY TEXTS (to supplement distributed handwritten notes):

Taylor and Wheeler, Spacetime Physics:
Good for spacetime, Lorentz transformations as spacetime rotations, the boost parameter, and rocket problems.

Rohlf, Modern Physics from alpha to Z0 (H7C required), chapter 4:
Fair for relativistic kinematics -- a Compton scattering derivation is given, though it is not the most elegant.

Purcell, Electricity and Magnetism (H7B text), appendix A and chapter 9:
Good for relativistic transformation of electric and magnetic fields.

OPTICS TEXTS:

Fowles, Introduction to Modern Optics (H7C required): Terse.

Hecht, Schaum's Outline of Theory and Problems of Optics (H7C recommended): Students have found this to be a useful summary of formulae and a place to find simple practice problems.

OPTICS TEXTS THAT PREVIOUSLY HAVE BEEN USED IN H7C:

Pedrotti & Pedrotti, Introduction to Optics, 2nd ed. (Prentice-Hall)

Hecht, Optics, 3rd ed. (Addison Wesley)

MODERN PHYSICS TEXT:

Rohlf, Modern Physics from alpha to Z0 (H7C required)

MODERN PHYSICS TEXTS THAT PREVIOUSLY HAVE BEEN USED IN H7C:
These are listed in descending (subjective) order of usefulness:

Blatt, Modern Physics (McGraw-Hill)

Beiser, Concepts of Modern Physics, 5th ed. (McGraw-Hill)

Ohanian, Modern Physics, 2nd ed. (Prentice-Hall)

Eisberg & Resnick, Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, 2nd ed. (Wiley)

Serway, Modern Physics, 2nd ed. (Saunders)

Tipler, Physics for Scientists and Engineers, Vol II, 3rd ed. (Worth)

Errata

Fowles, **Introduction to Modern Optics**, Second Edition

M. Strovink

This book was chosen for the optics portion of H7C because the selection and relative coverage of topics is excellent. As now reprinted by Dover, it is also bargain priced. For the most part Fowles tells you what you want to know, with brevity and insight. Perhaps more important for a general course like H7C, Fowles usually does not tell you what you do not want to know. This is in contrast to the pricey and encyclopædic Hecht, in which fundamental concept and arcane detail are mixed together without any obvious differentiation.

The main problem with the book is that Fowles, while a good physicist, seems as an author not to be a sufficiently concerned with correctness. The text is full of annoying lapses in accuracy of the drawings. Despite the existence of two editions, the book seems never to have been adequately proof-read or revised.

Therefore, to benefit from this otherwise good book, **you should correct the many errors in your copy**, using this guide.

Page 23, Eq. 2.16 should read

$$H = \frac{nE}{Z_0}\frac{\mu_0}{\mu}$$

Page 25, Eq. 2.23 should read

$$I = \frac{1}{2}E_0H_0 = \frac{n}{2Z_0}|E_0|^2\frac{\mu_0}{\mu}$$

Page 41, Eqs. 2.49, 2.50, and 2.51: these equations assume that μ is the same in both media.
Page 43 (an omission, not an error). To Eqs. 2.54 and 2.55 should be added:

$$t_s = \frac{2\cos\theta}{\cos\theta + n\cos\phi}$$

$$t_p = \frac{2\cos\theta}{n\cos\theta + \cos\phi}$$

Page 45. The details in Fig. 2.12 are reliable only to a factor of ≈ 2, due to bad registration of the curves (printed in color in the hardcover version) with respect to the axes.
Page 56, Problem 2.23: "Brewster window" should be "Brewster interface", *i.e.* one interface between $n = 1$ and $n = n$.
Page 64, last sentence, beginning "In this case the central fringe...", and continuing onto page 65, should be ignored. This sentence would be correct only if plate A were not (even half) silvered.
Page 75, Fig. 3.13: For your own increased comprehension, indicate the following on the figure: s is the distance between the sources S_a and S_b; l is the distance between the receivers P_1 and P_2; r is the distance between the average of S_a and S_b, and the average of P_1 and P_2.
Page 76, sentence following Eq. 3.39 should read: "... small in comparison with τ_0."
Page 76, sentence preceding Eq. 3.40 should read: "We then have $\tau_b - \tau_a = (r_{2b} - r_{2a})/c$, or approximately"
Page 76, Eq. 3.40 should read:

$$\tau_b - \tau_a \approx \frac{sl}{cr}$$

Page 77, Eq. 3.41 should read:

$$\omega(\tau_b - \tau_a) = \frac{\omega s l_t}{cr} = \pi$$

Page 77, Eq. 3.42 should read:

$$l_t = \frac{r\lambda}{2s}$$

Page 77, Eq. 3.43 should read:

$$l_t = \frac{\lambda}{2\theta_s}$$

Page 81, Eq. 3.46 should read:

$$G(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(x)e^{-ikx}dx$$

The factor 2π should appear to the power -1 rather than $-\frac{1}{2}$ because there is no corresponding factor of $1/\sqrt{2\pi}$ in Eq. 3.45. Conversely, in Eq. 3.28 on page 71 Fowles instead has adopted the neater convention of inserting a factor $1/\sqrt{2\pi}$ in the definitions of each of the Fourier transform pairs.

Page 97, Fig. 4.7: For consistency with Eqs. 4.23, 4.27, and 4.32, in the [central] region with index n_1 shown in the figure, k_1 and k_1' should be replaced by k and k'.

Page 98, last sentence: add at the end of this sentence ", with $R + n_T T/n_0 = 1$."

Page 137, Fig. 5.28; page 139, Fig. 5.29; and page 143, Fig. 5.32: The patterns should be centered on $y = 0$ or $\nu = 0$ (again a regisration problem).

Page 150, Problem 5.16: The last parenthesis should be "(Assume the trailing edge of the moon to be effectively straight.)"

Page 220, 4^{th} sentence should read: "For left circularly polarized light, the direction of spin of the photon is parallel to the direction of propagation, whereas for right circularly polarized light, it is antiparallel to the direction of propagation. [This is a convention that depends on the charge of the particle (electron or positron) to which the photon couples; our (more usual) convention takes that charge to be positive.]"

Page 297, Eq. 10.11 should read:

$$\frac{1}{f} = (n-1)\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{(n-1)t}{nr_1r_2}\right]$$

Page 297, Eq. 10.12 should read:

$$d_2 = -ft\left(\frac{1-n}{r_1}\right)$$

assuming that d_1 and d_2 are positive distances as drawn in Fig. 10.3.

Page 298, Fig. 10.3. The rays are so badly drawn as to be meaningless. The top ray should emanate from the object in a direction parallel to the axis, and it should be bent only at principal plane H'. The bottom ray should emanate from the image in a direction parallel to the axis, and it should be bent only at principal plane H.

Page 307, 1^{st} complete sentence should read: "The interference pattern can be made to shift by one fringe *i.e.* from a bright fringe to the adjacent dark fringe, or *vice versa*) by displacing either of the two mirrors M_1 or M_2 a distance of $\frac{1}{4}$ wavelength."

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 1

1. Two supernovæ are observed on earth in the direction of the north star, separated by 10 years. From the theory of supernovæ these are known to have identical ("standard candle") light output, yet the first is observed to have four times the light intensity of the second because it is closer.
(a.)
An astronomer theorizes that the two stars were at rest with respect to the earth, and that the first supernova triggered the second. What is the *maximum* distance between the earth and the first supernova under this hypothesis?
(b.)
A physicist theorizes that the two stars were traveling with the same (unspecified) velocity away from the earth, and that, in their common rest frame, the two supernovæ occurred at the same proper time. What is the *minimum* distance between the earth and the first supernova under this hypothesis?

2. Inertial reference frames $\mathcal{S}'$ and $\mathcal{S}$ coincide at $t' = t = 0$. You may ignore the z dimension, so that a point in spacetime is determined by only three quantities $r \equiv (ct, x, y)$. The Lorentz transformation between $\mathcal{S}$ and $\mathcal{S}'$ is given by

$$\begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct \\ x \\ y \end{pmatrix},$$

where $\mathcal{L}$ is a 3×3 matrix.
(a.)
Assume for this part that $\mathcal{S}'$ moves with velocity

$$\mathbf{V} = \beta c \hat{\mathbf{x}}$$

with respect to $\mathcal{S}$. Using your knowledge of Lorentz transformations (no derivation necessary), write $\mathcal{L}$ for this case.
(b.)
Assume for this part that $\mathcal{S}'$ moves with velocity

$$\mathbf{V} = \beta c \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}$$

with respect to $\mathcal{S}$. Find $\mathcal{L}$ for this case. (*Hint.* Rotate to a system in which $\mathbf{V}$ is along the $\hat{\mathbf{x}}$ axis, transform using your answer for part (**a.**), and then rotate back. Check that your result is symmetric under interchange of x and y, as is $\mathbf{V}$, and that it reduces to the unit matrix as $\beta \to 0$.)

3. Work out the Lorentz transformation matrix $\mathcal{L}$ for the general case in which $\boldsymbol{\beta}$ of frame $\mathcal{S}'$ is directed along an arbitrary unit vector $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ as seen in frame $\mathcal{S}$, *e.g.*

$$r' = \mathcal{L} r, \quad \mathcal{L} = ?$$

4. In a straight channel oriented along the $\hat{\mathbf{z}}$ axis there are two opposing beams:

- a beam of positrons (charge $+e$) with velocity $+\hat{\mathbf{z}}\beta c$.
- a beam of electrons (charge $-e$) with velocity $-\hat{\mathbf{z}}\beta c$.

Each beam is confined to a small cylindrical volume of cross sectional area A centered on the $\hat{\mathbf{z}}$ axis. Within that volume, there is a uniform number density $= n$ positrons/m^3 and n electrons/m^3.
(a.)
In terms of n, A, e, and β, calculate the total current I in the channel due to the sum of both beams (note $I \neq 0$).
(b.)
Use Ampère's Law to calculate the azimuthal magnetic field B_ϕ outside the channel a distance r from the $\hat{\mathbf{z}}$ axis.

Consider now a Lorentz frame $\mathcal{S}'$ travelling in the $\hat{\mathbf{z}}$ direction with velocity βc relative to the lab frame described above. (This β is the same β as above.)
(c.)
As seen in $\mathcal{S}'$, calculate the number density n'_+ of *positrons* within the cylindrical volume. (You may use elementary arguments involving

space contraction, or you may use the fact that $(c\rho, \mathbf{j})$ is a 4-vector, where ρ is the charge density (Coul/m^3) and $\mathbf{j}$ is the current density (amps/m^2).)

(d.)

As seen in $\mathcal{S}'$, calculate the number density n'_- of *electrons* within the cylindrical volume.

(e.)

Calculate the radial electric field E'_r seen in $\mathcal{S}'$. Do this both

- by using the results of (c.) and (d.) plus Gauss's law, and
- by using the results of (b.) plus the rules for relativistic $\mathbf{E}$ and $\mathbf{B}$ field transformations.

5. (Taylor and Wheeler problem 51)

The clock paradox, version 3.

Can one go to a point 7000 light years away – *and return* – without aging more than 40 years? "Yes" is the conclusion reached by an engineer on the staff of a large aviation firm in a recent report. In his analysis the traveler experiences a constant "1-g" acceleration (or deceleration, depending on the stage reached in her journey). Assuming this limitation, is the engineer right in his conclusion? (For simplicity, limit attention to the first phase of the motion, during which the astronaut accelerates for 10 years – then double the distance covered in that time to find how far it is to the most remote point reached in the course of the journey.)

(a.)

The acceleration is *not* $g = 9.8$ meters per second per second relative to the laboratory frame. If it were, how many times faster than light would the spaceship be moving at the end of ten years (1 year $= 31.6 \times 10^6$ seconds)? *If the acceleration is not specified with respect to the laboratory, then with respect to what is it specified?* Discussion: Look at the bathroom scales on which one is standing! The rocket jet is always turned up to the point where these scales read one's *correct* weight. Under these conditions one is being accelerated at 9.8 meters per second per second with respect to a spaceship that (1) instantaneously happens to be riding alongside with identical velocity, but (2) is *not* being accelerated, and, therefore (3) *provides the* (momentary) *inertial frame of reference relative to which the acceleration is g.*

(b.)

How much velocity does the spaceship have after a given time? This is the moment to object to the question and to rephrase it. *Velocity βc* is not the simple quantity to analyze. The simple quantity is the *boost parameter η*. This parameter is simple because it is *additive* in this sense: Let the boost parameter of the spaceship with respect to the imaginary instantaneously comoving inertial frame change from 0 to $d\eta$ in an astronaut time $d\tau$. Then the boost parameter of the spaceship with respect to the *laboratory* frame changes in the same astronaut time from its initial value η to the subsequent value $\eta + d\eta$. Now relate $d\eta$ to the acceleration g in the instantaneously comoving inertial frame. In this frame $g\, d\tau = c\, d\beta = c\, d(\tanh \eta) = c \tanh (d\eta) \approx c\, d\eta$ so that

$$c\, d\eta = g\, d\tau$$

Each lapse of time $d\tau$ on the astronaut's watch is accompanied by an additional increase $d\eta = \frac{g}{c} d\tau$ in the boost parameter of the spaceship. In the laboratory frame the total boost parameter of the spaceship is simply the sum of these additional increases in the boost parameter. Assume that the spaceship starts from rest. Then its boost parameter will increase linearly with *astronaut* time according to the equation

$$c\eta = g\tau$$

This expression gives the boost parameter η of the spaceship in the *laboratory* frame at any time τ in the *astronaut's* frame.

(c.)

What laboratory distance x does the spaceship cover in a given astronaut time τ? At any instant the velocity of the spaceship in the laboratory frame is related to its boost parameter by the equation $dx/dt = c \tanh \eta$ so that the distance dx covered in *laboratory* time dt is

$$dx = c \tanh \eta\, dt$$

Remember that the time between ticks of the astronaut's watch $d\tau$ appear to have the larger value dt in the laboratory frame (time dilation) given by the expression

$$dt = \cosh \eta \, d\tau$$

Hence the laboratory distance dx covered in *astronaut* time $d\tau$ is

$$dx = c \tanh \eta \cosh \eta \, d\tau = c \sinh \eta \, d\tau$$

Use the expression $c\eta = g\tau$ from part b to obtain

$$dx = c \sinh \left(\frac{g\tau}{c} \right) d\tau$$

Sum (integrate) all these small displacements dx from zero astronaut time to a final astronaut time to find

$$x = \frac{c^2}{g} \left[\cosh \left(\frac{g\tau}{c} \right) - 1 \right]$$

This expression gives the laboratory *distance* x covered by the spaceship at any time τ in the astronaut's frame.
(d.)
Plugging in the appropriate numerical values, determine whether the engineer is correct in his conclusion reported at the beginning of this exercise.

6. Electrons ($mc^2 = 0.5 \times 10^6$ eV) are accelerated over a distance of 3.2 km from rest to a total energy of 5×10^{10} eV at SLAC (Stanford).
(a.)
To what boost η are the electrons ultimately brought?
(b.)
Assuming that the electrons are subjected to a uniform acceleration as observed in their comoving inertial frame, how many g's of acceleration do they feel?
(c.)
As observed in the lab, for what time interval is each electron in flight? What is the corresponding *proper* time interval? Evaluate the ratio of the two intervals (a sort of average γ factor).

7. (Taylor and Wheeler problem 75)
Doppler equations.
A photon moves in the xy laboratory plane in a direction that makes an angle ϕ with the x axis, so that its components of momentum are $p_x = p \cos \phi$, $p_y = p \sin \phi$, and $p_z = 0$.
(a.)
Use the Lorentz transformation equations for the momentum-energy 4-vector and the relation $E^2/c^2 - p^2 = 0$ for a photon to show that, in the rocket frame $\mathcal{S}'$ (moving with velocity $\beta_r c$ along the x, x' direction, and coinciding with the laboratory frame at $t = t' = 0$), the photon has an energy E' given by the equation

$$E' = E \cosh \eta_r (1 - \beta_r \cos \phi)$$

and moves in a direction that makes an angle ϕ' with the x' axis given by the equation

$$\cos \phi' = \frac{\cos \phi - \beta_r}{1 - \beta_r \cos \phi}$$

(b.)
Derive the inverse equations for E and $\cos \phi$ as functions of E', $\cos \phi'$, and β_r.
(c.)
If the frequency of light in the laboratory is ν, what is the frequency ν' of light in the rocket frame? This difference in frequency due to relative motion is called the *relativistic Doppler shift*. Do these equations enable one to tell in what frame the *source* of the photons is at rest?

8. Consider the following situation. A star is known, by means of external data, to be located instantaneously a distance D from an observer on earth. The external data do not tell us the rate of change of D with time.

In her measurements, the observer corrects for aberration caused by the local velocity of the earth's surface, due both to its daily rotation and its yearly orbit. Therefore we do not need to take into account these boring local phenomena in what follows.

After making these corrections, the observer sees that the star is undergoing angular motion $d\psi/dt$ across the sky, such that $D\,d\psi/dt = c$, where c is the speed of light.

Finally, the observer measures the wavelength spectrum of light from this star, and finds its features not to be redshifted or blueshifted at all – they are exactly where they would be if the star were perfectly at rest with respect to the observer.

Is it possible that this situation is physically reasonable? If so, what might be the true motion of the star with respect to the observer? If not, why not?

"Let's get something straight here... e is real, 10 is just the number of fingers we have."

- Prof. Nima Arkani-Hamed, UC Berkeley

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

If you liked problem 1 and you're interested in astrophysics, general relativity, and cosmology, you should check out a paper by Saul Perlmutter, Michael S. Turner, and Martin White (Physical Review Letters, July 26, 1999, Volume **83**, Issue 4, pp. 670-673). This article and references therein describe an ongoing study of type Ia supernovae which have "standard candle" light output and have enabled these scientists to measure large-scale cosmological parameters. One of the most interesting results is that their data is consistent with a universe that is expanding at an *accelerating* rate! Saul Perlmutter's group is here at Berkeley and works at LBL.

Problem 1

First, let's consider the implications of the difference in light intensity of the two supernovae (SN1 and SN2). These particular supernovae are known to have identical "standard candle" light output, i.e. the total light power P emitted is the same for SN1 and SN2. A small solid angle $d\Omega$ of the total light is detected on earth, so the intensity of light I detected is given by:

$$I = \frac{d\Omega \cdot P}{4\pi R^2}, \tag{1}$$

where R is the distance from a supernova to the earth at the time the light is emitted. Therefore the ratio of light intensities tells us the ratio of distances:

$$\frac{I_1}{I_2} = \frac{R_2^2}{R_1^2} = 4. \tag{2}$$

(a)

An astronomer theorizes that SN1 causes SN2, and that they are both at rest with respect to the earth. Since the two events SN1 $= (ct_1, x_1)$ and SN2 $= (ct_2, x_2)$ are causally related, there must be a timelike or lightlike separation between the events:

$$c^2\Delta t^2 - \Delta x^2 \geq 0, \tag{3}$$

where $\Delta t = t_2 - t_1$ and $\Delta x = x_2 - x_1$.

From Eq. (2), we see that if the distance between SN2 and SN1 is Δx, the distance between SN1 and earth is also Δx. Then the elapsed time Δt_{earth} between detection of the two supernovae on earth, taking into account the propagation time of the light to the earth, is given by:

$$c\Delta t_{earth} = c\Delta t + c\left(\frac{2\Delta x}{c}\right) - c\left(\frac{\Delta x}{c}\right) = c\Delta t + \Delta x. \tag{4}$$

From Eq. (3), we know that $c\Delta t \geq \Delta x$, so we find that:

$$\Delta x \leq \frac{c\Delta t_{earth}}{2}, \tag{5}$$

or that $\Delta x_{max} = 5$ light years.

(b)

A physicist theorizes that the two supernovae were traveling away from the earth at some velocity and occurred at the same proper time. In this case the two events have a spacelike or lightlike separation:

$$c^2\Delta t^2 - \Delta x^2 \leq 0. \tag{6}$$

Thus, in the earth frame there is an observed time difference Δt_{obs} between SN1 and SN2, which from Eq. (6) must satisfy:

$$c\Delta t_{obs} \leq \Delta x_{obs}, \tag{7}$$

where Δx_{obs} is the distance between SN1 and SN2 as observed in the earth frame. As in part (a) we include the light propagation time, and find that:

$$c\Delta t_{earth} = c\Delta t_{obs} + \Delta x_{obs} \leq 2\Delta x_{obs}. \tag{8}$$

So in this case we find that $\Delta x_{obs,min} = 5$ light years.

Problem 2

(a)

This is just the traditional Lorentz matrix, only in 3D, so it is similar to the expression (1.12) in Prof. Strovink's notes on relativity,

$$\begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \end{pmatrix}. \tag{9}$$

(b)

The general idea is to rotate to a system where we know the correct transform (from part (a)), and then rotate back. So we begin with:

$$\boldsymbol{r}' = \mathfrak{L} \cdot \boldsymbol{r}. \tag{10}$$

Then we rotate the coordinate system with a rotation matrix R so that $\vec{\beta}$ is along $\hat{x}$:

$$R\boldsymbol{r}' = R\mathfrak{L} \cdot \boldsymbol{r} = R\mathfrak{L}R^{-1}(R\boldsymbol{r}). \tag{11}$$

In this frame we know the Lorentz transform Λ from part (a), so we find that:

$$R\mathfrak{L}R^{-1} = \Lambda. \tag{12}$$

In other words,

$$\mathfrak{L} = R^{-1}\Lambda R. \tag{13}$$

The math can be made a little easier in these cases because rotations are described by orthogonal matrices which satisfy $R^{-1} = R^T$ where R^T is the transpose of R.

Now for the actual math. From (a) and Eq. (13) we find:

$$\mathfrak{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix} \cdot \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix}, \tag{14}$$

where we use straightforward 3D extensions of the usual rotation matrices. Multiplying these matrices gives us:

$$\mathfrak{L} = \begin{pmatrix} \gamma & -\gamma\beta cos\theta & -\gamma\beta sin\theta \\ -\gamma\beta cos\theta & \gamma cos^2\theta + sin^2\theta & -cos\theta sin\theta + \gamma cos\theta sin\theta \\ -\gamma\beta sin\theta & -cos\theta sin\theta + \gamma cos\theta sin\theta & cos^2\theta + \gamma sin^2\theta \end{pmatrix}. \tag{15}$$

We are given that:

$$\boldsymbol{V} = \beta c \frac{\hat{x} + \hat{y}}{\sqrt{2}}, \tag{16}$$

so $\theta = \pi/4$. Then $\mathfrak{L}$ is given by:

$$\mathfrak{L} = \begin{pmatrix} \gamma & \frac{-\gamma\beta}{\sqrt{2}} & \frac{-\gamma\beta}{\sqrt{2}} \\ \frac{-\gamma\beta}{\sqrt{2}} & \frac{1+\gamma}{2} & \frac{1}{2}(\gamma - 1) \\ \frac{-\gamma\beta}{\sqrt{2}} & \frac{1}{2}(\gamma - 1) & \frac{1+\gamma}{2} \end{pmatrix} \tag{17}$$

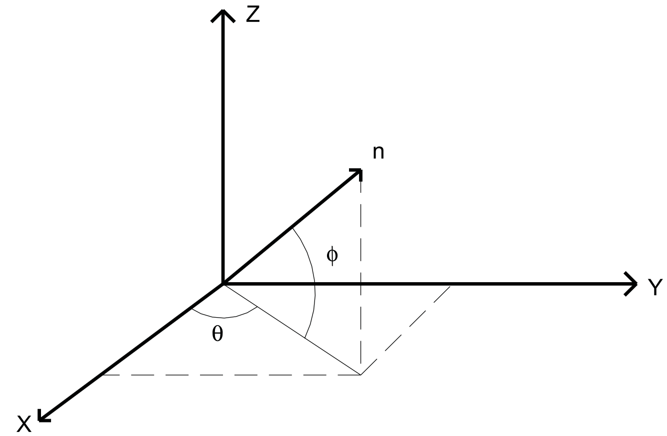


Figure 1: Relationship between $\hat{n}$ and the angles θ and ϕ employed in problem (3) in the first approach.

As you can see by inspection, this matrix is symmetric under interchange of x and y and reduces to the identity matrix as $\beta \to 0$.

Problem 3

Here are two common approaches to this problem. The first method involves matrix multiplication in a manner similar to that employed in problem 2. The second involves determining a general vector formula for the Lorentz transform.

Approach 1

First, we'll determine the rotation matrix R which will take us into the frame where $\vec{\beta}$ is along $\hat{x}$. For me, it's easier to think of this in terms of the angles θ and ϕ as defined in Fig. 1.

From Fig. 1, we notice that there is a natural correspondence between (n_x, n_y, n_z) and θ, ϕ given by

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} cos\phi cos\theta \\ cos\phi sin\theta \\ sin\phi \end{pmatrix}. \tag{18}$$

Later, these relations will be used to express $\mathfrak{L}$ in terms of n_x, n_y and n_z.

R is given by the multiplication of two rotation matrices A and B,

$$R = B \cdot A, \tag{19}$$

where A rotates the axes about $\hat{z}$ by θ

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

and B rotates the axes about a new $\hat{y}'$ (the y-axis after rotation by A) by ϕ so that $\hat{x}$ is along $\hat{n}$:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\phi & 0 & sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & -sin\phi & 0 & cos\phi \end{pmatrix}. \tag{21}$$

Applying Eq. (13), we can now solve for $\mathfrak{L}$:

$$\mathfrak{L} = A^{-1} \cdot B^{-1} \cdot \Lambda \cdot B \cdot A = A^T \cdot B^T \cdot \Lambda \cdot B \cdot A, \tag{22}$$

where Λ is given by Eq. (1.12) in Prof. Strovink's notes on relativity.

The result of this rather tedious matrix multiplication, after some simplification using basic trigonometric identities, is given by $\mathfrak{L} =$

$$\begin{pmatrix} \gamma & -\beta\gamma cos\theta cos\phi & -\beta\gamma sin\theta cos\phi & -\beta\gamma sin\phi \\ -\beta\gamma cos\theta cos\phi & 1+(\gamma-1)cos^2\theta cos^2\phi & (\gamma-1)sin\theta cos\theta cos^2\phi & (\gamma-1)cos\theta sin\phi cos\phi \\ -\beta\gamma sin\theta cos\phi & (\gamma-1)sin\theta cos\theta cos^2\phi & 1+(\gamma-1)sin^2\theta cos^2\phi & (\gamma-1)sin\theta sin\phi cos\phi \\ -\beta\gamma sin\phi & (\gamma-1)cos\theta sin\phi cos\phi & (\gamma-1)sin\theta sin\phi cos\phi & 1+(\gamma-1)sin^2\phi \end{pmatrix}. \tag{23}$$

If we then use the relations given in Eq. (18) to re-express Eq. (23) in terms of n_x, n_y and n_z we find that:

$$\mathfrak{L} = \begin{pmatrix} \gamma & -\beta\gamma n_x & -\beta\gamma n_y & -\beta\gamma n_z \\ -\beta\gamma n_x & 1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z \\ -\beta\gamma n_y & (\gamma-1)n_y n_x & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z \\ -\beta\gamma n_z & (\gamma-1)n_z n_x & (\gamma-1)n_z n_y & 1+(\gamma-1)n_z^2 \end{pmatrix}. \tag{24}$$

Approach 2

In this approach, we work out a vector formula for the Lorentz transformation using the fact that length contraction occurs only in the direction of $\vec{\beta}$. If, for example, $\hat{n}$ is along $\hat{x}$, Λ from Eq. (1.12) in Strovink's notes is applicable and we find:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta x) \\ x + (\gamma-1)x - \gamma\beta ct \\ y \\ z \end{pmatrix}. \tag{25}$$

Let $\vec{r} \equiv (x, y, z)$, then by analogy with Eq. (25) we find that:

$$ct' = \gamma(ct - \beta\vec{r}\cdot\hat{n}) \tag{26}$$

and

$$\vec{r}' = \vec{r} + \hat{n}((\gamma-1)\vec{r}\cdot\hat{n} - \gamma\beta ct). \tag{27}$$

We can then express these equations in terms of n_x, n_y and n_z:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma ct - \gamma\beta(n_x x + n_y y + n_z z) \\ x + n_x(\gamma-1)(n_x x + n_y y + n_z z) - \gamma\beta ct \\ y + n_y(\gamma-1)(n_x x + n_y y + n_z z) - \gamma\beta ct \\ z + n_z(\gamma-1)(n_x x + n_y y + n_z z) - \gamma\beta ct \end{pmatrix}. \tag{28}$$

These expressions can be re-written in matrix form, yielding $\mathfrak{L}$ from Eq. (10) to be:

$$\mathfrak{L} = \begin{pmatrix} \gamma & -\beta\gamma n_x & -\beta\gamma n_y & -\beta\gamma n_z \\ -\beta\gamma n_x & 1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z \\ -\beta\gamma n_y & (\gamma-1)n_y n_x & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z \\ -\beta\gamma n_z & (\gamma-1)n_z n_x & (\gamma-1)n_z n_y & 1+(\gamma-1)n_z^2 \end{pmatrix}, \tag{29}$$

which you will notice is the same result as the one obtained in approach 1.

Problem 4

(a) The current I is the charge per second traveling through the channel, given by:

$$I = nA(+e)(+\beta c) + nA(-e)(-\beta c) = 2nAe\beta c. \tag{30}$$

(b)

Ampere's law (in SI units, feel free to use whatever units you like of course) is:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enclosed}. \tag{31}$$

So in our case, assuming an infinitely long channel and using I from Eq. (30):

$$B_\phi \cdot 2\pi r = \mu_0 2nAe\beta c, \tag{32}$$

therefore

$$B_\phi = \frac{\mu_0 nAe\beta c}{\pi r}. \tag{33}$$

(c)

Let's solve this both suggested ways... first using length contraction. The density of positrons n_+ is given by:

$$n_+ = \frac{N_+}{A \cdot d}, \tag{34}$$

where d is a unit length of the channel in the lab frame S and N_+ is the total number of positrons contained in this volume. This new frame S' is the rest frame of the positrons, so $d' = \gamma d$ (sort of length un-contraction). Therefore the observed positron density in S' is given by:

$$n'_+ = \frac{N_+}{A\gamma d} = \frac{n_+}{\gamma}. \tag{35}$$

We can arrive at the same conclusion using the fact that $\left(c\rho, \vec{j}\right)$ is a four-vector. Considering only the z-direction, we have the relation:

$$\begin{pmatrix} c\rho' \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \cdot \begin{pmatrix} c\rho \\ j_z \end{pmatrix}, \tag{36}$$

where the charge density in the lab frame S satisfies $c\rho = n_+ec$ and the current density in S is given by $j_z = n_+e\beta c$. Thus from Eq. (36) we find that:

$$c\rho' = n'_+ec = \gamma\left(n_+ec - \beta^2 n_+ec\right). \tag{37}$$

Consequently,

$$n'_+ = \frac{n_+}{\gamma} \tag{38}$$

as above.

(d)

Here we'll just stick to the four-vector method. The relationship between the charge density of electrons seen in S' and S is given by:

$$\begin{pmatrix} c\rho' \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \cdot \begin{pmatrix} c\rho \\ j_z \end{pmatrix}, \tag{39}$$

from which we find the charge density:

$$c\rho' = \gamma c\rho - \beta\gamma j_z. \tag{40}$$

The electron charge density in S satisfies $c\rho = n_-(-e)c$ and the current density in S is given by $j_z = n_-(-e)(-\beta)c$. Plugging these into Eq. (40) allows us to solve for n'_-:

$$n'_- = \gamma\left(1 + \beta^2\right)n_-. \tag{41}$$

(e)

First we solve the problem using Gauss's law:

$$\int E'_r \cdot dA = \int \frac{\rho}{\epsilon_0} dV. \tag{42}$$

Choosing a cylindrical Gaussian surface centered on the z-axis with radius r and length d, we find:

$$E'_r \cdot 2\pi rd = Ad\left(\rho'_+ + \rho'_-\right)/\epsilon_0. \tag{43}$$

Using the relations for the charge density of positrons and electrons in S' from Eqs. (38) and (41), we find that:

$$\rho'_+ + \rho'_- = \frac{ne}{\gamma}\left(1 - \gamma^2(1+\beta^2)\right) = -2\beta^2\gamma ne. \tag{44}$$

Combining these results, we find for the radial electric field E'_r seen in S':

$$E'_r = -\hat{r}\frac{\beta^2\gamma neA}{\pi\epsilon_0 r}. \tag{45}$$

We can also solve this problem using the relativistic field transformations for E and B given in Prof. Strovink's notes (Eq. (1.33)), in particular:

$$E'_\perp = \gamma\left(E_\perp + c\vec{\beta} \times \vec{B}\right). \tag{46}$$

Employing B_ϕ from Eq. (33) and noting that $E_r = 0$ in S, we find that:

$$E'_r = -c\beta\gamma B_\phi \hat{r} = -\hat{r}\frac{\beta^2\gamma neA}{\pi\epsilon_0 r}. \tag{47}$$

Where we use the fact that $\epsilon_0\mu_0 = 1/c^2$. This, of course, agrees with our result from Eq. (45) using Gauss's law.

Problem 5

(a)

Note the interesting fact that the number of seconds in a year is approximately $\pi \times 10^7$, a useful fact at cocktail parties and for back-of-the-envelope calculations.

If you naively multiply the acceleration by the time, you find:

$$v = gt \approx 10c. \tag{48}$$

So, if you're a believer in relativity, this can't be right...

(b)

This part is basically worked out in the text of the problem, so there's nothing to say...

(c)

We start with

$$dx = c \cdot sinh(\eta)d\tau = c \cdot sinh\left(\frac{g\tau}{c}\right)d\tau, \tag{49}$$

where we use the expression

$$c\eta = g\tau. \tag{50}$$

Next we integrate the small displacements from $0 \to \tau_f$ where τ_f is the final "astronaut time."

$$\int_0^{x_f} dx = \int_0^{\tau_f} c \cdot sinh\left(\frac{g\tau}{c}\right)d\tau \tag{51}$$

We can make a straightforward change of variable $\xi = g\tau/c$:

$$x_f = \frac{c^2}{g}\int_0^{g\tau_f/c} sinh(\xi)d\xi. \tag{52}$$

Finally arriving at the solution:

$$x_f = \frac{c^2}{g}(cosh(g\tau_f/c) - 1). \tag{53}$$

(d)

If we plug in the numbers we find that $x_f \approx 10^{20}$ meters or 10^4 light years. This is just the first leg of the journey, so the furthest distance the astronaut can reach is twice this, or 20,000 light years away! So the engineer was right...

Problem 6

(a)

The relativistic expression for energy E of particles with non-zero mass is given by Eq. (1.23) in Strovink's notes:

$$E = \gamma mc^2, \tag{54}$$

where m is the rest mass of the particles. Since $\gamma = cosh(\eta)$, the boost η is given by:

$$\eta = cosh^{-1}\left(\frac{E}{mc^2}\right) \tag{55}$$

Knowing from the problem that $mc^2 = 0.5 \times 10^6$ eV and $E_{final} = 5 \times 10^{10}$ eV, we can solve for η:

$$\eta = 12.2 \tag{56}$$

(b)

We can use the result obtained in problem 5, namely Eq. (53), replacing g with some constant acceleration a. We also replace $cosh(g\tau_f/c)$ with γ_f, which from part (a) we find is $\gamma_f \approx 10^5$. This gives us:

$$x_f \approx \frac{c^2}{a}\gamma_f \tag{57}$$

Solving for a and making the appropriate substitutions yields:

$$a \approx 3 \times 10^{17} g \tag{58}$$

(c)

We can use the relation between proper time $d\tau$ and time in the laboratory frame dt from problem 5:

$$dt = cosh(\eta)d\tau. \tag{59}$$

If we apply the relation $c\eta = a\tau$, then integrating this expression yields:

$$t_{lab} = \frac{c}{a}sinh(\eta_f) = \frac{c}{a}\beta_f\gamma_f \approx 10^{-5}\ s \tag{60}$$

where t_{lab} is the time interval in the lab frame.

From $c\eta = a\tau$ we can quickly calculate the proper time interval:

$$\tau = \frac{c}{a}\eta_f \approx 10^{-9}\ s \tag{61}$$

So, taking the ratio gives an "average" γ factor of 10^4.

Problem 7

(a)

We know that photons satisfy $E^2 - p^2c^2 = 0$. Then, if we substitute the appropriate values from the problem into the equation describing the Lorentz transformation for the four-momentum (ignoring the z-direction), we find:

$$\begin{pmatrix} E'/c \\ (E'/c)cos\phi' \\ (E'/c)sin\phi' \end{pmatrix} = \begin{pmatrix} cosh(\eta) & -sinh(\eta) & 0 \\ -sinh(\eta) & cosh(\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E/c \\ (E/c)cos\phi \\ (E/c)sin\phi \end{pmatrix}. \tag{62}$$

Solving for E' gives us:

$$E' = E(cosh(\eta) - sinh(\eta)cos\phi) = Ecosh(\eta)(1 - \beta cos\phi). \tag{63}$$

If we then find the equation for p'_x we can solve for $cos\phi'$:

$$cos\phi' = \frac{E}{E'}(cosh(\eta)cos\phi - sinh(\eta)). \tag{64}$$

Substituting in the expression for E' from Eq. (63) yields:

$$cos\phi' = \frac{cos\phi - \beta}{1 - \beta cos\phi}. \tag{65}$$

(b)

Now we use the inverse Lorentz transform:

$$\begin{pmatrix} E/c \\ (E/c)cos\phi \\ (E/c)sin\phi \end{pmatrix} = \begin{pmatrix} cosh(\eta) & sinh(\eta) & 0 \\ sinh(\eta) & cosh(\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E'/c \\ (E'/c)cos\phi' \\ (E'/c)sin\phi' \end{pmatrix}. \tag{66}$$

If we perform calculations similar to those in part (a), we find:

$$E = E'(cosh(\eta) + sinh(\eta)cos\phi') = E'cosh(\eta)(1 + \beta cos\phi'). \tag{67}$$

and

$$cos\phi = \frac{E'}{E}(cosh(\eta)cos\phi' + sinh(\eta)) = \frac{cos\phi' + \beta}{1 + \beta cos\phi'}. \tag{68}$$

(c)

Here, we can employ the relationship between energy and frequency of a photon, namely:

$$E = h\nu, \tag{69}$$

where h is Planck's constant. Thus from Eq. (63) we solve for ν', finding the relativistic Doppler shift formula:

$$\nu' = \nu cosh(\eta)(1 - \beta cos\phi). \tag{70}$$

If an observer knows only the frequency as observed in a given frame, one cannot figure out what the frequency of light was in the rest frame of the source. Thus a measurement of light frequency in a particular frame does not directly tell us about the velocity of the source. However, if we have prior knowledge of what the frequency of light at rest should be (for example, well-known atomic transitions in hydrogen or helium), we can tell something about the motion of the source.

Problem 8

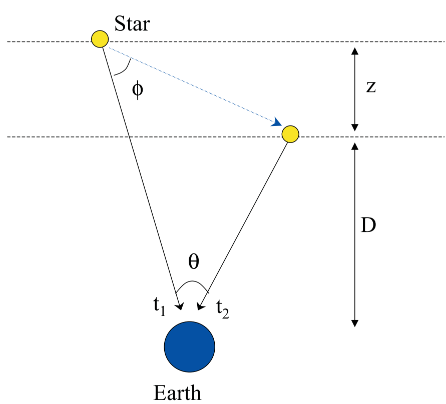
This situation is physically reasonable, here is one example of how it could happen...
Could the star be moving toward or away from us, even though the spectral features are not redshifted or blueshifted? The answer is yes, as can be seen from the relativistic Doppler shift given in Eq. (70). We demand that $\nu' = \nu$, and then

find a condition on the velocity of the source βc and the angle between $\vec{\beta}$ and the direction to earth ϕ:

$$cos\phi = \frac{\gamma - 1}{\beta\gamma}. \tag{71}$$

So long as this condition is satisfied, there is no restriction on the motion of the source (save that the source, if massive, cannot move at the speed of light!).

Next we consider if some particular type of motion could increase the apparent velocity of the star across the sky. Once again the answer is yes. Consider the situation depicted in the figure to the right. Of course, the drawing is greatly exaggerated in dimensions since $z \ll D$ and $t_2 - t_1$ is differentially small, but hopefully it will give you the basic idea. Suppose the astronomer makes two measurements with which she determines the motion of the star across the sky. The star is moving toward the earth in this case, so it takes the light detected in the first measurement longer to get to the earth. Suppose that the star gets closer to the earth by z between the times it emits the detected light. Then the time between the two light measurements on earth is:



$$t_2 - t_1 = \Delta t - z/c. \tag{72}$$

where Δt is the time it takes the star to move to the new location in the earth frame. Then the apparent angular motion is given by:

$$D\frac{d\theta}{dt} = \frac{D\Delta\theta}{\Delta t - z/c}. \tag{73}$$

So in fact (which is clear if you try some reasonable numbers), this apparent velocity can exceed the real velocity of the source by quite a bit, enough to make the star look like it's going c or faster. There are real cases of this in astronomy... for example at the center of the galaxy there are stars whose apparent velocity greatly exceeds c (of course they're redshifted and blueshifted all over the place, but you get the idea...)!

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 2

1.
A closely spaced circular parallel plate capacitor with long axial leads has a small (temporarily constant) current I passing through it (because the voltage across it is changing very slowly). The plates are perfectly conducting and have radius b.
(a.)
Use Gauss's law to find the time rate of change dE/dt of the electric field within the plates. You may assume that the charge densities on the inside surfaces of the plates do not vary appreciably across those surfaces.
(b.)
Use the Ampère-Maxwell equation to find the magnitude and direction of the magnetic field halfway between the plates, at a radius $r < b$ from the axis.
(c.)
Use Ampère's law to evaluate the magnetic field in the vicinity of one of the long axial leads, far from the capacitor. Compare it to the answer for (b.).
(d.)
Suppose instead that I varies slowly. Far from the fringe of the capacitor, would you expect the electric field to vary slightly with r? Explain.

2.
(based on *Purcell 10.14.*)
Consider three closely spaced parallel plate capacitors of the same square area and plate separation. The first capacitor C_1 consists only of those plates and vacuum. Both C_2 and C_3 are half-filled with an insulating material having dielectric constant ϵ, but the dielectric is arranged in different ways: C_2's dielectric extends over the full plate area, but fills only the half gap closest to one of the plates; C_3's dielectric extends from one plate to the other, but covers only half of the gap area. (The dielectric boundaries are always either parallel or perpendicular to the plates.)

Calculate the capacitances C_2 and C_3, expressed as a ratio to C_1.

3.
(based on *Purcell 10.23.*)
Consider an oscillating electric field, $E_0 \cos \omega t$, inside a dielectric medium that is not a perfect insulator. The medium has dielectric constant ϵ and conductivity σ. This could be the electric field of some leaky capacitor which is part of a resonant circuit, or it could be the electric field at a particular location in an electromagnetic wave. Work in SI units. Show that the Q factor, defined by

$$Q = \omega \, \frac{\text{energy stored}}{\text{average power dissipated}} \ ,$$

is $\epsilon\omega/\sigma$ for this system, and evaluate it for seawater at a frequency of 1000 MHz. The conductivity is 4 (ohm-m)$^{-1}$, and the dielectric constant may be assumed to be the same as that of pure water at the same frequency,

$$\frac{\epsilon}{\epsilon_0} \approx 78 \ .$$

What does your result suggest about the propagation of decimeter waves through seawater?

4.
(based on *Purcell 10.24.*)
A block of glass, refractive index $n = \sqrt{\epsilon/\epsilon_0}$, fills the space $y > 0$, its surface being the xz plane. A plane wave traveling in the positive y direction through the empty space $y < 0$ is incident upon this surface. The electric field in this wave is $\hat{\mathbf{z}}\, E_i \sin{(ky - \omega t)}$. There is a wave inside the glass block, described exactly by

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{z}}\, E_0 \sin{(ky - \omega t)} \\ \mathbf{B} &= \hat{\mathbf{x}}\, B_0 \sin{(ky - \omega t)} \ . \end{aligned}$$

There is also a reflected wave in the space $y < 0$, traveling away from the glass in the negative y direction. Its electric field is $\hat{\mathbf{z}}\, E_r \sin{(ky + \omega t)}$. Of course, each wave has its magnetic field of amplitude, respectively, B_i, B_0, and B_r. The total magnetic field must be continuous at $y = 0$, and the total electric field, being parallel to the surface, must be continuous also. Show that this requirement, and the relation of B_0 to E_0 given in the equation

$$B_0 = \sqrt{\epsilon\mu_0}\, E_0 \quad ,$$

suffice to determine the ratio of E_r to E_i. When a light wave is incident normally on a vacuum-glass interface, what fraction of the energy is reflected if the index n is 1.6?

5.
(based on *Purcell 11.11.*)
Write out Maxwell's equations as they would appear if we had magnetic charge and magnetic charge currents as well as electric charge and electric currents. Invent any new symbols you need and define carefully what they stand for. Be particularly careful about + and − signs. Work in SI units.

6.
(based on *Purcell 11.17.*)
An iron plate 0.2 m thick is magnetized to saturation in a direction parallel to the surface of the plate. A "10 GeV/c" muon having momentum p, with $pc = 10^{10}$ eV, moving perpendicular to the plate's surface, enters the plate and passes through it with relatively little loss of energy. (This is possible because the muon, of mass m with $mc^2 \approx 10^8$ eV, is ≈ 200 times heavier than an electron, so it radiates 200^2 times fewer photons.) Calculate approximately the angular deflection of the muon's trajectory. Take the saturation magnetization of iron to be equivalent to 1.5×10^{29} electron magnetic moments per m^3 (the electron magnetic moment is $\mu_B \approx 6 \times 10^{-5}$ eV per Tesla).

7.
Fowles 1.4.

8.
Fowles 1.6.

"The purpose of physics is to understand the universe... the purpose of mathematics is, well, obscure to me..."

- Prof. Seamus Davis, UC Berkeley

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

If you're interested in the possibility of magnetic monopoles, you might want to look up a paper by Blas Cabrera (Physical Review Letters, vol. **48**, no. 20, 1982 pp. 1378-81), where the possible detection of a single magnetic monopole is discussed. There have been no further monopoles detected since that time, so this report remains unconfirmed. There is also an excellent discussion of magnetic monopoles in J.D. Jackson's *Classical Electrodynamics*.

A discussion of the additional problem presented in discussion section this week can be found in a paper by Robert Romer (American Journal of Physics vol. **50**, no. 12, 1982 pp. 1089-93).

Problem 1

(a)

We use Gauss's law and choose a cylindrical surface of radius r centered on the axis (we'll call it $\hat{z}$) of the parallel plate capacitor, far from the edges of the capacitor ($r \ll b$). Then:

$$\int \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} = \kappa\pi r^2/\epsilon_0, \tag{1}$$

where κ is the surface charge density of the capacitor. We find directly from Eq. (1) that:

$$\vec{E} = (\kappa/\epsilon_0)\hat{z}. \tag{2}$$

Since there is a current I, the surface charge density changes with time by an amount:

$$\frac{d\kappa}{dt} = \frac{I}{\pi b^2}, \tag{3}$$

where we assume the current is flowing in the $\hat{z}$ direction. So from Eqs. (2) and (3), we find that:

$$\frac{d\vec{E}}{dt} = \frac{I}{\epsilon_0 \pi b^2}\hat{z}. \tag{4}$$

(b)

The Ampere-Maxwell equation, since there is no real (conduction) current J between the plates of the capacitor, reduces to:

$$\nabla \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{5}$$

Then using our result from part (a) and integrating (we choose an Amperian loop centered on the z-axis of radius r), we find:

$$\oint \vec{B} \cdot d\ell = 2\pi r B_\phi = \frac{\mu_0 I r^2}{b^2} \tag{6}$$

Thus we find the magnetic field in the $\hat{\phi}$ direction to be:

$$B_\phi = \frac{\mu_0 I r}{2\pi b^2}. \tag{7}$$

(c)

Far from the capacitor, there is no changing electric field and therefore only conduction current, so this is the familiar Ampere's law:

$$\oint \vec{B} \cdot d\ell = \mu_0 I_{encl}, \tag{8}$$

from which we find a magnetic field in the $\hat{\phi}$ direction:

$$B_\phi = \frac{\mu_0 I}{2\pi r}. \tag{9}$$

which you will note is equivalent to Eq. (7) when $r \to b$. Also note that inside the capacitor, the magnetic field grows with r while far from the capacitor the field falls as $1/r$.

(d)

Let's consider the electric field in two different regions. First, we'll consider $\vec{E}$ far from the capacitor in the vicinity of one of the long axial leads (as in part (c)). The changing current produces a changing magnetic field, and from Maxwell's equations we know this creates an electric field:

$$\oint \vec{E} \cdot d\ell = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}. \tag{10}$$

From Eq. (9), we see that $\frac{\partial \vec{B}}{\partial t}$ is given by:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\mu_0}{2\pi r}\frac{\partial I}{\partial t}\hat{\phi}. \tag{11}$$

We choose an Amperian loop as indicated in Fig. 1. There is no electric field perpendicular to the wire (along $\hat{r}$). This can be deduced from symmetry considerations. Suppose there was an electric field in the $\hat{r}$ direction. How does it know whether to point in the $+\hat{r}$ or $-\hat{r}$ direction? That has to be decided by either the direction of the current or the change in current. If we reverse these quantities, the electric field in the $\hat{r}$ direction should reverse. But on the opposite sides of the wire, these quantities have opposite signs! The only way this can be true is if the electric field in the $\hat{r}$ direction is zero.

Furthermore, we know that the electric field must go to zero as $r \to \infty$. But since $\oint \vec{E} \cdot d\ell \neq 0$, it must be the case that we have an electric field in the $\hat{z}$ direction which varies with r. In other words, it is apparent that the electric field is larger closer to the wire (z-axis).

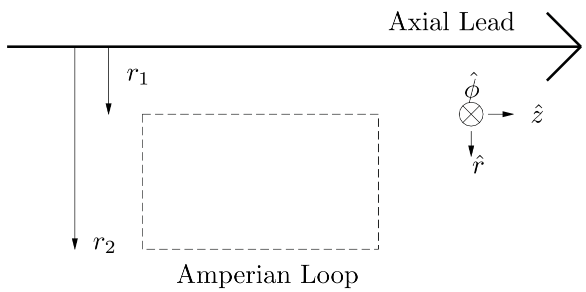


Figure 1

This can be done explicitly, of course, from Eqs. (10) and (11):

$$E(r_2) - E(r_1) = \frac{\mu_0}{2\pi}\frac{\partial I}{\partial t}\ln r_1/r_2. \tag{12}$$

Let's now consider the electric field inside the capacitor, far from the fringe (as in part (b)). Once again we apply Eq. (10), but in this case:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\mu_0 r}{2\pi b^2}\frac{\partial I}{\partial t}\hat{\phi}, \tag{13}$$

We see that there is also a component of the electric field in the $\hat{z}$ direction which varies with r by utilizing similar arguments as those presented above:

$$E(r) = -\frac{\mu_0 r^2}{4\pi b^2}\frac{\partial I}{\partial t}. \tag{14}$$

Problem 2

We can simplify the problem by thinking of C_2 and C_3 as two capacitors in series or in parallel, respectively (Fig. 1). The capacitance C of a parallel plate capacitor is given by:

$$C = \frac{\epsilon A}{d} \tag{15}$$

where A is the area of the plates and d is the plate separation. So for C_1:

$$C_1 = \frac{\epsilon_0 A}{d} \tag{16}$$

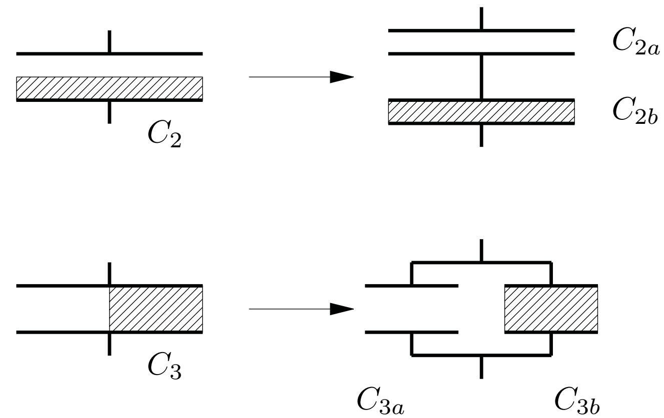


Figure 2

For C_2 we break up the problem into two parts, solving for C_{2a} and C_{2b} (shown in Fig. 1), then determining C_2 using:

$$C_2 = \left(\frac{1}{C_{2a}} + \frac{1}{C_{2b}}\right)^{-1}. \tag{17}$$

From Eq. (15) we can find C_{2a} and C_{2b}, where:

$$C_{2a} = \frac{\epsilon_0 A}{d/2} = 2C_1 \tag{18}$$

and

$$C_{2b} = \frac{\epsilon A}{d/2} = 2\frac{\epsilon}{\epsilon_0}C_1. \quad (19)$$

So with a wee bit of algebra, we find that:

$$C_2 = \frac{2C_1}{\epsilon_0/\epsilon + 1}. \quad (20)$$

Similarly for C_3, we break up the capacitor into two parts C_{3a} and C_{3b}, and then solve for C_3 using:

$$C_3 = C_{3a} + C_{3b}. \quad (21)$$

We use Eq. (15) to solve for C_{3a} and C_{3b}, finding:

$$C_{3a} = \frac{\epsilon_0 A/2}{d} = \frac{1}{2}C_1 \quad (22)$$

and

$$C_{3b} = \frac{\epsilon A/2}{d} = \frac{\epsilon}{2\epsilon_0}C_1. \quad (23)$$

So here the overall capacitance is given by:

$$C_3 = \frac{C_1}{2}(\epsilon/\epsilon_0 + 1). \quad (24)$$

Problem 3

The energy per unit volume U stored in an electromagnetic wave is given by:

$$U = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) = \epsilon E^2. \quad (25)$$

If we then time average the energy, we find that the average energy stored is:

$$\langle U \rangle = \epsilon E_0^2 \int \cos^2(\omega t)dt = \frac{\epsilon E_0^2}{2}. \quad (26)$$

The average power P dissipated per unit volume is given by the relation:

$$P = \frac{J^2}{\sigma}, \quad (27)$$

where the current density is given by Ohm's law:

$$J = \sigma E. \quad (28)$$

Once again taking the time average, we find:

$$\langle P \rangle = \frac{\sigma E_0^2}{2}. \quad (29)$$

The Q-factor is the ratio of these two quantities, $\langle U \rangle$ and $\langle P \rangle$, multiplied by the frequency:

$$Q = \frac{\epsilon\omega}{\sigma}. \quad (30)$$

If we plug in the numbers for seawater, we find that $Q \approx 1.1$. This suggests that decimeter waves cannot propagate very far in seawater, since the energy in the wave falls to $1/e$ its initial value in about one decimeter!

Problem 4

First, we can write down the the electric and magnetic fields of the incident, transmitted and reflected waves:

$$\begin{gathered}\hat{z}E_i \sin(ky - \omega t)\\ \hat{x}B_i \sin(ky - \omega t)\\[1em] \hat{z}E_0 \sin(k_0 y - \omega t)\\ \hat{x}B_0 \sin(k_0 y - \omega t)\\[1em] \hat{z}E_r \sin(ky + \omega t)\\ \hat{x}B_r \sin(ky + \omega t)\end{gathered} \quad (31)$$

We note that $k_0 = nk$ since the transmitted wave is in glass. Then we can impose the condition

$$|B| = |\sqrt{\epsilon\mu}E| \quad (32)$$

on each of the waves, and demand that the Poynting vector, $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$, is along the direction of propagation of the waves. This fixes the amplitudes and signs of the magnetic fields with respect to the electric fields:

$$\begin{gathered}\hat{z}E_i \sin(ky - \omega t)\\ \hat{x}\sqrt{\epsilon_0\mu_0}E_i \sin(ky - \omega t)\\ \\ \hat{z}E_0 \sin(k_0 y - \omega t)\\ \hat{x}\sqrt{\epsilon\mu_0}E_0 \sin(k_0 y - \omega t)\\ \\ \hat{z}E_r \sin(ky + \omega t)\\ -\hat{x}\sqrt{\epsilon_0\mu_0}E_r \sin(ky + \omega t)\end{gathered} \tag{33}$$

Now we consider the fields at $y = 0$, the interface between the block of glass and vacuum. We require that the electric and magnetic fields parallel to the surface of the glass satisfy:

$$\begin{aligned} E_{\shortparallel} &= E'_{\shortparallel}\\ \frac{B_{\shortparallel}}{\mu} &= \frac{B'_{\shortparallel}}{\mu'}.\end{aligned} \tag{34}$$

After substitution, this leaves us with two equations:

$$\begin{aligned} -E_i + E_r &= -E_0\\ E_i + E_r &= \sqrt{\frac{\epsilon}{\epsilon_0}}E_0.\end{aligned} \tag{35}$$

We can then eliminate E_0 from these equations yielding the ratio of E_r to E_i:

$$\frac{E_r}{E_i} = \frac{\sqrt{\epsilon/\epsilon_0} - 1}{\sqrt{\epsilon/\epsilon_0} + 1}. \tag{36}$$

The energy is proportional to E^2 (as can be readily seen by considering the Poynting vector $\vec{S}$), and in this case the index of refraction $n = \sqrt{\epsilon/\epsilon_0}$. Thus the ratio of reflected to incident energy U_r/U_i is given by:

$$\frac{U_r}{U_i} = \left(\frac{E_r}{E_i}\right)^2 = \left(\frac{n-1}{n+1}\right)^2. \tag{37}$$

For $n = 1.6$, 5% of the energy is reflected.

Problem 5

If there were magnetic charges, a magnetic charge density ρ_m and a magnetic current density $\vec{J}_m$ would appear in Maxwell's equations. To avoid confusion, let's denote the traditional electric charge density ρ_e and electric current density $\vec{J}_e$. We can place both of these, with some constants c_1 and c_2 which will be defined later, in Maxwell's equations to make them nice and symmetric:

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho_e/\epsilon_0\\ \vec{\nabla}\cdot\vec{B} &= c_1\rho_m\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} + c_2\vec{J}_m\\ \vec{\nabla}\times\vec{B} &= \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}_e\end{aligned} \tag{38}$$

We can go further and work out a relationship between magnetic charge density and current density. We begin by demanding that magnetic charges and currents satisfy the continuity equation, namely:

$$\vec{\nabla}\cdot\vec{J}_m + \frac{\partial\rho_m}{\partial t} = 0. \tag{39}$$

Then if we take the divergence of the new third Maxwell's equation, we get:

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{E} = -\vec{\nabla}\cdot\frac{\partial\vec{B}}{\partial t} + c_2\vec{\nabla}\cdot\vec{J}_m. \tag{40}$$

There is a vector derivative rule that states for any vector field $\vec{A}$, $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$. So the left-hand side of (40) is 0. The derivatives on the right hand side, $\vec{\nabla}$ and $\frac{\partial}{\partial t}$, can be swapped and we get:

$$-\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B} + c_2\vec{\nabla}\cdot\vec{J}_m = 0. \tag{41}$$

From the second Maxwell equation we know that $\vec{\nabla}\cdot\vec{B} = c_1\rho_m$, so we find:

$$-c_1\frac{\partial\rho_m}{\partial t} + c_2\vec{\nabla}\cdot\vec{J}_m = 0. \tag{42}$$

If we then apply the continuity equation, Eq. (39), we find that $c_1 = -c_2 \equiv c$.

Thus the final form of Maxwell's equations is:

$$\vec{\nabla}\cdot\vec{E} = \rho_e/\epsilon_0$$
$$\vec{\nabla}\cdot\vec{B} = c\rho_m$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - c\vec{J}_m$$
$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}_e, \quad (43)$$

where c is a constant of proportionality between the magnetic charge unit and the magnetic field it produces (the equivalent of $1/\epsilon_0$ for electric fields).

Problem 6

The first part of this problem is to calculate the magnetic field $\vec{B}$ inside the magnetized iron. We can use the auxiliary field $\vec{H}$ to make our job a little easier. We know that:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}. \quad (44)$$

Also, we have the relation:

$$\oint \vec{H}\cdot d\vec{\ell} = I_{free}, \quad (45)$$

where in our problem $I_{free} = 0$ everywhere. We choose an Amperian loop as pictured in Fig. 3 ($\vec{M}$ is in the $\hat{z}$ direction), taking advantage of the planar symmetry of the problem (we can assume the iron plate is infinite). Since the component of $\vec{H}$ perpendicular to the surface of the iron plate must be zero based on symmetry, and outside the iron plate $\vec{H}\to 0$ as $y\to\pm\infty$, we conclude that in fact $\vec{H} = 0$ everywhere.

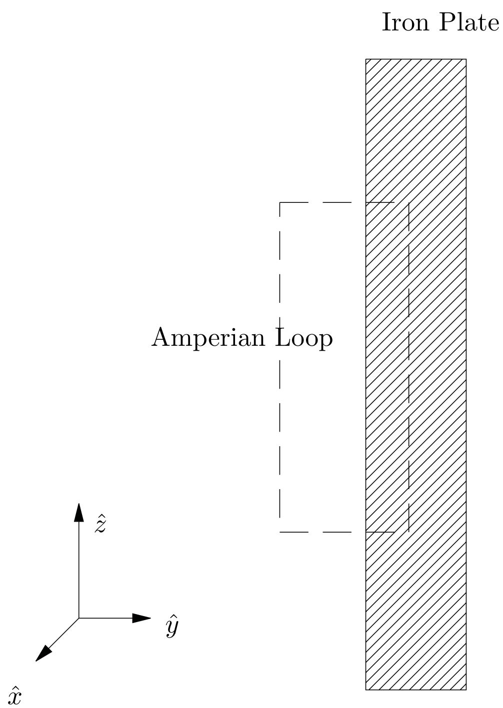


Figure 3

I would like to pause here and point out that this conclusion is not entirely trivial. If there is no free current I_{free}, that does not necessarily mean that $\vec{H} = 0$ everywhere. The fundamental reason for this is that in order to completely determine a vector field you must know both its curl and divergence. Only in cases where we have planar, cylindrical, toroidal or solenoidal symmetry can we conclude that $\vec{\nabla}\cdot\vec{H} = 0$, and get $\vec{H}$ quickly. This is different from Ampere's law with $\vec{B}$ where we always know $\vec{\nabla}\cdot\vec{B} = 0$. So, be careful when using $\vec{H}$!

Anyhow, in this case it's no problem, we find that:

$$\vec{B} = \mu_0\vec{M}. \quad (46)$$

inside the iron plate and zero outside the plate.

This problem now reduces to the traditional problem of solving for the cyclotron orbit of a moving charged particle in a magnetic field. An important difference, as pointed out by Paul Wright in section (thanks!), is that in this case we need to be careful about relativistic corrections to the radius of the cyclotron orbit.

To find the radius of the cyclotron orbit R, we balance the Lorentz force qvB with the relativistic centrifugal force $\gamma mv^2/R$. This tells us:

$$R = \frac{\gamma mv}{qB}. \quad (47)$$

where R is the radius of the circular orbit. If you take a look at Fig. 4, hopefully the simple geometric arguments suggested convince you that in fact:

$$\sin\theta = d/R = \frac{dqBc}{\gamma mvc} = \frac{dqBc}{pc}, \quad (48)$$

where θ is the deflection angle and d is the thickness of the plate. The rest of the problem is working out the correct units...

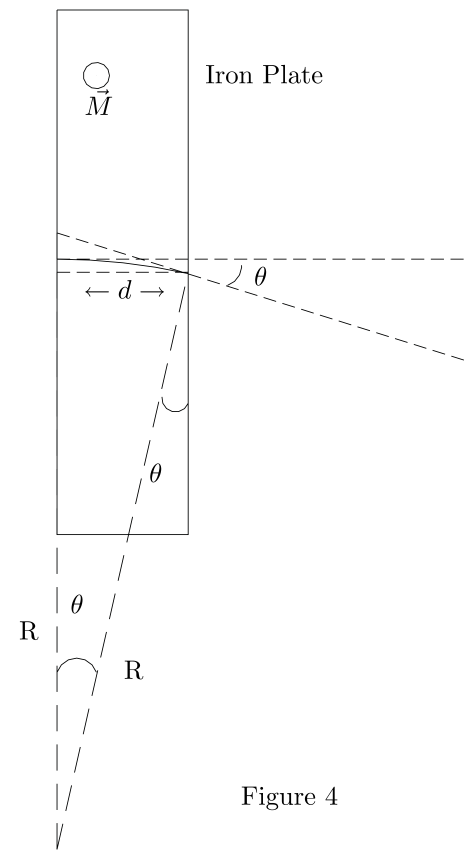


Figure 4

First let's get B in SI units. $B = \mu_0 M = 4\pi\times10^{-7}$ N/A^2 $\cdot 1.5\times10^{29}$ electron magnetic moments per m^3 $\cdot 9\times10^{-24}$ J/T, or about 1.7 T. Then $dqcB = 10^8$ eV, so $\sin\theta \approx \theta = \frac{dqBc}{pc} = 10^{-2}$ rad. That's only about half a degree, so not too big of a deflection...

Problem 7

Fowles 1.4

The 3D wave equation is:

$$\nabla^2 f = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} \tag{49}$$

We employ spherical coordinates, and since our wavefunction is a function only of r, ∇^2 is also a function only of r:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right). \tag{50}$$

Plugging in the spherical harmonic wavefunction $f = \frac{1}{r}e^{\imath(kr-\omega t)}$, we get:

$$\nabla^2 f = -\frac{k^2}{r}e^{\imath(kr-\omega t)} = -k^2 f. \tag{51}$$

If we evaluate the right-hand side of Eq. (49), and use the fact that $k = \omega/u$, we find that:

$$\frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = -k^2 f. \tag{52}$$

which verifies that f is a solution to the 3D wave equation.

Problem 8

Fowles 1.6

(a)

Let's begin by deriving

$$u_g = u - \lambda\frac{\partial u}{\partial\lambda}. \tag{53}$$

We can begin by using Fowles (1.33), the definition of the group velocity:

$$u_g = \frac{d\omega}{dk} = \frac{d\omega}{d\lambda}\cdot\frac{d\lambda}{dk}, \tag{54}$$

where $k = 2\pi/\lambda$ is the wave vector. We can also express ω in terms of u and λ:

$$\omega = ku = \frac{2\pi u}{\lambda}. \tag{55}$$

If you take the derivative of ω with respect to λ:

$$\frac{d\omega}{d\lambda} = -\frac{2\pi u}{\lambda^2} + \frac{2\pi}{\lambda}\frac{du}{d\lambda}. \tag{56}$$

Now we calculate $\frac{d\lambda}{dk}$:

$$\frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi}. \tag{57}$$

If we then substitute the expressions in Eqs. (56) and (57) into Eq. (54), we arrive at our result:

$$u_g = u - \lambda\frac{\partial u}{\partial\lambda}. \tag{58}$$

(b)

We use similar tricks to derive the result:

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}. \tag{59}$$

We begin by noting that

$$\frac{1}{u_g} = \frac{dk}{d\omega} = \frac{dk}{d\lambda_0}\cdot\frac{d\lambda_0}{d\omega} \tag{60}$$

Let's write the wave vector in terms of λ_0 and n:

$$k = \frac{2\pi n}{\lambda_0} \tag{61}$$

We can then take some derivatives, and find that:

$$\frac{dk}{d\lambda_0} = -\frac{2\pi n}{\lambda_0^2} + \frac{2\pi}{\lambda_0}\frac{dn}{d\lambda_0} \tag{62}$$

and

$$\frac{d\lambda_0}{d\omega} = -\frac{\lambda_0^2}{2\pi c}. \tag{63}$$

Substituting these results back into Eq. (60) gives us the answer we were looking for:

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}. \tag{64}$$

That's all folks!

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 3

1.
(based on *Purcell B.1.*)
An electron of rest mass m_e and charge e, moving initially at a constant velocity v, is brought to rest with a uniform deceleration a that lasts for a time $t = v/a$. Compare the electromagnetic energy radiated during the deceleration with the electron's initial kinetic energy. Express this ratio in terms of two lengths: the distance that light travels in time t, and the classical electron radius r_0, defined as

$$r_0 \equiv \frac{e^2}{4\pi\epsilon_0 m_e c^2} \ .$$

To carry out this calculation, you need a formula like Purcell Eq. (B.6) that relates the instantaneous radiated power $P_{\rm rad}$ to the instantaneous acceleration a. In SI units, this formula is

$$P_{\rm rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3} \ .$$

(Note that, as far as one has been able to tell experimentally, the electron actually is consistent with having zero radius, and it must have a radius at least several orders of magnitude smaller than the "classical radius" r_0.)

2.
(based on *Purcell B.3.*)
A plane electromagnetic wave with angular frequency ω and electric field amplitude E_0 is incident on an isolated electron. In the resulting sinusoidal oscillation of the electron, the maximum acceleration is $|e|E_0/m$, where e is the electron's charge.

Averaged over many cycles, how much power is radiated by this oscillating charge? (Note that, when the maximum *acceleration* of the electron rather than its maximum *amplitude* is held fixed, the power radiated is independent of the frequency ω.)

Divide this average radiated power by $\epsilon_0 E_0^2 c/2$, the average power density (per unit area of wavefront) in the incident wave. The quotient σ has the dimensions of area and is called a *scattering total cross section.* The energy radiated, or scattered, by the electron, and thus lost from the plane wave, is equivalent to the energy falling on an area σ. (The case considered here, involving a free electron moving nonrelativistically, is often called *Thomson scattering*, after J.J. Thomson, the discoverer of the electron, who first calculated it.)

3.
(based on *Purcell B.4.*)
The master formula

$$P_{\rm rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3} \ .$$

is useful for particles moving relativistically, even though $v \ll c$ was assumed in Purcell's derivation of it. To apply it to a relativistic situation, all we have to do is (i) transform to a comoving inertial frame F' in which the particle in question is, at least temporarily, moving slowly; (ii) apply the master formula in that frame; and (iii) transform back to any frame we choose.

Consider a highly relativistic electron ($\gamma \gg 1$) moving perpendicular to a magnetic field $\mathbf{B}$. It is continually accelerated (in a direction perpendicular both to its velocity and to the field), so it must radiate. At what rate does it lose energy? To answer this, transform to a frame F' moving momentarily along with the electron, find E' in that frame, and thereby find $P'_{\rm rad}$.

Now show that, because power is *energy/time*, $P_{\rm rad}$ must be equal to $P'_{\rm rad}$.

This radiation generally is called *synchrotron radiation.* It is both a blessing and a curse. The blessing is that intense beams of UV and X-ray photons are created at synchrotrons designed for that purpose, such as Berkeley Lab's Advanced Light Source. These beams are essential for many studies and uses such as semiconductor lithography. The curse is that synchrotron radiation prevents circular electron accelerators of practical size (up to tens of km in circumference) from exceeding about 10^{11} eV in energy, much weaker than the 10^{12} eV proton beams that have been available at Fermilab for a decade.

4.
Fowles 2.4.
The solution to this problem was sketched in lecture on 7 Sep. In Fowles' notation, $\mathbf{E}_0$ and $\mathbf{B}_0$ are the same as the $\mathbf{E}_1$ and $\mathbf{B}_1$ discussed in class.

5.
Fowles 2.7.

6.
Fowles 2.10.

7.
(a.)
For an ideal linear polarizer with its transmission axis at an arbitrary angle ϕ with respect to the x axis, calculate the Jones matrix. (As usual, the beam direction is z, ϕ is an angle in the xy plane, and ϕ is positive as one rotates from x toward y.)
(b.)
For a linear polarizer, show that its Jones matrix $\mathcal{M}$ is not *unitary*, *i.e.* $\mathcal{M}^*_{ij} \neq \left(\mathcal{M}^{-1}\right)_{ji}$. This means that the action of the wave plate violates *time-reversal invariance*. This makes sense because, for general polarization, the irradiance of a light beam is reduced after passing through the plate.

8.
A *wave plate* is made out of a birefringent crystal whose lattice constants are different in the "fast" and "slow" directions of polarization. This leads to different indices of refraction for the two polarizations. If the x axis is along the "slow" direction of the plate, x polarized light accumulates a phase shift δ with respect to light polarized in the "fast" or y direction, with

$$\delta = \frac{\omega D}{c}(n_x - n_y) \ .$$

Here $n_x > n_y$ if the x direction is "slow", ω is the (fixed) angular frequency of the light, and D is the thickness of the plate. Because the absolute phase of the light is of no experimental interest, the effect of the wave plate is equivalent to multiplying the x component of (complex) $\mathbf{E}_1$ by $\exp\left(i\delta/2\right)$ and the y component by $\exp\left(-i\delta/2\right)$.
(a.)
Write the Jones matrix for the ideal wave plate just described.
(b.)
Calculate the Jones matrix for the general case in which the "slow" plate axis lies at an angle ϕ from the x axis, where, as in Problem 7, ϕ is an angle in the xy plane.
(c.)
For the wave plate in (b), show that its Jones matrix $\mathcal{M}$ *is unitary*, *i.e.* $\mathcal{M}^*_{ij} = \left(\mathcal{M}^{-1}\right)_{ji}$. This means that the action of the wave plate *is time-reversal invariant*. For any polarization, the irradiance of a light beam is unaffected by traversing the plate (though its polarization may change dramatically).

"Niels Bohr (not for the first time) was ready to abandon the law of conservation of energy. It is interesting to note that Bohr was an outspoken critic of Einstein's light quantum (prior to 1924), that he discouraged Dirac's work on the relativistic electron theory (telling him, incorrectly, that Klein and Gordon had already succeeded), that he opposed Pauli's introduction of the neutrino, that he ridiculed Yukawa's theory of the meson, and that he disparaged Feynman's approach to quantum electrodynamics."

- Prof. David Griffiths, Reed College, excerpted from *Introduction to Elementary Particles*

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

An interesting (to me) point concerning the physical meaning of orthogonal polarization was raised after discussion section the other day. Fowles says that two waves $\mathbf{E}_1$ and $\mathbf{E}_2$ whose complex electric field amplitudes satisfy:

$$\mathbf{E}_1 \cdot \mathbf{E}_2^* = 0 \tag{1}$$

are orthogonally polarized.

For linear polarization, there is a simple geometric analogy. Linearly polarized light (e.g., in the $\hat{x}$ direction) is orthogonal to light with a perpendicular linear polarization (e.g., in the $\hat{y}$ direction). So no light will get through two linear polarizers which are "orthogonal" in the Euclidean geometry sense. However, this picture breaks down for more complicated polarization states, e.g. circular polarizations. For example, two circular polarizations whose electric fields are always at right angles to each other are *not* orthogonal!

Orthogonality for polarization states can be understood using notions from linear algebra. In this sense the complex vector space of polarization states can be spanned by two linearly independent, complex vectors - any two linearly independent complex vectors are said to be orthogonal. This is what Fowles means when he says two polarization states are orthogonal.

Another interesting question raised after discussion section, although it is a bit beyond the scope of the course, was whether or not a free-falling (in a gravitational field) charged particle radiates. You would expect that it might not based on the equivalence principle, which basically states that a free-falling frame is equivalent to an inertial frame. However, if you observe the charge from the surface of some planet, you would see a charged particle undergoing acceleration. This is a little funny, since you would expect that the charge either loses energy or it doesn't...

After thinking about it a little and consulting some wise general relativity texts (e.g., Wald's *General Relativity* or Misner, Thorne and Wheeler's *Gravitation*), I believe that the answer is frame dependent (as is the case for most relativity paradoxes). If you're free-falling with the particle, you don't see any radiation. If the charge is accelerating with respect to you, then you see radiation. This can be shown with the general relativistic field transformations. What about energy conservation? Well, I'm no Niels Bohr, so I think that if you change your acceleration into the frame of the particle, everything works out... but to prove this seems a bit complicated... anyhow, good stuff to think about, keep up the great work! Thanks!

Problem 1

First we'll calculate the electromagnetic energy radiated $P_{rad}\Delta t$ during the deceleration lasting for $\Delta t = v/a$, which is given by:

$$P_{rad}\Delta t = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2 av}{c^3}. \tag{2}$$

The electron's initial kinetic energy K is just:

$$K = \frac{1}{2}mv^2, \tag{3}$$

and the ratio $P_{rad}\Delta t/K$ is given by:

$$\frac{P_{rad}\Delta t}{K} = \frac{4}{3}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{mc^2}\right)\left(\frac{a}{cv}\right). \tag{4}$$

The distance d traveled by light in Δt is cv/a, and the classical radius of the electron r_0 is given by the formula in the problem set: $r_0 \equiv e^2/\left(4\pi\epsilon_0 mc^2\right)$. Therefore:

$$\frac{P_{rad}\Delta t}{K} = \frac{4}{3}\left(\frac{r_0}{d}\right). \tag{5}$$

Here's an interesting fact that might help you remember some important lengths in physics:

One of the most important constants in physics is the fine-structure constant α, which sets the strength scale for the electromagnetic force:

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \tag{6}$$

where e is in CGS units. You may recognize the upper division physics course, Physics α^{-1}.

You can guess the the classical radius of the electron by setting the rest energy of the electron equal to the potential energy stored in a spherical shell of radius r_0 with charge e on the surface.

$$mc^2 = \frac{e^2}{r_0}. \tag{7}$$

In a few weeks, you'll learn about Compton scattering (photon-electron scattering). The Compton wavelength of the electron is given by r_0/α. The Bohr radius, the radius of an electron's orbit in the hydrogen atom, is given by r_0/α^2.

Problem 2

The electron oscillates sinusoidally with the acceleration given by:

$$a = \frac{|e|E_0}{m}\sin\omega t. \tag{8}$$

The power P_{rad} radiated is given by:

$$P_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^4}{c^3}\frac{E_0^2}{m^2}\sin^2\omega t, \tag{9}$$

and of course if we average over many cycles...

$$\langle P_{rad}\rangle = \frac{1}{4\pi\epsilon_0}\frac{1}{3}\frac{e^4}{c^3}\frac{E_0^2}{m^2}. \tag{10}$$

Next if we divide this result by the average power density $\langle U\rangle$ in the incident wave we get the scattering cross section σ:

$$\sigma = \frac{1}{6\pi\epsilon_0^2}\frac{e^4}{m^2c^4}. \tag{11}$$

You might notice that $\sigma = (8/3)\pi r_0^2$, r_0 being the classical electron radius from problem 1...

Problem 3

We transform to a comoving inertial frame F' in which the electron is temporarily at rest. The electric and magnetic fields in F' are given by:

$$\begin{aligned} \vec{E}'_\perp &= \gamma\Big(\vec{E}_\perp + c\vec{\beta}\times\vec{B}\Big) \\ \vec{B}'_\perp &= \gamma\Big(\vec{B}_\perp - \frac{1}{c}\vec{\beta}\times\vec{E}\Big) \\ \vec{E}'_{\shortparallel} &= \vec{E}_{\shortparallel} \\ \vec{B}'_{\shortparallel} &= \vec{B}_{\shortparallel} \\ &. \end{aligned} \tag{12}$$

The force on the electron is the Lorentz force given by:

$$\vec{F} = e\Big(\vec{E} + \vec{v}\times\vec{B}\Big), \tag{13}$$

but in the comoving frame F' the electron's velocity is zero. Since in the lab frame F the electric field is zero, the force acting on the electron in F' is:

$$F = eE'_\perp = e\gamma c\beta B. \tag{14}$$

Thus, the acceleration a is:

$$a = \frac{e\gamma c\beta B}{m} \tag{15}$$

This acceleration can be used in our old pal which describes the power radiated:

$$P'_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3}, \tag{16}$$

which gives us:

$$P'_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^4\gamma^2\beta^2B^2}{m^2c}. \tag{17}$$

When we transform back to frame F, the energy transforms as $\Delta E' \to \gamma\Delta E$ and the time transforms as $\Delta t' \to \gamma\Delta t$. Therefore, since power is just $\Delta E/\Delta t$, $P'_{rad} = P_{rad}$.

Problem 4

Fowles 2.4

For this problem we employ the complex exponential form of the wave functions for $\vec{E}$ and $\vec{H}$:

$$\vec{E} = Re\Big(\vec{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big)$$
$$\vec{H} = Re\Big(\vec{H_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big). \quad (18)$$

The Poynting vector is given by:

$$\vec{S} = \vec{E} \times \vec{H}. \quad (19)$$

Using the expressions from Eq. (18), we find the Poynting vector is:

$$\vec{S} = Re\Big(\vec{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big) \times Re\Big(\vec{H_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big) \quad (20)$$

We can expand the exponentials in terms of sines and cosines and find the real parts:

$$\vec{S} = \Big(Re(\vec{E_0})\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) - Im(\vec{E_0})\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\Big)$$
$$\times\Big(Re(\vec{H_0})\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) - Im(\vec{E_0})\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\Big). \quad (21)$$

If we expand this expression and time average (i.e. we set $\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)$ and $\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)$ equal to $\frac{1}{2}$ and $\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$ equal to 0), then we get:

$$\langle\vec{S}\rangle = \frac{1}{2}\Big(Re(\vec{E_0}) \times Re(\vec{H_0}) + Im(\vec{E_0}) \times Im(\vec{H_0})\Big). \quad (22)$$

This expression, by inspection, is equivalent to:

$$\langle\vec{S}\rangle = \frac{1}{2}Re\Big(\vec{E_0} \times \vec{H}_0^*\Big), \quad (23)$$

which verifies the claim.

Problem 5

Fowles 2.7

This is pretty straightforward. Here's the prescription:

Given the electric field of the wave, $\vec{E}$, in the form:

$$\vec{E} = E_0\Big(\hat{i} + \hat{j}be^{i\theta}\Big)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (24)$$

the Jones vector is given by:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = E_0\begin{bmatrix} 1 \\ be^{i\theta} \end{bmatrix}. \quad (25)$$

You can normalize the Jones vector if you want, but if you didn't feel like doing that in this problem that's okay too. So pretty much we can just write down the answers, here they are:

(a)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sqrt{2}E_0\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (26)$$

(b)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sqrt{5}E_0\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}. \quad (27)$$

(c)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sqrt{2}E_0\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}. \quad (28)$$

(d)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = 2E_0\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{1+i}{2} \end{bmatrix}. \quad (29)$$

Problem 6

Fowles 2.10

We'll start with an arbitrary polarization:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} a \\ be^{i\theta} \end{bmatrix}. \tag{30}$$

Now we'll send it through a linear polarizer. Let's orient the linear polarizer at 45^o, so our resultant polarization is given by:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ be^{i\theta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a+be^{i\theta} \\ a+be^{i\theta} \end{bmatrix} = \frac{a+be^{i\theta}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{31}$$

We can ignore the amplitude out front. Now we'll sent it through a quarter-wave plate with the fast axis horizontal.

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \tag{32}$$

which is indeed circular polarization! What happens if we change the order?

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} a \\ be^{i\theta} \end{bmatrix} = \frac{a+be^{i\theta}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{33}$$

which is linear polarization. So circular polarized light is created only by placing the optical elements in the proper order.

Problem 7

(a)

For this problem, we can use the technique for changing the basis of a matrix operator that we employed in the first problem set to find Lorentz transforms in rotated frames (see PS1 solutions, problem 2). Namely,

$$\mathcal{M}' = R^{-1}\mathcal{M}R. \tag{34}$$

where $\mathcal{M}$ is the Jones matrix and R is the appropriate rotation matrix. For a linear polarizer with transmission axis at an arbitrary angle ϕ:

$$\mathcal{M} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}. \tag{35}$$

(b)

A quick way to check if the matrix is unitary is to multiply $\mathcal{M}$ by $\left(\mathcal{M}^T\right)^*$ and see if it equals the identity matrix:

$$\begin{aligned} \mathcal{M}\cdot\left(\mathcal{M}^T\right)^* &= \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}. \end{aligned} \tag{36}$$

Well that's not the identity matrix, so the Jones matrix of a linear polarizer is not unitary.

Problem 8

(a)

The Jones matrix for the ideal wave plate is:

$$\mathcal{M} = \begin{bmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{bmatrix}. \tag{37}$$

(b)

For the general case we just do the matrix multiplication as in problem 7 (a):

$$\begin{aligned} \mathcal{M}' &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \\ &= e^{-i\delta/2}\begin{bmatrix} e^{i\delta}\cos^2\phi - \sin^2\phi & \left(1+e^{i\delta}\right)\sin\phi\cos\phi \\ -\left(1+e^{i\delta}\right)\sin\phi\cos\phi & \cos^2\phi - e^{i\delta}\sin^2\phi \end{bmatrix}. \end{aligned} \tag{38}$$

(c)

Now we check for unitarity just as in problem 7 (b):

$$\begin{aligned} \mathcal{M}'\cdot\left(\mathcal{M}'\right)^{T*} = e^{-i\delta/2}&\begin{bmatrix} e^{i\delta}\cos^2\phi - \sin^2\phi & \left(1+e^{i\delta}\right)\sin\phi\cos\phi \\ -\left(1+e^{i\delta}\right)\sin\phi\cos\phi & \cos^2\phi - e^{i\delta}\sin^2\phi \end{bmatrix}. \\ e^{i\delta/2}&\begin{bmatrix} e^{-i\delta}\cos^2\phi - \sin^2\phi & \left(1+e^{-i\delta}\right)\sin\phi\cos\phi \\ -\left(1+e^{-i\delta}\right)\sin\phi\cos\phi & \cos^2\phi - e^{-i\delta}\sin^2\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{39}$$

So the Jones vector for the waveplate is a unitary operator!

Bye for now!

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 4

1.
A plane wave polarized in the $\hat{\mathbf{x}}$ direction is normally incident upon an ideal quarter-wave plate which has its slow axis oriented at $+45°$ with respect to the $\hat{\mathbf{x}}$ axis. Next it is reflected at normal incidence by a perfectly conducting mirror. Finally the wave passes back through the same quarter-wave plate. After completing this journey...
(a.)
What is the final wave's irradiance, relative to the incident wave?
(b.)
What is the final wave's state of polarization? (If linearly polarized, specify the polarization direction; if circularly polarized, state whether right- or left-handed.)
Both of your answers should be justified.

2.
A fiber-optic cable consists of a cylindrical core with refractive index n_1 surrounded by a sheath with refractive index n_2, where $1 < n_2 < n_1$. The ends of the cable are cut perpendicular to the cable axis and polished. A point source of light is placed on the cable axis a negligible distance away from one end. What fraction of the total light emitted by the point source is transmitted by the core to the (distant) end of the cable? (You may ignore losses due to reflection at the cable ends.)

3.
A vector field $\mathbf{F}(\mathbf{r})$ is equal to the curl of a vector potential $\mathbf{A}$ which is a continuous function of $\mathbf{r}$: $\mathbf{F}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$.

Prove that $\mathbf{F}_\perp$ is continuous across any surface, where "$\perp$" refers to the component of $\mathbf{F}$ which is perpendicular to the surface.

4.
A transverse electromagnetic wave in vacuum is normally incident on a semi-infinite slab of electrically insulating material which has $\epsilon = \epsilon_0$ but $\mu = \kappa_m \mu_0$. That is, the material has trivial dielectric but nontrivial magnetic properties. Consider the ferromagnetic limit $\kappa_m \to \infty$.

In that limit, what is the ratio of peak electric field $|\mathbf{E}''|_{\max}$ in the transmitted wave to the same quantity $|\mathbf{E}|_{\max}$ in the incident wave?

5.
Plane waves propagating in the $\pm z$ directions bounce between two semi-infinite nonconducting materials. The first material occupies the region $z < 0$, and the second occupies $z > L$, where L is a positive fixed distance. These (hypothetical) materials have no unusual magnetic properties ($\mu = \mu_0$), but they have *infinite* dielectric constant, $\epsilon/\epsilon_0 = \infty$.
(a.)
What components of $\mathbf{E}$ and $\mathbf{H}$ vanish, and where do they vanish? Justify each answer that you give.
(b.)
What angular frequencies for the light are possible?

6.
A nonrelativistic particle of mass m in one (x) dimension can be represented by a (complex) wavefunction

$$\Psi(x,t) \propto \exp\left(i(kx - \omega t)\right)$$

where

$$\hbar^2 k^2(x) = 2m(E - V(x)),$$

$E = \hbar\omega$ is the total and $V(x)$ is the potential energy, and $2\pi\hbar$ is Planck's constant. If $V(x)$ is piecewise flat, at the discontinuities in V the Schrödinger equation demands that *both* Ψ and $\partial\Psi/\partial x$ remain continuous.

Consider the reflection of a particle of initial kinetic energy T by a potential barrier of height $\Delta V < T$. Show that the ratio of reflected to incident amplitudes is given by the same formula as for the normal reflection of an electromagnetic wave at a dielectric interface with $\mu = \mu_0$ everywhere, provided that the refractive index in the quantum mechanical case is taken to be proportional to k.

7.
Fowles 3.1.

8.
Fowles 3.6.

"Physicists are like 3% of rats."
-Max Zolotorev, Lawrence Berkeley National Laboratory

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

As demonstrated in last week's problem set (problem 8), an ideal quarter-wave plate is described by a unitary Jones matrix. This means that the irradiance of the light beam is unaffected by traversing the plate. Also, the mirror is a perfect conductor, so 100% of the light is reflected. So the final wave's irradiance must be the same as the incident wave's.

(b)

The wave, initially polarized along the $\hat{x}$ direction, first passes through the quarter wave plate, whose fast axis is oriented at -45^o with respect to the initial light polarization:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} E_0 \\ 0 \end{bmatrix} = \frac{E_0}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{1}$$

Then the light bounces off a perfectly conducting mirror. This reverses the sign of the Poynting vector, which in turn changes the sign of the B-field relative to the E-field, since (as can be shown from Maxwell's equations):

$$\begin{aligned} \vec{E} &= E_0 \hat{\eta} e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} \\ \vec{B} &= \frac{1}{v}\hat{k}\times\vec{E}, \end{aligned} \tag{2}$$

where $v = c/n$ is the phase velocity of light in a medium and $\hat{\eta}$ is the direction of the light's electric field. The light electric field undergoes a phase shift of π upon relection (as can be deduced from the boundary conditions), but for circularly polarized light the direction of rotation (clockwise or counter-clockwise) of the electric field with respect to a fixed coordinate system is preserved. However, we are now viewing it from the opposite direction (since $\vec{k}$ changed sign). Therefore the handedness of polarization has changed upon reflection:

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \frac{E_0}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{3}$$

This result can also be arrived at using Fowles's reflection matrix (page 52).
The beam now passes back through the quarter waveplate, but now the wave sees the fast-axis oriented at +45^o. So we find that:

$$\begin{bmatrix} E''_x \\ E''_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \frac{E_0}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = -iE_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

In other words the resultant light is linearly polarized in the $\hat{y}$ direction.

Problem 2

First let's derive Fowles' result regarding the acceptance angle α for a fiber-optic cable (pages 46-47).
At the first phase transition as the light enters the fiber-optic cable, we have from Snell's law:

$$\sin\alpha = n_1 \sin\beta. \tag{5}$$

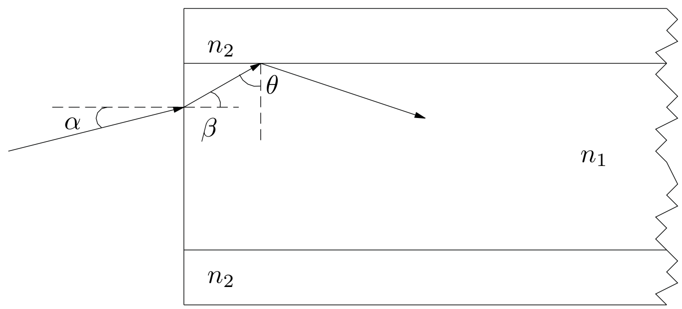


Figure 1

We want $\theta = \pi/2 - \beta$ to be greater than or equal to the critical angle, $\sin^{-1} n$ for total internal reflectance, where $n = n_2/n_1$. With a little trigonometry it can be shown that for these conditions,

$$\sin\beta = \frac{\sqrt{n_1^2 - n_2^2}}{n_1}. \tag{6}$$

Combining Eqs. (5) and (6) yields:

$$\sin\alpha = \sqrt{n_1^2 - n_2^2}, \tag{7}$$

which proves Fowles's assertion.
The next step is to compute the solid angle of light accepted by the fiber-optic cable, given by:

$$\Delta\Omega = \int_0^{2\pi}\int_0^{\alpha} \sin\theta d\theta d\phi = 2\pi(1-\cos\alpha). \tag{8}$$

Divided by the total solid angle (4π), this yields the fraction T of the total light transmitted by the core to the end of the cable:

$$T = \frac{(1-\cos\alpha)}{2}. \tag{9}$$

Problem 3

A vector field $\mathbf{F}(\vec{r})$ is equal to the curl of a vector potential $\mathbf{A}(\vec{r})$, so we know that the divergence of $\mathbf{F}(\vec{r})$ is zero:

$$\nabla\cdot\mathbf{F}(\vec{r}) = \nabla\cdot(\nabla\times\mathbf{A}(\vec{r})) = 0. \tag{10}$$

Then we know that:

$$\int \nabla\cdot\mathbf{F}(\vec{r})dV = \oint \mathbf{F}(\vec{r})\cdot d\vec{A} = 0 \tag{11}$$

Choose a volume of vanishing thickness $\vec{\delta}$ about a surface of area A (where $\vec{\delta}$ is always normal to the surface), then from Eq. (11) we have that:

$$\mathbf{F}_\perp\left(\vec{r}+\vec{\delta}\right)A - \mathbf{F}_\perp(\vec{r})A + O(\delta) = 0 \tag{12}$$

where $O(\delta)$ indicates a term of order δ, which arises from some finite amount of "flux" of $\mathbf{F}(\vec{r})$ out the sides of the volume.

The condition for continuity of $\mathbf{F}_\perp(\vec{r})$ is that for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|\mathbf{F}_\perp\left(\vec{r}+\vec{\delta}\right) - \mathbf{F}_\perp(\vec{r})| < \epsilon$. From Eq. (12) we know that:

$$|\mathbf{F}_\perp\left(\vec{r}+\vec{\delta}\right) - \mathbf{F}_\perp(\vec{r})| = \frac{O(\delta)}{A}. \tag{13}$$

Given any $\epsilon > 0$, clearly we can choose δ to make:

$$\frac{O(\delta)}{A} < \epsilon. \tag{14}$$

Therefore $\mathbf{F}_\perp(\vec{r})$ is continuous.

Problem 4

From Strovink's treatment of reflection/refraction at a plane interface between insulators, we have for normal incidence:

$$\frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_1 + Z_2}$$
$$\frac{E_t}{E_i} = \frac{2Z_2}{Z_1 + Z_2}, \tag{15}$$

where E_r, E_i, E_t are the reflected, incident and transmitted electric field amplitudes, respectively, and

$$Z_{1,2} = \sqrt{\frac{\mu_{1,2}}{\epsilon_{1,2}}}. \tag{16}$$

In particular, for the ferromagnetic material described, $Z_2 \to \infty$ while Z_1 is finite, so:

$$\frac{E_t}{E_i} = \frac{2Z_2}{Z_1 + Z_2} \to 2. \tag{17}$$

Problem 5

(a)

Plane waves propagating in the $\pm z$ directions must satisfy Eqs. (15) at the interfaces ($z = 0$ and $z = L$). At either interface we have, since $Z_2 \to 0$,

$$\frac{E_t}{E_i} = \frac{2Z_2}{Z_1 + Z_2} \to 0. \tag{18}$$

Therefore $\mathbf{E}$ vanishes in the material, i.e. $\mathbf{E}$=0 for $z < 0$ and $z > L$. So at the interfaces, $\mathbf{E}$=0.

The magnetic fields of the plane waves propagating in the $\pm z$ directions between the materials must satisfy $\mathbf{E}_+\times\mathbf{H}_+ = -\mathbf{E}_-\times\mathbf{H}_-$ (i.e. the Poynting vectors of right- and left-traveling waves must be oriented in opposite directions). So while the electric fields cancel ($\mathbf{E}_+ = -\mathbf{E}_-$) at the interfaces the magnetic fields must add ($\mathbf{H}_+ = \mathbf{H}_-$)! Also we have the boundary condition $\mathbf{H}_{\parallel} = \mathbf{H}'_{\parallel}$, so that if $\mathbf{H}$ is finite on one side of the interface, it must also exist on the other side. So there can be components of $\mathbf{H}$ everywhere.

(b)

Our requirements from part (a) set up a standing wave, where the components of $\mathbf{E}$ and $\mathbf{H}$ are π out of phase. The wave is time independent in order to assure that $\mathbf{E}$=0 at the interfaces for all times t, so we can postulate:

$$\mathbf{E} = \vec{E_0}\sin kz, \tag{19}$$

which works so long as $k = N\pi/L$ where N is an integer. So we get the condition on angular frequency from $k = \omega/c$, which implies

$$\omega = \frac{N\pi c}{L}. \tag{20}$$

Problem 6

We have two regions as shown in Figure 2, with k_1 and k_2 in each (defined as in the problem, where they are dependent on the potential V and the particle's total energy U, which is conserved, U = T + V). In region 1 we have the wavefunctions:

$$Ae^{i(k_1 x-\omega t)} + Be^{-i(k_1 x+\omega t)}, \tag{21}$$

and in region 2 we have a transmitted wavefunction:

$$Ce^{i(k_2 x-\omega t)}. \tag{22}$$

Continuity of the wavefunctions across the boundary (x=0) demands:

$$A + B = C. \tag{23}$$

Since $\frac{\partial\psi}{\partial x}$ is also continuous:

$$k_1(A - B) = k_2 C. \tag{24}$$

Substituting Eq. (23) into (24), we get:

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2} \tag{25}$$

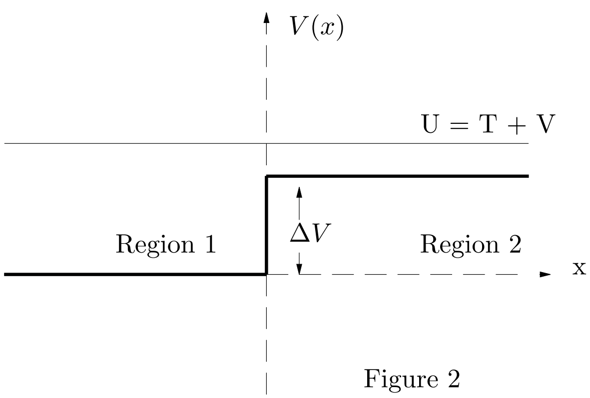


Figure 2

If we assume $n \propto k$, then we get the formula for normal reflection of an EM wave at a dielectric interface with $\mu = \mu_0$:

$$\frac{B}{A} = \frac{n_1 - n_2}{n_1 + n_2} \tag{26}$$

Problem 7

Fowles 3.1

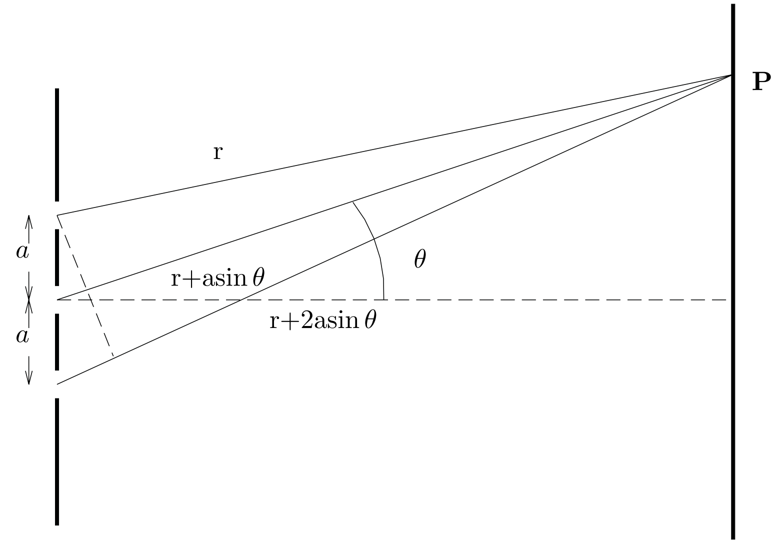


Figure 3

The sum of the amplitudes of the waves A coming from the slits at point P (see Figure 3) are given by the proportionality:

$$A \propto e^{ikr} + e^{ik(r+a\sin\theta)} + e^{ik(r+2a\sin\theta)} \tag{27}$$

or,

$$A \propto e^{ikr}\left(1 + e^{ika\sin\theta} + e^{2ika\sin\theta}\right) \tag{28}$$

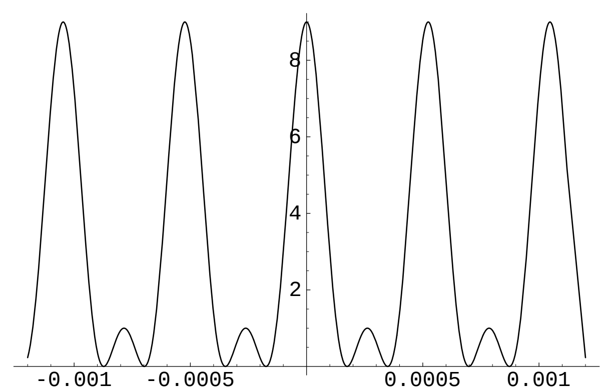


Figure 4

The interference pattern is given by the norm square of the amplitude:

$$I(\theta) \propto |A|^2 \propto (1 + \cos(ka \sin\theta))^2 \tag{29}$$

The pattern that you get when you plot this function depends on what value you choose for ka. Let's take $a = 1$ mm and $k = 12{,}000$ mm^{-1}, then our pattern is shown in Figure 4 as a function of θ.

Problem 8

Fowles 3.6

Light passes through the gas cell twice, so the optical path difference d_{op} is given by:

$$d_{op} = c\Delta t = \frac{2l}{c/n} - \frac{2l}{c} = 2l(n-1) \tag{30}$$

n changes as the gas fills the cell, and since $I \propto 1 + \cos(2\pi d_{op}/\lambda)$, a new fringe appears every time $d_{op} = 1/2$. Thus the total number of fringes N is given by:

$$N = 2\frac{2l(n-1)}{\lambda} = \frac{4l(n-1)}{\lambda}. \tag{31}$$

Plugging in the suggested values gives us $N = 203$ fringes.

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 5

1.
It is known that, in Region 1 ($y > 0$, $\epsilon_1 = 4\epsilon_0$, $\mu_1 = \mu_0$), there exists a plane wave propagating in the $\hat{\mathbf{x}}$ direction.
a.
What is μ_2 in Region 2 ($y < 0$, $\epsilon_2 = \epsilon_0$)?
b.
What state(s) of polarization ($\mathbf{E}$ along $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, or both) can the plane wave be in? Why?

2.
Fowles 2.23. Substitute "interface" for "window" – that is, consider just one glass-air interface, not two. The "degree of polarization" is defined by Fowles' Eq. (2.26).

3.
A beam of light travelling in the plane $y = 0$ is refracted by the interface $z = 0$ between two insulators: vacuum ($z < 0$) and a material ($z > 0$) with $\mu = \mu_0$ and with dielectric constant ϵ, where $\epsilon > \epsilon_0$. In the semi-infinite region $z < 0$, the angles of incidence and reflection, with respect to the z axis, are $60°$. In the semi-infinite region $z > 0$, the angle of refraction, with respect to the same axis, is $30°$.
(a.)
Taking the reflected and refracted angles to be as given, calculate ϵ/ϵ_0 for the material.
(b.)
The incident beam is right-hand circularly polarized. What is the state of polarization of the reflected beam? Explain your answer.
(c.)
Calculate the ratio R of irradiances

$$R = \frac{I_{\text{reflected}}}{I_{\text{incident}}} \ .$$

4.
Fowles 3.11.

5.
Fowles 3.13.

6.
Fowles 4.4.

7.
A camera lens is purplish because it is *optically coated* to minimize reflection at the center of the visible spectrum. (The coating parameters therefore are not optimized for red or blue light.)

Consider a plane EM wave in vacuum with wavelength λ normally incident on a semi-infinite piece of glass with refractive index $n > 1$ and unit permeability $\mu/\mu_0 = 1$. Choose the thickness and the refractive index of a coating on the glass in order to force the reflected wave to vanish.

8.
Show that the matrix equation (Fowles 4.24) for a single-layer film in fact is an equation that merely transforms the total (complex) E_T and H_T just to the right of the right hand interface to the total (complex) E_0 and H_0 just to the left of the left hand interface.

Show, therefore, that, for a multilayer film, the overall transfer matrix is equal to the product of the transfer matrices for the individual films, as (Fowles 4.28) asserts without proof.

"Guys, don't worry about midterms. They're not the best measure of your worth as a physicist. In fact, I did rather poorly on my first physics midterm, I got something like 3 out of 40. Of course, everyone else got 1 out of 40, but that's not really the point..."

- Prof. Nima Arkani-Hamed, UC Berkeley

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

Boundary conditions at the interface ($y = 0$) between Region 1 and Region 2 imply there will be EM waves propagating in the $\hat{x}$-direction in both regions. Without knowing their polarizations just yet, let the electric fields of the waves be given by:

$$\begin{aligned} &\mathbf{E}_1 e^{i(k_1 x - \omega t)} \\ &\mathbf{E}_2 e^{i(k_2 x - \omega t)}. \end{aligned} \tag{1}$$

At the interface between Region 1 and Region 2, since the electric and magnetic field amplitudes in both regions are independent of x and t, the only way to satisfy boundary conditions at all positions and times is if:

$$e^{i(k_1 x - \omega t)} = e^{i(k_2 x - \omega t)} \tag{2}$$

For example, at $t = 0$, this requires that:

$$k_1 = k_2 \tag{3}$$

which means the indices of refraction in the two regions must be the same. Thus we have,

$$\sqrt{\frac{\epsilon_1 \mu_1}{\epsilon_0 \mu_0}} = \sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_0 \mu_0}} \tag{4}$$

which gives us $\mu_2 = 4\mu_0$.

(b)

We have the following boundary conditions at the interface:

$$\begin{aligned} \epsilon_1 E_\perp^{(1)} &= \epsilon_2 E_\perp^{(2)} \\ \mu_1 H_\perp^{(1)} &= \mu_2 H_\perp^{(2)} \\ E_\parallel^{(1)} &= E_\parallel^{(2)} \\ H_\parallel^{(1)} &= H_\parallel^{(2)} \end{aligned} \tag{5}$$

First let's see if $\vec{E}$ can be along $\hat{y}$. In this case, from Eqs. (5), we have that:

$$\begin{aligned} \epsilon_1 E_y^{(1)} &= \epsilon_2 E_y^{(2)} \Rightarrow \\ 4E_y^{(1)} &= E_y^{(2)} \end{aligned} \tag{6}$$

and we also know that for the wave in Region 1:

$$\begin{aligned} H_z^{(1)} &= \sqrt{\frac{\epsilon_1}{\mu_1}} E_y^{(1)} \Rightarrow \\ H_z^{(1)} &= 2\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(1)}, \end{aligned} \tag{7}$$

and for the wave in Region 2:

$$\begin{aligned} H_z^{(2)} &= \sqrt{\frac{\epsilon_2}{\mu_2}} E_y^{(2)} \Rightarrow \\ H_z^{(2)} &= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(2)} = 2\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(1)} \end{aligned} \tag{8}$$

We see that these relations verify that $H_z^{(1)} = H_z^{(2)}$ as demanded by the boundary conditions in Eqs. (5). The relations are consistent, so therefore such a polarization is allowed.

Can $\vec{E}$ be along $\hat{z}$? We have from Eqs. (5) that

$$E_z^{(1)} = E_z^{(2)} \tag{9}$$

For the wave in Region 1:

$$\begin{aligned} H_y^{(1)} &= \sqrt{\frac{\epsilon_1}{\mu_1}} E_y^{(1)} \Rightarrow \\ H_y^{(1)} &= 2\sqrt{\frac{\epsilon_0}{\mu_0}} E_z^{(1)}. \end{aligned} \tag{10}$$

In Region 2:

$$H_y^{(2)} = \sqrt{\frac{\epsilon_2}{\mu_2}} E_y^{(2)} \Rightarrow$$
$$H_y^{(2)} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(1)}. \tag{11}$$

We see that this verifies the boundary condition on H_y from Eqs. (5):

$$\mu_1 H_y^{(1)} = 2\sqrt{\epsilon_0\mu_0} E_y^{(1)} =$$
$$\mu_2 H_y^{(2)} = 2\sqrt{\epsilon_0\mu_0} E_z^{(1)}. \tag{12}$$

So in fact the wave can be in either polarization state.

Problem 2

Reflection at the Brewster angle transmits all TM light and reflects part of the TE light. The degree of polarization P (for linear polarized light) is given by Fowles Eq. (2.27):

$$P = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{13}$$

The light is initially unpolarized, so it is 50% TE and 50% TM. The light transmitted has reduced TE intensity (since some is reflected at the interface), but the TM intensity remains the same. Consequently, the transmitted light is partially polarized:

$$P = \frac{I_{TM} - I_{TE''}}{I_{TM} + I_{TE''}} = \frac{I_{TE'}}{I_{TM} + I_{TE''}}. \tag{14}$$

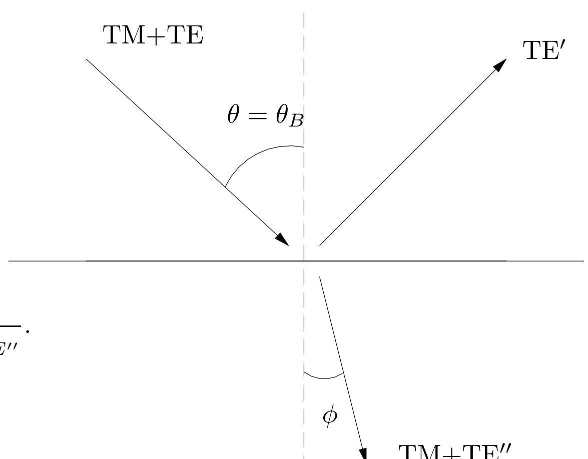


Figure 1

Now we can apply the Fowles formalism for reflection and transmission of TM and TE waves, using the definitions on page 43 of Fowles:

$$r_s = \left[\frac{E'}{E}\right]_{TE}$$
$$t_s = \left[\frac{E''}{E}\right]_{TE}$$
$$r_p = \left[\frac{E'}{E}\right]_{TM}$$
$$t_p = \left[\frac{E''}{E}\right]_{TM} \tag{15}$$

From the boundary conditions for **E** and **H** at the interface (Eqs. (5)), we find some basic relations between the quantities in Eqs. (15):

$$t_s = r_s + 1$$
$$nt_p + r_p = 1 \tag{16}$$

At the Brewster angle $r_p = 0$, so $t_p = 1/n$. Employing the fact that $I \propto nE^2$ (thanks to Paul Wright!), we have for the partial polarization:

$$P = \frac{1}{n}\frac{r_s^2}{t_s^2 + t_p^2} = \frac{1}{n}\frac{(t_s - 1)^2}{t_s^2 + 1/n^2} \tag{17}$$

Fowles works out the general formulae (Fresnel's equations) for reflection/refraction at a plane interface, and in particular for t_s we have from Fowles (2.56):

$$t_s = \frac{2\cos\theta\sin\phi}{\sin(\theta+\phi)}. \tag{18}$$

If we combine Fowles Eq. (2.64)

$$\tan\theta_B = n \tag{19}$$

with Snell's Law

$$n = \frac{\sin\theta}{\sin\phi}, \tag{20}$$

we have the Brewster condition:

$$\theta + \phi = \pi/2, \tag{21}$$

which follows from the fact that $\sin\phi = \cos\theta$. Knowing also then from Eq. (21) that $\sin(\theta+\phi) = 1$, we find that:

$$t_s = 2\cos^2\theta \tag{22}$$

Using the trigonometric identity $\sec^2 = \tan^2 + 1$ and Eq. (19) we find that $\cos^2\theta = 1/(n^2+1)$ so

$$t_s = \frac{2}{n^2+1} \tag{23}$$

Plugging Eq. (23) into Eq. (17), with a little algebra, gives us:

$$P = \frac{n[(n^2-1)]^2}{1+6n^2+n^4}. \tag{24}$$

For glass, where $n = 1.5$, we have that $P \approx 12\%$.

Problem 3

(a)

Note that the Brewster condition (Eq. (21)) is met. Thus from Eq. (19) we have that $n = \tan 60^o = \sqrt{3}$. Since $\mu = \mu_0$, we find that $\epsilon = 3\epsilon_0$.

(b)

Since the Brewster condition is met, the reflected light is 100% TE. Therefore reflected light is linearly polarized along $\hat{y}$.

(c)

The light is initially circularly polarized, so it is an equal superposition of linear polarizations (TE and TM). Since the TM component has 100% transmission, it suffices to consider the transverse electric case ($\vec{E}$ along $\hat{y}$) where all reflection occurs. We have the boundary condition that $E_\parallel$ is continuous, so

$$E_i + E_r = E_t. \tag{25}$$

Also $H_\parallel$ is continuous, and we have for the incident, reflected and transmitted waves the following components of H in the $\hat{x}$ direction:

$$H_x^{(i)} = -E_i\sqrt{\frac{\epsilon_0}{\mu_0}}\cos 60^o$$

$$H_x^{(r)} = E_r\sqrt{\frac{\epsilon_0}{\mu_0}}\cos 60^o$$

$$H_x^{(t)} = -E_t\sqrt{\frac{3\epsilon_0}{\mu_0}}\cos 30^o \tag{26}$$

From which we have the condition:

$$-E_i + E_r = -3E_t. \tag{27}$$

Adding Eq. (25) to Eq. (27) gives us $(E_r/E_i = 1/2)_{TE}$, or $(I_r/I_i = 1/4)_{TE}$. The intensity in the TE component is half the initial intensity, so in total $I_r/I_i = 1/8$.

Problem 4

Here we treat the tungsten filament as a relatively long straight wire of thickness $s = 0.1$ mm. The distance between the filament and an aperture is r. We want a transverse coherence width $l_t \geq 1$ mm. Then from Fowles Section 3.7, and in particular Eq. (3.42), we find that:

$$l_t = \frac{r\lambda}{s} \geq 1mm \tag{28}$$

If we assume that the tungsten lamp has a central wavelength of 5000 $\AA$, then Eq. (28) demands that $r \geq 200$ mm.

If a double-slit aperture is used, the slits should be oriented parallel to the lamp filament, otherwise the thickness of the wire s would have to be replaced with the length of the wire, which is naturally much greater than s. This would force r to be much greater.

Problem 5

The power spectrum of the Gaussian pulse $f(t)$ is given by $G(\omega) = |g(\omega)|^2$, where in our case $g(\omega)$ is:

$$g(\omega) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt \exp\left(-at^2 + i(\omega-\omega_0)t\right) \tag{29}$$

It is first useful to derive a result about Gaussian integrals. It turns out that $\int_{-\infty}^{\infty} e^{-ax^2}dx$ converges, so let's set it equal to some constant c. Now consider the integral over the entire plane:

$$\int_{-\infty}^{\infty} e^{-ax^2}dx \int_{-\infty}^{\infty} e^{-ay^2}dy = c^2. \tag{30}$$

Next we convert the integral into polar coordinates ($r^2 = x^2 + y^2$).

$$\int_0^{\infty} e^{-ar^2} r dr \int_0^{2\pi} d\phi = c^2. \tag{31}$$

Which is relatively straightforward with the substitution $u = ar^2$, $du = 2ardr$. From this integral we find that $c^2 = \pi/a$.

Thus all we need to do is to convert the integral in Eq. (29) to a form resembling a Gaussian. This can be done by completing the square in the exponent:

$$-at^2 + i(\omega - \omega_0)t = -a\left[\left(t^2 - 2i\frac{\omega-\omega_0}{2a}t - \frac{(\omega-\omega_0)^2}{4a^2}\right) + \frac{(\omega-\omega_0)^2}{4a^2}\right]$$
$$= -a\left(t - i\frac{\omega-\omega_0}{2a}\right)^2 - \frac{(\omega-\omega_0)^2}{4a} \quad (32)$$

Now the integral in Eq. (29) is just a Gaussian integral, which is no longer a problem...

Working through the constants gives us:

$$g(\omega) = \frac{A}{\sqrt{2a}}\exp\left[-\frac{(\omega-\omega_0)^2}{4a}\right]. \quad (33)$$

$G(\omega) = |g(\omega)|^2$ is clearly of the same form, and so $G(\omega)$ is a Gaussian function centered at ω_0.

Problem 6

The condition for a fringe maximum to occur is given by Fowles Eq. (4.10):

$$2N\pi = \frac{4\pi}{\lambda}nd\cos\theta + \delta_r. \quad (34)$$

Use the small angle approximation:

$$2N\pi = \frac{4\pi}{\lambda}nd\left(1 - \frac{\theta^2}{2}\right) + \delta_r. \quad (35)$$

We are told that the zeroth order fringe ($N = 0$) has zero radius ($\therefore \theta = 0$), so we have:

$$\delta_r = -\frac{4\pi}{\lambda}nd \quad (36)$$

and subsequently:

$$2N\pi = \frac{4\pi}{\lambda}nd\frac{\theta^2}{2}. \quad (37)$$

So $\theta \propto \sqrt{N}$. Since the radius of the fringes r is proportional to θ, it follows immediately that $r \propto \sqrt{N}$.

Problem 7

This solution follows directly from the discussion of antireflecting films in Fowles (page 99). We want to choose the thickness of the film to be $\frac{\lambda}{4}$. Then the reflectance is zero if the index of refraction of the coating $n = \sqrt{\frac{\epsilon}{\epsilon_0}}$.

Problem 8

Fowles Eq. (4.24) states:

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} r = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} t \quad (38)$$

which is equivalent to:

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} E_0 + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} E_0' = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} E_T \quad (39)$$

The total **E** and **H** just to the right of the right-hand interface are:

$$E_{RH} = E_T$$
$$H_{RH} = n_T E_T. \quad (40)$$

The total **E** and **H** just to the left of the left-hand interface are:

$$E_{LH} = E_0 + E_0'$$
$$H_{LH} = n_0 E_0 - n_0 E_0'. \quad (41)$$

When the relations in Eqs. (40) and (41) are substituted into Eq. (39), we find that:

$$\begin{bmatrix} E_{LH} \\ H_{LH} \end{bmatrix} = M \begin{bmatrix} E_{RH} \\ H_{RH} \end{bmatrix} \quad (42)$$

Therefore the overall transfer matrix M_{tot} is merely the product of transfer matrices for the individual films M_i. This follows from induction. Suppose there are n films with transfer matrices $M_1, M_2, ..., M_n$. Let the fields to the right of the last film be E_n and H_n. The fields just to the left of the nth film are:

$$\begin{bmatrix} E_{n-1} \\ H_{n-1} \end{bmatrix} = M_n \begin{bmatrix} E_n \\ H_n \end{bmatrix} \quad (43)$$

the fields just to the left of the (n-1)th film are:

$$\begin{bmatrix} E_{n-2} \\ H_{n-2} \end{bmatrix} = M_{n-1} \begin{bmatrix} E_{n-1} \\ H_{n-1} \end{bmatrix} = M_{n-1} M_n \begin{bmatrix} E_n \\ H_n \end{bmatrix} \tag{44}$$

and so on...

This argument leads to Fowles Eq. (4.28) as stated.

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Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 6

Problems 1-4 are suitable review problems for Midterm 1. Problems 5-8 involve new material beyond the range covered by Midterm 1.

1.
Model an animal's eye as a sphere composed of vitreous humor, backed by a retina. If the eye is focused at infinity, what is the refractive index of the humor?

2. A point source of isotropic light is located at the center of a small hemispherical hole in the plane end face of a cylindrical light guide with refractive index $n = 2$, permeability $\mu = \mu_0$.

What fraction of the light emitted can be transmitted a long distance (relative to its radius) by the light guide? (A *number* is required.)

3.
Using a combination of optical devices (polarizers, wave plates...), design an optical system that will pass right-hand circularly polarized light without changing its polarization, but will completely block left-hand circularly polarized light. This system is called a "right-hand circular analyzer". Use Jones matrices to prove that your design will work.

4.
A Michelson interferometer produces fringes on its screen (which is not quite perfectly aligned). It is fed by laser light polarized out of the interferometer plane along $\hat{\mathbf{z}}$. With equal path lengths the fringe visibility $\mathcal{V} \equiv (I_{\max} - I_{\min})/(I_{\max} + I_{\min})$ is unity.
(a.)
An ideal linear polarizer, with transmission axis oriented at 90° to $\hat{\mathbf{z}}$, is placed in one leg of the interferometer. What is $\mathcal{V}$? Explain.
(b.)
Same as (a.) except that a second linear polarizer, with transmission axis oriented at 45° to $\hat{\mathbf{z}}$, is added in the same leg *upstream* of the first (*i.e.* closer to the half-silvered mirror). Show your calculation for $\mathcal{V}$.

5.
Two identical horizontal thin slits in a black plate are centered at $y = \pm\frac{h}{2}$, where y is the vertical coordinate. A screen with vertical coordinate Y is located a distance D downstream. If an analyzer is present, it is located just upstream of the screen. Fraunhofer conditions apply, *i.e.* $kh^2 \ll D$, and small-angle approximations can be made, *i.e.* $|y| \ll D$, $|Y| \ll D$. Plane wave A is normally incident on the top slit and plane wave B is normally incident on the bottom slit, with

$$\mathbf{E}_A \propto \Re[(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{i(kz-\omega t)}]$$

and

$$\mathbf{E}_B \propto \Re[(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i(kz-\omega t)}] \ ,$$

with $\Re$ denoting the real part. When either slit is blocked and no analyzer is in place, the intensity $I(Y = 0) \equiv I_0$. When neither slit is blocked, find $I(Y)/I_0$, where I_0 is defined above, for the following cases:
(a.)
No analyzer is in place.
(b.)
The analyzer accepts only $\hat{\mathbf{y}}$ polarized light.
(c.)
The analyzer accepts only right-hand circularly polarized light $\mathbf{E} \propto \Re[(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i(kz-\omega t)}]$.

6.
Prove that

$$\sum_{n=1}^{N} \exp{(i\phi_n)} = \frac{\sin N\Delta\phi/2}{\sin \Delta\phi/2} \exp{(i\bar{\phi})},$$

where

$$\Delta\phi \equiv \phi_{n+1} - \phi_n,$$

and $\bar{\phi}$ is the average of the ϕ_n.

7.
A plane wave of wavelength λ is incident on (A)

no screen, *i.e.* all the light passes through; (B) a black disk of radius R; (C) a black screen with a circular hole of the same radius R. The relative intensities seen by an observer on the axis at distance D downstream are in the ratio $I_A : I_B : I_C = 1 : 1 : 0$. Fraunhofer conditions do not apply to this geometry, although the obliquity and $1/r$ factors do not vary appreciably across the screen.

(a.)

Find the smallest possible value of R that is consistent with the above conditions, expressed in terms of D and λ.

(b.)

In this problem the screen aperture functions g_B for case (B) and g_C for case (C) sum together to give the aperture function g_A for case (A). For the particular R that you obtained for part (a), the intensities I_B and I_C also sum together to give I_A. For what other choices of R would that be true? Explain.

8.

A plane electromagnetic wave propagates in the $\hat{\mathbf{z}}$ direction within a good conductor ($\lambda_{\text{EM wave}} \gg \delta(=\text{skin depth}) \gg \lambda_{\text{plasma}}$). Evaluate the *total* power lost per *square* meter due to Joule (ohmic) heating in the region $0 < z < \infty$. Show that this is equal to the average value of $|\mathbf{S}|$ at $z = 0$. You may take $\mu = \mu_0$ for this conductor.

At the question period after a Dirac lecture at the University of Toronto, somebody in the audience remarked: "Professor Dirac, I do not understand how you derived the formula on the top left side of the blackboard."
"This is not a question," snapped Dirac, "it is a statement. Next question, please."

- George Gamow, excerpted from *Thirty Years that Shook Physics*, a very fun book on the people involved in the early development of quantum mechanics.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

The eye is focused at infinity, so we assume the rays incident on the eye are all parallel (Fig. 1). From the geometry of the diagram in Fig. 1 it is clear that $\theta = 2\phi$. Snell's law demands that $\sin\theta = n\sin\phi$ where n is the refractive index of the humor. We can make the small angle approximation (making the realistic assumption that light passes only through a small iris in the center of the front of the eye) and just say that $\theta = n\phi$, which gives us $n = 2$.

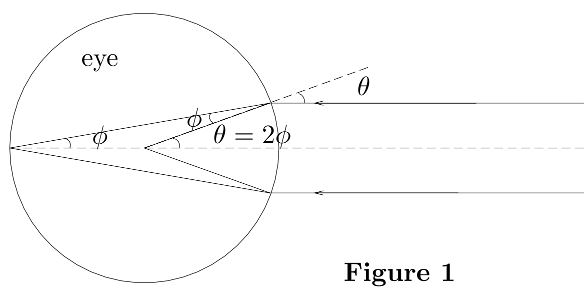

Figure 1

Problem 2

All light coming from the point source is normally incident on the surface of the hemispherical hole in the end of the light guide. Thus the amplitude of the electric field of transmitted light E_t is given by (from Strovink and/or Fowles):

$$\frac{E_t}{E_0} = \frac{2Z_2}{Z_2 + Z_1}, \tag{1}$$

where $Z_{1,2} = \sqrt{\mu_{1,2}/\epsilon_{1,2}}$. Thus the transmission coefficient $t = \frac{E_t}{E_0}$ is given by:

$$t = \frac{2}{1+n}, \tag{2}$$

where $n = 2$ is the refractive index. The percent of light transmitted (intensity) is $T = |t|^2 = 4/9$ in our case.

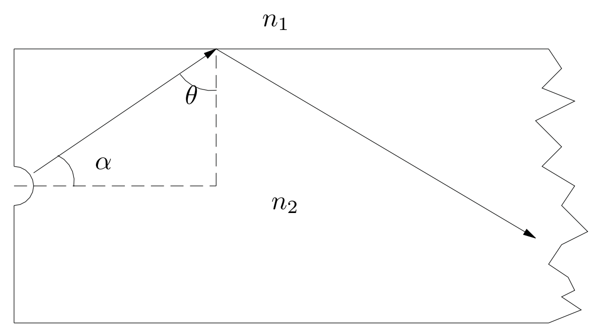

Figure 2

Next consider the diagram in Fig. 2. We require that $\varphi \geq \sin^{-1}(1/n)$ for total internal reflection. From geometry this demands that $\alpha \leq \cos^{-1}(1/n)$. We can now integrate to find the total solid angle $\Delta\Omega$ of light accepted into the light guide:

$$\Delta\Omega = \int_0^{2\pi}\int_0^{\alpha} \sin\theta d\theta d\phi = 2\pi(1-\cos\alpha) = 2\pi(1-1/n) \tag{3}$$

The percent of light accepted is then $\Delta\Omega/4\pi$, or 1/4. So then the fraction of light that travels an appreciable distance is given by the fraction of light transmitted through the interface in the correct direction which is 1/9.

Problem 3

We start out with right-hand circularly polarized light and send it through a quarter-wave plate with the fast axis vertical:

$$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{4}$$

Next, we send the light through a linear polarizer with the transmission axis at 45^o:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{5}$$

So we have 100% transmission for right-circularly polarized light.

For left circularly polarized light, no light is transmitted:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

So the right hand circular analyzer works as claimed.

Problem 4

(a)

If a linear polarizer at 90^o to $\hat{z}$ (the direction of light polarization) is placed in a leg of the Michelson interferometer, no light travels in one leg of the interferometer. Then there will be no fringes and no interference, so $\mathcal{V} \equiv (I_{\max} - I_{\min})/(I_{\max} + I_{\min}) = 0$. This is clear since $I_{\min} = I_{\max}$ if there are no fringes.

(b)

Now, with a linear polarizer at 45^o to $\hat{z}$ upstream of the first linear polarizer, there is light transmitted in both legs of the interferometer. Interference will not occur for light of orthogonal polarizations, so only light polarized in the $\hat{z}$ direction contributes to the fringes.

The linear polarizer at 45^o transmits:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot E_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{E_0}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{7}$$

The next polarizer only transmits light in the orthogonal direction, so the transmitted light is given by

$$\frac{E_0}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The light bounces off the mirror, which preserves polarization, passes through the second polarizer with no loss of amplitude, then passes through the polarizer at 45^o (which now appears to be at -45^o with respect to the direction of light propagation):

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \frac{E_0}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{E_0}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{8}$$

So now at the output we have two waves from the different arms of the interferometer (supposing light is propagating in the $\hat{x}$ direction):

$$\begin{aligned} \vec{E}_1 &= \frac{E_0}{4}\hat{z} + \frac{E_0}{4}\hat{y} \\ \vec{E}_2 &= E_0 e^{i\phi}\hat{z}, \end{aligned} \tag{9}$$

where ϕ is the phase difference induced by the differing path lengths for the arms of the interferometer.

Thus the intensity of light at the output is given by:

$$I = 2\left(\frac{E_0}{4}\right)^2 + E_0^2 + 2\frac{E_0^2}{4}\cos\phi \tag{10}$$

so we see that for $I_{\max}$ and $I_{\min}$:

$$I_{\max,\min} = \frac{9}{8}E_0^2 \pm \frac{E_0^2}{2}. \tag{11}$$

Using these results in our equation for fringe visibility we find that $\mathcal{V} \equiv (I_{\max} - I_{\min})/(I_{\max} + I_{\min}) = 4/9$

Problem 5

(a)

We have two light beams, whose electric field amplitudes are given by:

$$\mathbf{E}_A \propto \mathrm{Re}[(\hat{x} + i\hat{y})e^{i(kz-\omega t)}]$$

and

$$\mathbf{E}_B \propto \mathrm{Re}[(\hat{x} - i\hat{y})e^{i(kz-\omega t)}]\ ,$$

with Re denoting the real part. We are not given the slit widths, so let's assume that they are small enough to be ignored in our analysis...

With no analyzer in place, the two beams are orthogonally polarized so there is no interference. Thus the intensity at the screen is simply the sum of the two individual intensities, which with small angle approximations is roughly $I \approx 2I_0$.

(b)

With an analyzer that accepts only $\hat{y}$ polarized light, the two light beams have the same polarization after the analyzer and then can interfere. The interference is that of a typical double slit experiment (Young's experiment), as solved in Fowles pp. 59-61. Of course, half the amplitude of each wave has been removed by the analyzer, so the resulting interference pattern is given by:

$$I(Y) = \frac{I_0}{4}\left(1 + \cos\left(\frac{\pi h Y}{\lambda D}\right)\right). \tag{12}$$

(c)

The analyzer blocks out left-hand circularly polarized light so there is contribution only from $\mathbf{E}_B$. Therefore the intensity at the screen is $I \approx I_0$.

Problem 6

We want to prove that

$$\sum_{n=1}^{N} \exp(i\phi_n) = \frac{\sin N\Delta\phi/2}{\sin \Delta\phi/2} \exp(i\bar{\phi}),$$

where

$$\Delta\phi \equiv \phi_{n+1} - \phi_n,$$

and $\bar{\phi}$ is the average of the ϕ_n.

Recall that for a geometric series $\sum_0^\infty ar^n = \frac{a}{1-r}$, where $|r| < 1$. Let us rewrite the sum above as the sum of two infinite series. In order to use the geometric series formula, let's multiply each term in the series by an amplitude $\alpha_n = (1-x)^n$ which ensures that $|\alpha_n \exp(i\phi_n)| < 1$. We can require that the α_n's are so close to unity that they are very well approximated by 1 for the first N terms. We are only worried about the convergence of the tail of the series, which is taken care of with this postulate. We could even take the limit as $x \to 0$ to make this argument more mathematically sound.

Now let's write the series

$$\sum_{n=1}^{N} \exp(i\phi_n)$$

as the difference of two infinite geometric series:

$$\begin{aligned} \sum_{n=1}^{N} \exp(i\phi_n) &\to \sum_{n=0}^{N} \alpha_n \exp(i\phi_1)\exp(in\Delta\phi) \\ &= \sum_{n=0}^{\infty} \alpha_{n+1} \exp(i\phi_1)\exp(in\Delta\phi) - \sum_{n=0}^{\infty} \alpha_{N+n+1} \exp(i\phi_1 + N\Delta\phi)\exp(in\Delta\phi) \\ &= \frac{(1-x)\exp(i\phi_1)}{1-(1-x)\exp(i\Delta\phi)} - \frac{(1-x)^{(N+1)}\exp(i\phi_1 + N\Delta\phi)}{1-(1-x)\exp(i\Delta\phi)} \end{aligned} \tag{13}$$

We now let x go to 0, and we then have:

$$\begin{aligned} \sum_{n=1}^{N} \exp(i\phi_n) &= \frac{\exp(i\phi_1)}{1-\exp(i\Delta\phi)} - \frac{\exp(i\phi_1 + N\Delta\phi)}{1-\exp(i\Delta\phi)} \\ &= \exp\left(\phi_1 + \frac{N-1}{2}\Delta\phi\right)\left(\frac{\exp(-iN\Delta\phi/2) - \exp(+iN\Delta\phi/2)}{\exp(-i\Delta\phi/2) - \exp(+i\Delta\phi/2)}\right) \end{aligned} \tag{14}$$

from which we deduce that:

$$\sum_{n=1}^{N} \exp(i\phi_n) = \frac{\sin N\Delta\phi/2}{\sin \Delta\phi/2} \exp(i\bar{\phi}) \tag{15}$$

proving the original conjecture.

Problem 7

As discussed in Fowles pp. 126-128, this problem can be solved using Fresnel zones. The radius of the Nth Fresnel zone in this case is given by:

$$R = \sqrt{N\lambda D} \tag{16}$$

since the factor $L = (1/h + 1/h')^{-1} = D$ in our case (see Fowles Eq. 5.36).

(a)

(C) The smallest radius for which the light from Fresnel zones cancel is that which includes the first two. In this case the optical disturbance is given by

$$U_p = |U_1| - |U_2| \approx 0.$$

(A) With no screen, the optical disturbance is half that due to the first Fresnel zone,

$$U_p = \frac{1}{2}|U_1|.$$

(B) If we block out the first two Fresnel zones, the optical disturbance is approximately half that due to the third zone, or

$$U_p = \frac{1}{2}|U_3| \approx \frac{1}{2}|U_1|.$$

So as you can see, the choice of

$$R = \sqrt{2\lambda D}$$

satisfies all the required conditions.

(b)

This will hold whenever a similar situation occurs, i.e. an even number of Fresnel zones are blocked by the black disk. The optical disturbances U_A and U_B will always sum to equal the optical disturbance without the screen U_0 because they are complementary apertures. However, only when either U_A or U_B is zero can the squares ($\propto I$) be equal.

So whenever $R = \sqrt{2n\lambda D}$ where n is an integer, this is the case.

Problem 8

From Maxwell's equations in conducting media, we get the wave equations:

$$\begin{aligned} \nabla^2 \vec{E}(z,t) &= \tilde{E}_0 e^{i(\kappa z - \omega t)} \\ \nabla^2 \vec{B}(z,t) &= \tilde{B}_0 e^{i(\kappa z - \omega t)} \end{aligned} \tag{17}$$

where

$$\kappa^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega. \tag{18}$$

For a good conductor,

$$\kappa = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i). \tag{19}$$

Also from Maxwell's equations for a conducting medium, we find that:

$$B_0 = \frac{|\kappa|}{\omega}. \tag{20}$$

The power lost per square meter due to ohmic heating is given by the relation:

$$\begin{aligned} \langle P \rangle &= \sigma \int \langle E^2 \rangle dV \\ &= \frac{\sigma}{2} \int |\tilde{E}_0|^2 \exp\left(-\sqrt{\frac{\mu\sigma\omega}{2}} z\right) = \frac{\sigma}{2}\sqrt{\frac{2}{\mu\sigma\omega}} E_0^2 \end{aligned} \tag{21}$$

The average value of $|\vec{S}|$ is given by $\langle \frac{1}{\mu}\vec{E} \times \vec{B} \rangle$. Plugging in values from above, we find

$$\langle |\vec{S}| \rangle = \frac{\sigma}{2}\sqrt{\frac{2}{\mu\sigma\omega}} E_0^2 \tag{22}$$

which confirms the conjecture stated in the problem.

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 7

1.
Fowles 6.1.

2.
Fowles 6.2.

3.
Fowles 6.6.

4.
Rohlf 2.13.
Note that this problem is more difficult than it first appears to be. You may *not* assume that the *average* number of unexpected events found, per 10^6 interactions, is equal to unity!

5.
Rohlf 2.38.

6.
Assume that the sun is a blackbody of temperature 5800 K and radius 7×10^8 m, located 1.5×10^{11} m from the earth. Assume further that the earth is a gray body which absorbs part of the radiation incident upon it from the sun, and then reradiates it isotropically. Neglect any other effects which could heat the earth. Calculate the surface temperature of the earth under these assumptions.

7.
Greenhouse effect.
Take the sun's blackbody spectrum to have its peak in the yellow ($\lambda = 0.58\ \mu$m). Take $T_{\text{sun}} = 5800$ K and $T_{\text{earth}} = 300$ K.
(a.)
Use Wien's Law ($\lambda_{\text{max}} \propto T^{-1}$) to estimate the wavelength at the peak of the earth's blackbody spectrum.
(b.)
Imagine that the (visible) sun's rays pass through the clear glass of a greenhouse. On the floor of the greenhouse is black dirt with $\epsilon \approx 1$, which absorbs these rays. In contrast, the (infrared) blackbody radiation from the dirt is totally absorbed by the glass and reradiated (half in and half out). Assuming that the temperature outside the greenhouse is 300 K, estimate the temperature inside it.

8.
Nuclear winter (inverse Greenhouse effect).
According to some experts (though this is controversial), after nuclear war a thin layer of dust would remain in the upper atmosphere of the earth. To a first approximation, the dust absorbs all light from the sun, which is near visible in wavelength. The dust then reradiates that energy in the infrared, to which it is transparent: half in toward the earth, half out to space. The dust is nearly transparent to the earth's outward radiation, also in the infrared.

If the peacetime surface temperature of the earth is 300 K, what would that temperature become after nuclear war? To relate to it physiologically, express this latter temperature in °F.

Above the front door of Niels Bohr's cottage was nailed a horseshoe. A visitor who saw it exclaimed: "Being as great a scientist as you are, do you really believe that a horseshoe above the entrance to a home brings good luck?"
"No," answered Bohr, "I certainly do not believe in this superstition. But you know," he added with a smile, "they say that it does bring luck even if you don't believe in it!"

- George Gamow, excerpted from *Thirty Years that Shook Physics.*

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

This problem makes sense only if you make some rather poorly motivated approximations. In particular, we must assume that we are far away from resonance (namely that $\omega_0^2 - \omega^2 \gg \gamma\omega$). We also can assume that there are very few electrons (namely that $\frac{Ne^2}{m\epsilon_0} \ll 1$, which is well-motivated by the fact that κ is much less than n, i.e. few absorbers). If we make these approximations, the results follow almost immediately. If you don't make these assumptions, then the results are clearly incorrect (see Figure 6.1 in Fowles, which is nothing like the equations Fowles asks us to derive). Thus, we'll make these assumptions!

Then we can apply these approximations to equations 6.34 and 6.35 in Fowles. We find that:

$$n^2 - \kappa^2 \approx n^2 \approx 1 + \frac{Ne^2}{m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2} \tag{1}$$

Using a first order Taylor expansion, we then find that:

$$n \approx 1 + \frac{Ne^2}{2m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2} \tag{2}$$

Since n is approximately 1, κ is given by:

$$\kappa \approx \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2}. \tag{3}$$

Problem 2

Once again Fowles attempts to confuse us by implying the above results can be applied in the solution to this problem when they can't. This is because the above equations break down in the vicinity of the resonance, which is where we must work to solve this problem. So here we bring back the $\gamma^2\omega^2$ terms, but continue to assume there are few electrons. In this case our formulas for κ and n are given by:

$$n \approx 1 + \frac{Ne^2}{2m\epsilon_0}\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{4}$$

and

$$\kappa \approx \frac{Ne^2}{2m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{5}$$

If we take the derivative of n with respect to ω and set it equal to zero, we find two positive roots yielding the values for the max and min of the the function n, namely

$$\omega = \omega_0\sqrt{1 \pm \gamma/\omega_0}.$$

It is safe to assume, since damping is small, that this value can be approximated by the first order Taylor expansion:

$$\omega = \omega_0 \pm \gamma/2.$$

If we plug these values into our expression for κ (Eq. 5), we see that these values are those where κ attains half its maximum value.

Problem 3

We are given that $\sigma = 6.8 \times 10^7$ mho/m and that $N_e = 1.5 \times 10^{28}$ electrons/m^3. Using these values in the appropriate Fowles formulas gives us the desired answers...

(a)

Plasma frequency

$$\omega_p = \sqrt{\frac{Ne^2}{m\epsilon_0}} = 6.9 \times 10^{15}\mathrm{s}^{-1}$$

(b)

Relaxation time

$$\tau = \frac{\mu_0\sigma c^2}{\omega_p^2} = 1.6 \times 10^{-13}\mathrm{s}$$

(c)

Our frequency with a wavelength of 10^{-6} m is given by

$$\omega = \frac{2\pi c}{\lambda} = 1.9 \times 10^{15}\mathrm{s}^{-1}.$$

Real and imaginary parts of the index of refraction can be derived from from Fowles Eqs. 6.55 and 6.56:

$$n^2 - \kappa^2 = 1 - \frac{\omega_p^2}{\omega^2 + \tau^{-2}}$$
$$2n\kappa = \frac{1}{\omega\tau}\frac{\omega_p^2}{\omega^2 + \tau^{-2}}$$

Clearly, $\omega_p, \omega \gg \tau^{-1}$, so we get

$$n^2 - \kappa^2 \approx 1 - \left(\frac{\omega_p}{\omega}\right)^2 = -12.2$$

$$2n\kappa \approx \frac{1}{\omega\tau}\frac{\omega_p^2}{\omega^2} = 0.044.$$

We can then solve these equations for n and κ, and with a little algebra we get:

$$n = 0.006$$

$$\kappa = 3.5.$$

(d)

The reflectance is given by the Hagen-Rubens formula,

$$R = 1 - \sqrt{\frac{8\omega\epsilon_0}{\sigma}} \approx 1.$$

Problem 4

This problem, as Prof. Strovink pointed out, is a little bit tricky. Our experimenter finds 1 event in 10^6 interactions. Now we want to be 90% sure we find a second event. How many interactions do we need? The basic problem is that we don't really know the average number of events we should see per 10^6 interactions, which is needed to calculate how many more interactions are necessary to be 90% sure we'll see a second event. So we'll have to try to figure out some function describing our confidence in the value of a we have measured and then convolve it with the probability for seeing a second event.

The probability density function, in this case that for Poisson statistics, is given by:

$$f_p(x) = \frac{e^{-a}a^x}{x!} \tag{6}$$

where x is a non-negative integer and a is the average value of x. The probability density function (with known parameter a) allows us to predict the frequency with which random data x will take on some particular value.

We first want to calculate a value for $a = n{\cdot}p$ (where n is the number of interactions and p is the probability for an event), a distribution function based on the most likely values for a and our confidence in those values. The most likely value of p from the data is 10^{-6}.
For a conservative upper limit (without assuming very much about the prior probability distribution), we can estimate that the lower limit of p (at a 95% confidence level) must be that for which the probability of seeing one event in 10^6 interactions is at least 5%:

$$f_p(1) = e^{-np}np \geq 0.05, \tag{7}$$

which tells us that $np \geq 0.05$, or $p \geq 5 \times 10^{-8}$. Furthermore, we know that the upper limit at 95% confidence level on p can be found from:

$$f_p(1) = e^{-np}np \leq 0.95, \tag{8}$$

which tells us that $p \leq 5.14 \times 10^{-6}$.

We want to be 90% sure we'll see a second event. We could guess that if we're 95% sure p is bigger than $p_{\min} = 5 \times 10^{-8}$ and 95% sure that we'll see at least one more event after n_2 interactions using this value for p, we'll be 90% sure to see a second event. We're 95% sure we'll see at least one more event if the probability to see zero events is less than 0.05:

$$f_p(0) = e^{-n_2 p_{\min}} \leq 0.05,$$

which gives us

$$n_2 \geq 6 \times 10^7 \text{ interactions.}$$

This is a conservative upper limit on the number of interactions we need before we'll see another event.

The above analysis gives us some idea of what our "likelihood distribution" $\tilde{L}$ for p looks like... it is peaked at 1×10^{-6} and nears zero at both 5×10^{-8} and 5.14×10^{-6}. It is well described by the function

$$\tilde{L}(p) = \frac{p}{p_0} e^{-p/p_0}$$

where $p_0 = 10^{-6}$ is the most likely value for p. It is not an accident that this looks exactly like $f_p(1)$. This, as Prof. Strovink explained to me, is just Bayes' assumption of a uniform prior probability distribution – meaning that we assume we found the most likely value of p_0 in our experiment and the distribution of probabilities is that from, in our case, Poisson statistics.

A little more mathematical rigor can be applied if we take our "likelihood distribution" $\tilde{L}$ for p and convolve it with the restriction that the probability for zero events must be less than 10%. This approach yields more or less the same result as Rohlf's answer, which is reasonable since in the Bayesian approach we assume a prior probability distribution with $p_0 = 10^{-6}$ as the central value. Rohlf just assumed that *a priori* we knew the probability for an event to occur would be $p_0 = 10^{-6}$. If we try this approach, we find that:

$$\frac{\int_0^\infty \tilde{L}(p) e^{n_2 p} dp}{\int_0^\infty \tilde{L}(p) dp} \le 0.1$$

Plugging in our assumed prior probability distribution or "likelihood distribution" and using the substitution $u = (1/p_0 + n_2)p$ in the numerator's integral, we find:

$$\frac{1}{p_0} \int_0^\infty \frac{p}{p_0} e^{-p/p_0} e^{-n_2 p} dp = \frac{1}{p_0^2 \cdot (1/p_0 + n_2)^2} \le 0.1.$$

From which we can calculate $n_2 = 2.16 \times 10^6$ for 90% CL that we will see a second event. This is far smaller than our original rough estimate, but assumes a prior probability distribution. This is probably the more correct approach.

Well, as you can probably tell, this problem was quite difficult for me, so don't feel too bad if you had some trouble as well...

Problem 5

As evidenced by the sampling of problems from Rohlf, we can guess his two main interests are particle physics and beer.

The expression for the number of particles from the ideal gas law is:

$$N = \frac{PV}{kT}. \tag{9}$$

We can take the derivative of N with respect to time to obtain:

$$\frac{dN}{dt} = \frac{P}{kT}\frac{dV}{dt}. \tag{10}$$

We know from the statement of the problem that the number of CO_2 molecules is proportional to the surface area of the beer bubbles,

$$\frac{dN}{dt} = Cr^2.$$

Also, assuming a spherical bubble, we know that

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Plugging these into Eq. (10), we find that:

$$\frac{dr}{dt} = \frac{CkT}{4\pi P}, \tag{11}$$

which indicates the radius of the bubble increases linearly with time.

Problem 6

The power given off by the sun is S (power radiated per unit area) times the surface area of the sun, which is:

$$P_S = (\sigma T_S^4) \cdot (4\pi R_S^2) \tag{12}$$

The portion of this power received by the earth is scaled down by the emissivity factor ϵ (earth is treated as a gray body) and the cross-sectional area of the earth over the surface area of a sphere with a radius equal to the distance between the earth and the sun:

$$P_E^{(in)} = \epsilon(\sigma T_S^4) \cdot (4\pi R_S^2) \frac{\pi R_E^2}{4\pi R_{ES}^2} \tag{13}$$

The power re-radiated by the earth is given by:

$$P_E^{(out)} = (\epsilon\sigma T_S^4)(4\pi R_E^2). \tag{14}$$

In equilibrium, $P_E^{(in)} = P_E^{(out)}$. If we equate these expressions, we find that:

$$\frac{T_E^4}{T_S^4} = \frac{1}{4}\frac{R_S^2}{R_{ES}^2}, \tag{15}$$

from which we deduce that

$$T_E = 290 \text{ K}.$$

Problem 7

(a)

We apply Wien's law to get the peak of the earth's blackbody spectrum. First, the constant can be derived from plugging in the known parameters for the sun:

$$\lambda_{\max} = \frac{C}{T}$$

$$C = \lambda_{\max,\text{sun}} T_S = (0.58\ \mu\text{m})(5800\text{ K}) = 3364 \mu\text{m K}.$$

Applying Wien's law to the earth, we get:

$$\lambda_{\max,\text{earth}} = \frac{C}{T_E} = 11.2\ \mu\text{m}.$$

(b)

If half the power re-radiated from the dirt is radiated back into the greenhouse at every interface with the walls, in equilibrium the power radiated by the dirt should be twice that incident from the sun. The power from the sun hitting the dirt of the greenhouse is:

$$P_S^{(dirt)} = (\sigma T_S^4)(4\pi R_S^2)\frac{\text{Area of dirt}}{4\pi R_{ES}^2}, \tag{16}$$

The power re-radiated by the dirt is:

$$P_{out}^{(dirt)} = \sigma T_{\text{dirt}}^4 \cdot (\text{Area of dirt}) \tag{17}$$

Setting $2P_{out}^{(dirt)} = P_S^{(dirt)}$, we find that:

$$T_{\text{dirt}}^4 = \frac{T_S^4}{2}\frac{R_S^2}{R_{ES}^2}$$

$$T_{\text{dirt}} = 333 \text{ K}.$$

Problem 8

Since we only get half the light power from the sun that we used to down on the earth, the new temperature on earth T_E' is given simply by:

$$T_E' = \left(\frac{1}{2}\right)^{1/4} T_E = 252\text{ K} = -5.3\ ^o\text{F}.$$

Brrrr......

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 8

1.
(a.)
Consider a distant star with the same luminosity and surface temperature as the sun. A person (who is as "efficient" as one of Rutherford's graduate students) can see the star if 250 visible photons per second pass through her pupil, which has a radius of 2 mm when nearly fully dilated. What is the maximum distance at which the star is visible to the naked eye?
(b.)
How many cosmic photons per second per square cm were incident on the Nobel-Prizewinning microwave antenna of Penzias and Wilson?

2.
(a.)
The maximum energy of photoelectrons from aluminum is 2.3 eV for incident radiation of 0.2 μm and 0.9 eV for radiation of 0.313 μm. Use these data to calculate Planck's constant and the work function of aluminum.
(b.)
An aluminum photocathode receives incident radiation of 0.313 μm. When the intensity of this radiation is 1 mW, a current of 1 μA is observed in a circuit that detects the photoelectrons that are liberated. Estimate the quantum efficiency of the photocathode.

3.
Rohlf 3.20.

4.
(a.)
The power radiated by an accelerated charge e is given in classical physics by the formula

$$P = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}a^2 \quad \text{(SI units)},$$

where a is the acceleration. Using this formula, calculate the power radiated by an electron in a Bohr orbit characterized by the quantum number n. No numbers are required. (According to the correspondence principle, when n is very large this should agree with a proper quantum mechanical calculation.)
(b.)
The decay rate for an electron in an orbit may be defined to be the power radiated, P, divided by the energy emitted in the decay. (The decay rate is the inverse of the lifetime). Use the Bohr theory expression for the energy radiated, and the expression for P from part (a.) to calculate the "correspondence" value of the decay rate when the electron makes a transition from orbit n to orbit $n-1$. What is the value of this decay rate when $n = 2$? (This will not agree exactly with the true quantum theory, since the correspondence principle will not hold when n is not $\gg 1$.) What is the decay rate when the transition is from an orbit n to an orbit $n-m$?
(c.)
Use the value of the "lifetime" of an electron in an $n = 2$ Bohr orbit, calculated in part (b.), to estimate the uncertainty in the energy of the $n = 2$ energy level. How does it compare with the energy of that level?

5.
Rohlf 3.42.
Do the first part of the problem. Then answer the final question posed in this problem for two extreme cases:
(a.)
The muon capture probability is of the same order of magnitude as the decay probability.
(b.)
The muon capture probability is many orders of magnitude smaller than the decay probability.

6.
There exists a fundamental constant of nature $\mathcal{O}$ whose value is 25,813 ohms. Calculate the ratio between the numerical value of $\mathcal{O}$ and the numerical value of Z_0, the characteristic impedance of free space. Using any hint that you can obtain from this ratio, determine the *algebraic* value

of $\mathcal{O}$, expressed in terms of other fundamental constants.

7.
Rohlf 3.56.

8.
Rohlf 4.54.

Stop talking and write down the Hamiltonian!

- I. B. Khriplovich, Novosibirsk State University (Russia), during a seminar at Berkeley.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

First we determine how many visible photons are emitted from the surface of the sun-like star. The radiated power per unit area per unit wavelength $\frac{dR}{d\lambda}$ is given by the Planck distribution:

$$\frac{dR}{d\lambda} = \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda kT} - 1)}. \tag{1}$$

We want to convert this quantity into the number of photons per second per unit area per unit wavelength $\frac{dn}{d\lambda}$, which can be done by dividing $\frac{dR}{d\lambda}$ by the energy per photon:

$$E_{\text{photon}} = \hbar\omega = \frac{hc}{\lambda}. \tag{2}$$

We integrate over the visible spectrum to get photons per second per unit area, using $T = 5800\ K$ for the temperature of the sun-like star:

$$n = \int_{400\ \text{nm}}^{700\ \text{nm}} d\lambda \frac{2\pi c}{\lambda^4(e^{hc/\lambda kT} - 1)} = 6.3 \times 10^{25} \text{ photons s}^{-1} \text{ m}^{-2}. \tag{3}$$

The total number of photons given off by the sun per second N is n times the surface area of the star, which is roughly $4\pi R_S^2 = 6 \times 10^{18}\ m^2$. This gives us 3.8×10^{44} photons per second from the star! We scale this by the solid angle subtended by the observer's eye,

$$\frac{\pi r_{\text{eye}}^2}{4\pi R_{SE}^2},$$

and solve for R_{SE} such that the eye receives at least 250 photons. This gives us

$$R_{SE}^{\max} \approx 10^{19} \text{ m} \approx 1300 \text{ light years.}$$

(b)

The cosmic background radiation fills the universe roughly isotropically (there is no solid angle suppression), and the temperature of the radiation is $T = 2.74$ K. The incident number of photons per square cm on Penzias and Wilson's antenna is given by an equation similar to Eq. (3):

$$n = \int_0^\infty d\lambda \frac{2\pi c}{\lambda^4(e^{hc/\lambda kT} - 1)} = 2.6 \times 10^{12} \text{ photons s}^{-1} \text{ cm}^{-2}. \tag{4}$$

Problem 2

(a)

The maximum energy of a electron ejected by the photoelectric effect is given by:

$$\frac{hc}{\lambda} - \Phi, \tag{5}$$

where Φ is the work function. We have two data points to use in this relation, with which we can determine h:

$$\frac{hc}{0.2\mu m} - \Phi = 2.3 \text{ eV}$$

$$\frac{hc}{0.313\mu m} - \Phi = 0.9 \text{ eV}$$

Subtracting these equations and solving for h gives us

$$h \approx 2.6 \times 10^{-21} \text{ MeV} \cdot \text{s}.$$

Substituting this value of h back into one of the photoelectric effect equations gives us the work function

$$\Phi = 1.58 \text{ eV}.$$

(b)

Quantum efficiency of the photocathode is just the ratio of emitted electrons to incident photons. The number of photons is the power of light over the average energy per photon at this wavelength:

$$N_\gamma = \frac{P\lambda}{hc} = 1.6 \times 10^{15} \text{ photons/sec.}$$

The number of electrons is just the current over the charge per electron:

$$N_e = \frac{I}{q_e} = 6.2 \times 10^{12} \text{ electrons/sec.}$$

Taking the ratio gives us the quantum efficiency:

$$\text{QE} = 0.0039 = 0.39\%.$$

This was using the correct value for h. If you used the value of h you obtained from part (a) of this problem, you would find

$$\text{QE} = 0.0025 = 0.25\%.$$

Problem 3

(a)

For thermal radiation, the average energy per photon is given by:

$$\langle E \rangle = \frac{1}{n} \int_0^\infty dE \cdot E \frac{dn}{dE} \tag{6}$$

The energy per unit volume u is given by:

$$u = n\langle E \rangle = \int_0^\infty dE \cdot E \frac{dn}{dE} \tag{7}$$

We can employ the fact that from Eq. (7), $\frac{du}{dE} = E\frac{dn}{dE}$, which gives us

$$\frac{du}{dE}\frac{dE}{d\lambda} = E\frac{dn}{dE}\frac{dE}{d\lambda} \tag{8}$$

This reduces to our desired result, simply that:

$$\frac{du}{d\lambda} = E\frac{dn}{d\lambda} = \frac{hc}{\lambda}\frac{dn}{d\lambda}. \tag{9}$$

(b)

The total photon density n is given by:

$$n = \int_0^\infty d\lambda \frac{dn}{d\lambda} = \int_0^\infty d\lambda \frac{\lambda}{hc}\frac{du}{d\lambda}, \tag{10}$$

which gives us:

$$n = \int_0^\infty d\lambda \left(\frac{\lambda}{hc}\right) \frac{8\pi hc}{\lambda^5(e^{hc/\lambda kT} - 1)} = 8\pi \int_0^\infty \frac{d\lambda}{\lambda^4(e^{hc/\lambda kT} - 1)}. \tag{11}$$

Making the appropriate substitution $x \equiv hc/\lambda kT$ and $dx = -\frac{hc}{kT}\frac{d\lambda}{\lambda^2}$, we get the desired result:

$$n = 8\pi\left(\frac{kT}{hc}\right)^3 \int_0^\infty dx \frac{x^2}{e^x - 1} \approx (3.17 \times 10^{19} \text{ eV}^{-3} \cdot \text{m}^{-3})(\text{kT})^3. \tag{12}$$

(c)

The average energy per photon is u/n. $u = 4R/c$ can be found from the Stefan-Boltzmann law (the factor of 4 comes from averaging over all angles, Rohlf Eqs. (3.17) and (3.18)),

$$u = 4R/c = \frac{4\sigma'(kT)^4}{c}. \tag{13}$$

Therefore, the average energy per photon is found from the ratio of Eqs. (13) and (12) to be:

$$\langle E \rangle = \frac{u}{n} = \frac{4\sigma'(kT)}{c \cdot (3.17 \times 10^{19} \text{ eV}^{-3} \cdot \text{m}^{-3})}. \tag{14}$$

Plugging in the constants gives us:

$$\langle E \rangle \approx 2.7\ kT. \tag{15}$$

(d)

For the number density of photons, we obtain from Eq. (12):

$$n = (3.17 \times 10^{19} \text{ eV}^{-3} \cdot \text{m}^{-3})[(8.62 \times 10^{-5} \text{ eV/K})(2.74 \text{ K})]^3 \approx 4 \times 10^8 \text{ m}^{-3}.$$

For the energy density, we from Eq. (15) we find that

$$\langle E \rangle \approx 0.64\text{meV}.$$

Problem 4

(a)

Acceleration a for a circular orbit is given by:

$$a = \frac{v^2}{r}. \tag{16}$$

Angular momentum L, applying the Bohr quantization condition, is given by:

$$L = mvr = n\hbar. \tag{17}$$

Solving for v from Eq. (17) and substituting into Eq. (16), we obtain:

$$a = \frac{L^2}{m^2r^3} = \frac{n^2\hbar^2}{m^2r^3}. \tag{18}$$

The Bohr radius r is given by:

$$r = (4\pi\epsilon_0)\frac{n^2\hbar^2}{me^2}. \tag{19}$$

Using Eq. (19) in conjunction with Eq. (18), we find an expression for a in terms of fundamental constants:

$$a = \frac{n^2\hbar^2}{m^2r^3} = \left(\frac{1}{4\pi\epsilon_0}\right)^3\frac{me^6}{n^4\hbar^4}. \tag{20}$$

Using Eq. (20) in the classical expression for radiated power gives us:

$$P = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}a^2 = \left(\frac{1}{4\pi\epsilon_0}\right)^7\frac{2}{3}\frac{m^2e^{14}}{c^3n^8\hbar^8}. \tag{21}$$

(b)

The energy E_n in the n^{th} level of the Bohr atom is given by:

$$E_n = -\frac{\alpha^2mc^2}{2n^2} = -\frac{-13.6\text{ eV}}{n^2}, \tag{22}$$

so the energy radiated in a $n \to n-1$ transition is:

$$\Delta E = -13.6\text{ eV}\left(\frac{1}{n^2} - \frac{1}{(n-1)^2}\right). \tag{23}$$

The decay rate γ is then the ratio of the radiated power (we assume that the electron is in the n^{th} orbit until the moment it decays) to the energy difference between the levels:

$$\gamma = \frac{P}{\Delta E}. \tag{24}$$

If we plug in our result from part (a) and use $n = 2$, we get

$$\gamma = 10^8\text{ s}^{-1}.$$

If the decay is from the n^{th} level to the $(n-m)^{\mathrm{th}}$ level, we merely adjust the energy difference in Eq. (23):

$$\Delta E = -13.6\text{ eV}\left(\frac{1}{n^2} - \frac{1}{(n-m)^2}\right). \tag{25}$$

and employ this equation in Eq. (24). Qualitatively, we see that if the energy difference is greater and the power radiated is the same, the decay rate will decrease. This is an example of the limitations of the Bohr model, since although it correctly predicts the order of magnitude of the transition rates it does not correctly predict the dependence of transition rates on the energy difference between levels, which actually scales as ω^3.

(c)

There is an energy time uncertainty principle, which can be derived from $\Delta x\Delta p \geq \hbar/2$ in the following hand-waving fashion:

$$\Delta E\Delta t = \left(\frac{p}{m}\Delta p\right)\left(\frac{m}{p}\Delta x\right) = \Delta x\Delta p.$$

We use the lifetime $(1/\gamma)$ as the uncertainty in time, and then find for the uncertainty in energy:

$$\Delta E = \frac{\gamma\hbar}{2} \tag{26}$$

The value of ΔE is $(10^8\text{ s}^{-1})\cdot(197.3\text{ MeV fm}\cdot(3\times10^{23}\text{fm/s})^{-1})$, or 6×10^{-8} eV. The energy difference between the first and second levels in hydrogen is 10 eV, so the linewidth is smaller than the energy difference by nine orders of magnitude!

Problem 5

In calculations involving the Bohr model, the electron mass is replaced by the reduced mass of the muon-proton system, which is near the mass of the muon:

$$m = \frac{m_\mu m_p}{m_\mu + m_p} = 95 \text{ MeV}.$$

Energy levels in the Bohr model are linear with respect to the electron mass, and are given by

$$E_n = -\frac{13.6 \text{ eV}}{n^2} \cdot \frac{95 \text{ MeV}}{0.5 \text{ MeV}} \tag{27}$$

for the muon-proton system.

(a)

A free muon decays with a characteristic lifetime 2.2×10^{-6} s, primarily in the mode:

$$\mu \to e^- + \bar{\nu}_e + \nu_\mu.$$

If the capture probability was large, it would make the lifetime shorter compared to the lifetime of a muon (at rest), since the two rates would add.

(b)

If the capture probability was small enough to be neglected, then the effect of time dilation would lengthen the lifetime of the muon (since in the muon-proton system, the muon has a characteristic velocity αc), compared to a muon at rest.

Problem 6

If $\mathcal{O}$ is divided by the characteristic impedance of free space Z_0, we get $(2\alpha)^{-1}$. So

$$\mathcal{O} = \frac{Z_0}{2\alpha},$$

probably.

Problem 7

(a)

The typical electron velocity in the Bohr model is $v = \alpha c$, for the deuteron we replace α by α_s. So we have $v = \alpha_s c = 3 \times 10^7$ m/s for both the proton and neutron.

(b)

The reduced mass in the deuteron is roughly $m_p/2$, so the nuclear "Bohr radius" r is given by:

$$r = \frac{2\hbar c}{m_p c^2 \alpha_s} \approx \frac{2 \cdot 197.3 \text{ MeV} \cdot \text{fm}}{0.1 \cdot 938 \text{ MeV}} = 4 \text{ fm}.$$

(c)

The binding energy of the deuteron is roughly $\frac{1}{2}\alpha_s^2 mc^2 = 2$ MeV.

Problem 8

First, we equate the relativistic centripetal force to the electrostatic force acting on the electron:

$$\frac{ke^2}{r^2} = \frac{\gamma m v^2}{r}, \tag{28}$$

and proceed to solve for the radius:

$$r = \frac{ke^2}{\gamma m v^2}. \tag{29}$$

We can also use the Bohr quantization condition

$$pr = n\hbar$$

in conjunction with the relativistic expression for the momentum

$$p = \gamma m v$$

to find the radius:

$$r = \frac{\hbar}{\gamma m v}. \tag{30}$$

Setting Eqs. (29) and (30) equal, we can solve for the velocity, and we obtain the desired result $v = \alpha c$.

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 9

1.
Rohlf 5.31.

2.
Rohlf 6.12.

3.
Rohlf 6.29.

4.
Rohlf 6.32.

5.
Rohlf 7.4.

6.
Rohlf 7.5.

7.
Show that the "conservation of probability" law

$$\frac{\partial}{\partial t}\big(\psi^*\psi\big) + \frac{\partial}{\partial x} j = 0, \quad \text{where}$$
$$\frac{\hbar}{2mi}\Big(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\Big) \equiv j,$$

and j is the *probability current* in one dimension x, holds if $\psi(x,t)$ is a solution of the Schrödinger equation with a potential $V(x)$, provided that $V(x)$ is real. Therefore absorptive potentials are *imaginary*.

8. A particle of mass m is bound in an infinitely deep one-dimensional potential well extending from $x = 0$ to $x = L$. At $t = 0$ it is described by a wavefunction of the form

$$u(x) \propto \sin(\pi x/L) + \sin(2\pi x/L).$$

(a.)
Normalize $u(x)$.
(b.)
When the particle's energy E is measured for the first time, what value(s) could be obtained?
(c.)
What is the expectation value $\langle\mathcal{H}\rangle$ of the particle's kinetic energy? (Cogent arguments can substitute for some algebra here, and are encouraged.)
(d.)
At what time t_0 is the probability for the particle to be located on the *right-hand* side of the well ($L/2 < x < L$) a maximum? Reasoning rather than detailed calculation is needed. Please supply it.

"When the rules of quantum mechanics were formulated in the 1920's they represented a revolutionary break with the past, and an enormous extrapolation from experience. Since they were something very new, they could not be derived from something old and incorrect, that is, classical physics. Instead they had to be formulated by guessing, intuition, and inspiration. Their ultimate justification was, and is, logical consistency and agreement with experiment."

- Prof. Eugene D. Commins, U.C. Berkeley.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

The uncertainty principle yields an estimate for the minimum momentum of a proton trapped in the nucleus:

$$\Delta p \approx \frac{\hbar}{2\Delta r}. \tag{1}$$

For the kinetic energy of the proton, we obtain:

$$E_k = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{8m(\Delta r)^2} = \frac{\hbar^2 c^2}{8mc^2(\Delta r)^2} = \frac{197.3\text{ MeV}\cdot\text{fm}}{8(938\text{ MeV})(2\text{ fm})^2}, \tag{2}$$

from which we find

$$E_k = 1.4\text{ MeV}$$

(b)

We'll suppose, for the sake of this "back of the envelope" calculation, that the kinetic energy found in part (a) can be set equal to the potential energy at maximum displacement in a classical harmonic oscillator:

$$E_k = \frac{1}{2}kx_0^2 \tag{3}$$

The magnitude of the restoring force at maximum displacement is given by kx_0. So we find:

$$F = \frac{2E_k}{x_0} = 1.4\text{ MeV/fm}.$$

The strength of the electric force is given by

$$F_e = \frac{ke^2}{x_0^2} = \frac{1.44\text{ MeV}\cdot\text{fm}}{(2\text{ fm})^2} = 0.36\text{ MeV/fm}.$$

(c)

The maximum acceleration of the proton is given by:

$$a_{\max} = \frac{F}{m} = \frac{kx_0c^2}{mc^2} = 1.3\times 10^{29}\text{ m/s}^2 \sim 10^{28}g. \tag{4}$$

Problem 2

It is convenient to write the differential cross section as $\frac{d\sigma}{d\cos\theta}$ instead of $\frac{d\sigma}{d\theta}$ because it makes integration over solid angles a little easier, since the integral always involves $\cos\theta$ and the differential solid angle contains the term $\sin\theta d\theta = -d(\cos\theta)$.

It is relatively straightforward to show that either method of solving for the total cross section gives the same result, since

$$\frac{d\sigma}{d\theta} = \frac{d\sigma}{d\cos\theta}\cdot\frac{d\cos\theta}{d\theta} = -\sin\theta\frac{d\sigma}{d\cos\theta}.$$

If we integrate over all angles, we obtain for the total cross section:

$$\sigma = \int_{-1}^{+1} d(\cos\theta)\frac{d\sigma}{d\cos\theta} = \int_0^{\pi} d\theta\sin\theta\frac{1}{\sin\theta}\frac{d\sigma}{d\theta}.$$

Problem 3

(a)

The maximum kinetic energy that can be transferred to a gold nucleus in a collision with a 6 MeV α-particle would be when the collision is head-on and the α-particle bounces straight back. Because the gold nucleus is very massive compared to the α-particle, the amount of kinetic energy transferred to the gold nucleus should be small, so roughly $v_i = -v_f$ where v_i and v_f are the initial and final velocities of the α-particle. Thus, $M_{\text{gold}}V = 2m_\alpha v_i$. Using this result in the equation for kinetic energy, we find:

$$\frac{1}{2}M_{\text{gold}}V^2 = \frac{1}{2}M_{\text{gold}}\left(\frac{2m_\alpha v_i}{M_{\text{gold}}}\right)^2 = \left(\frac{4m_\alpha}{M_{\text{gold}}}\right)\left(\frac{1}{2}m_\alpha v_i^2\right) = \frac{4\cdot 4}{197}\cdot 6\text{ MeV} \approx 0.49\text{ MeV}$$

(b)

If we go into the rest frame of the α-particle ($\mathcal{S}'$), we find that because the α-particle is very massive compared to the electron, the energy transferred to the α-particle (to the electron in the lab frame) is small. Therefore we can approximate that in $\mathcal{S}'$, for a head-on collision $m_\alpha V' = 2m_e v_i$, where v_i is the speed of the α-particle in the lab frame and V' is the recoil speed of the α-particle in $\mathcal{S}'$. Transforming into the lab frame, we find that the recoil speed of the electron $v_e = 2v_i$. So the kinetic energy transferred to the electron is

$$E_k = \frac{1}{2}m_e(2v_i)^2 = \frac{4m_e}{m_\alpha}\left(\frac{m_\alpha v_i^2}{2}\right) = \frac{4 \cdot (0.511 \text{ MeV})}{3730 \text{ MeV}} 6 \text{ MeV} \approx 3.3 \text{ keV}.$$

Problem 4

(a)

The relationship between differential scattering cross section $d\sigma$ and the impact parameter b is given by Rohlf (6.18):

$$d\sigma = 2\pi b db \tag{5}$$

The total scattering cross section is derived from this expression:

$$\sigma = 2\pi \int_{b_2}^{b_1} b db = \pi(b_1^2 - b_2^2). \tag{6}$$

Using Rohlf (6.40)

$$\left(\frac{kq_1q_2}{mv^2}\right)^2 \left(\frac{1+\cos\theta}{1-\cos\theta}\right) = b^2$$

we find that the total scattering cross section is given by:

$$\sigma = 2\pi\left(\frac{kq_1q_2}{mv^2}\right)^2 \left(\frac{\cos\theta_1 - \cos\theta_2}{(1-\cos\theta_1)(1-\cos\theta_2)}\right). \tag{7}$$

(b)

Integrating explicitly gives us the same result:

$$\sigma = \int_{\cos\theta_2}^{\cos\theta_1} d(\cos\theta) \frac{d\sigma}{d\cos\theta}$$

$$\sigma = 2\pi\left(\frac{kq_1q_2}{mv^2}\right)^2 \int_{\cos\theta_2}^{\cos\theta_1} d(\cos\theta) \frac{1}{(1-\cos\theta)^2}$$

$$\sigma = 2\pi\left(\frac{kq_1q_2}{mv^2}\right)^2 \left(\frac{\cos\theta_1 - \cos\theta_2}{(1-\cos\theta_1)(1-\cos\theta_2)}\right)$$

Problem 5

A particle is confined to the region $-L/2 < x < L/2$. As discussed in section, this means that any state (wavefunction) of the particle can be described as a superposition of eigenfunctions of the energy operator (the Hamiltonian). These eigenfunctions "span" the Hilbert space corresponding to our system (a Hilbert space is an infinite dimensional vector space which is a subspace of the vector space of all continuous complex functions). Since inside the infinite potential well the particle is free, our Hamiltonian $\mathcal{H}$ is given by:

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}.$$

Eigenfunctions of $\mathcal{H}$ are

$$\psi_n = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right)$$

$$\psi_m = \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi x}{L}\right)$$

where $n = 1, 3, 5...$ and $m = 2, 4, 6....$ They have the eigenvalues

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

where $n = 1, 2, 3....$ These eigenfunctions are orthonormal, meaning that

$$\int_{-L/2}^{L/2} \psi_i^*(x)\psi_j(x)dx = 0$$

if $i \neq j$, and

$$\int_{-L/2}^{L/2} \psi_i^*(x)\psi_i(x)dx = 1.$$

(a)

Assume the particle is in an eigenstate of energy. The probability that the particle is found in the region $0 < x < L/2$ is $1/2$ by symmetry. This is because the potential is symmetric about $x = 0$, so every eigenfunction is symmetric or antisymmetic about $x = 0$. The square of any eigenfunction is symmetric about $x = 0$.

It is clear that probability does not depend on n because all of the eigenfunctions are symmetric or antisymmetric.

(b)

The probability P_c that a particle in the ground state is in the central half of the box is given by the integral:

$$P_c \int_{-L/4}^{L/4} dx|\psi_1|^2 = \frac{2}{L}\int_{-L/4}^{L/4} dx \cos^2(\pi x/L). \tag{8}$$

From which we find:

$$P_c = \frac{2}{L}\left[\frac{x}{2} + \frac{\sin(2\pi x/L)}{(4\pi/L)}\right]_{x=-L/4}^{x=L/4} = 0.82.$$

The probability decreases with n, and at large n approaches the classical limit $P_c = 0.5$.

Problem 6

(a)

The average value (expectation value) of x^2 as a function of n is given by:

$$\langle x^2\rangle = \int_{-L/2}^{L/2} dx\psi_n^* \cdot x^2 \cdot \psi_n. \tag{9}$$

We use the eigenfunctions discussed in problem (5), solving first for even n. In this case $\langle x^2\rangle$ is given by:

$$\langle x^2\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2 \sin^2(n\pi x/L)dx.$$

Making the u-substitution $u = n\pi x/L$, we obtain:

$$\langle x^2\rangle = \frac{2L^2}{m^3\pi^3}\int_{-m\pi/2}^{m\pi/2} u^2 \sin^2 u du$$

$$\langle x^2\rangle = \frac{2L^2}{m^3\pi^3}\left[\frac{u^3}{6} - \frac{u^2\sin(2u)}{4} + \frac{\sin(2u)}{8} - \frac{4\cos(2u)}{4}\right]_{-m\pi/2}^{m\pi/2}.$$

$$\langle x^2\rangle = \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}.$$

For odd n, the procedure is pretty much the same... you even end up with the same result.

$$\langle x^2\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2 \cos^2(n\pi x/L)dx.$$

Making the u-substitution $u = n\pi x/L$, we obtain:

$$\langle x^2\rangle = \frac{2L^2}{m^3\pi^3}\int_{-m\pi/2}^{m\pi/2} u^2 \cos^2 u du$$

$$\langle x^2\rangle = \frac{2L^2}{m^3\pi^3}\left[\frac{u^3}{6} + \frac{u^2\sin(2u)}{4} - \frac{\sin(2u)}{8} - \frac{4\cos(2u)}{4}\right]_{-m\pi/2}^{m\pi/2}.$$

$$\langle x^2\rangle = \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}.$$

Taking the limit as $n \to \infty$, we see that the rms value of x approaches $L/\sqrt{12}$.

Problem 7

We intend to prove the conservation of probability law:

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\partial}{\partial x}j = 0, \text{ where}$$

$$\frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right) \equiv j,$$

and j is the *probability current* in one dimension x, where $\psi(x,t)$ is a solution of the Schrödinger equation with a real potential $V(x)$.

We can start with the time-dependent Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t). \tag{10}$$

Then take the complex conjugate of (10):

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi^*(x,t) = -i\hbar\frac{\partial}{\partial t}\psi^*(x,t). \tag{11}$$

Now multiply (10) by $\psi^*(x,t)$ and (11) by $\psi(x,t)$, then subtract the equations. We obtain:

$$\psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi - \psi\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi^* = i\hbar\left(\psi^*\frac{\partial}{\partial t}\psi + \psi\frac{\partial}{\partial t}\psi^*\right). \tag{12}$$

From which we can deduce:

$$-\frac{\hbar}{2mi}\left(\psi^*\frac{\partial^2}{\partial x^2}\psi - \psi\frac{\partial^2}{\partial x^2}\psi^*\right) = \frac{\partial(\psi\psi^*)}{\partial t}. \tag{13}$$

Now consider $\frac{\partial}{\partial x}j$:

$$\frac{\partial}{\partial x}j = \frac{\partial}{\partial x}\left(\frac{\hbar}{2mi}\right)\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) \tag{14}$$

$$\frac{\partial}{\partial x}j = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial^2}{\partial x^2}\psi - \psi\frac{\partial^2}{\partial x^2}\psi^*\right) \tag{15}$$

If we use Eq. (15) in Eq. (13), we obtain the conservation of probability law:

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\partial}{\partial x}j = 0. \tag{16}$$

Problem 8

There was a correction to this problem, specifically that the initial wavefunction of the particle in the box is supposed to be

$$u(x) \propto \sin(\pi x/L) + \sin(2\pi x/L)$$

instead of

$$u'(x) \propto \exp(i\pi x/L) + \exp(i2\pi x/L).$$

It is useful to consider the problem with $u'(x)$. If we solve for the energy eigenfunctions of the Hamiltonian for this problem

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2},$$

we obtain:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \tag{17}$$

which correspond to the energy eigenvalues:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}. \tag{18}$$

From the postulates of quantum mechanics (discussed in section), we know that any state of an isolated system corresponds to a function in the corresponding Hilbert space. This Hilbert space is spanned by the eigenfunctions of a Hermitian operator (which corresponds to an observable, in this case energy). Therefore, if $u'(x)$ were a state in our system, it could be represented as a superposition of different eigenfunctions of energy:

$$u'(x) = \sum_{n=1}^{\infty} c_n\psi_n(x). \tag{19}$$

From equation (17), it is clear that all of the ψ_n's vanish at zero, whereas $u'(x)$ does not. Thus $u'(x)$ is a wavefunction that extends beyond our Hilbert space, or in other words is a particle not confined in the infinite square well, which creates a dilemma... one which is easily solved by use of $u(x)$ as the initial state of the particle. $u(x)$, by the way, is the wavefunction that would be obtained if the potential suddenly (which can be quantitatively defined) sprung up from nowhere and captured a particle formerly in $u'(x)$.

(a)

The normalization condition is:

$$\int_0^L dx|u(x)|^2 = 1. \tag{20}$$

Let

$$u(x) = C(\sin(\pi x/L) + \sin(2\pi x/L)),$$

then normalization implies:

$$C^2\int_0^L dx\left(\sin^2(\pi x/L) + 2\sin(\pi x/L)\sin(2\pi x/L) + \sin^2(2\pi x/L)\right) = 1$$

Note that $u(x)$ is a superposition of the first two energy eigenfunctions (given by Eq. (17)). Since eigenfunctions are orthonormal (discussed in problem (5)), the normalization condition reduces to:

$$C^2\int_0^L dx\frac{L}{2}\left(|\psi_1|^2 + |\psi_2|^2\right)$$

from which we conclude

$$C = \sqrt{\frac{2}{L}}.$$

The same result can be obtained through explicit integration.

(b)

Since $u(x)$ is a superposition of the first two energy eigenfunctions, a measurement of the energy of the particle will yield either E_1 or E_2 (given in Eq. (18)), each with a 50% probability. A measurement of an observable will always yield an eigenvalue of the corresponding Hermitian operator. Prof. Strovink mentions that this is the first measurement. This is important, since from another postulate of quantum theory we know that after measuring the energy, the wavefunction of the particle is subsequently described by the energy eigenfunction corresponding to the eigenvalue of energy obtained in the measurement.

(c)

The expectation value of the energy is

$$\langle E \rangle = \frac{E_1 + E_2}{2}.$$

This follows from the fact that $u(x)$ is a superposition of the first two energy eigenfunctions with equal probability to be found in either state. Thus repeated measurements on identical systems will yield E_1 half the time and E_2 the other half.

(d)

There are some subtle and important points in this part of the problem. As you saw in problem (5), a particle in an energy eigenstate of a symmetric potential always has an equal probability to be found on either the left- or right-hand side of the potential. This is not true for a superposition of energy eigenstates. This is readily seen by evaluating the expectation value of x for $u(x)$:

$$\langle x \rangle = \int_0^L x|u(x)|^2 dx \tag{21}$$

$$\langle x \rangle = \frac{2}{L}\int_0^L \left[x\sin^2\left(\frac{\pi x}{L}\right) + x\sin^2\left(\frac{2\pi x}{L}\right) + 2x\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right] dx \tag{22}$$

$$\langle x \rangle = \frac{L}{2} - \frac{16L}{9\pi^2}.$$

If we include the time evolution of $u(x)$ as described by the time-dependent Schrödinger equation, the constants in front of ψ_1 and ψ_2 acquire a time-dependence:

$$u(x,t) = c_1(t)\psi_1(x) + c_2(t)\psi_2(x). \tag{23}$$

The phase between the two wavefunctions ψ_1 and ψ_2 oscillates at a frequency:

$$\omega = \frac{\Delta E}{\hbar}, \tag{24}$$

where $\Delta E = E_2 - E_1$. Since $u(x,0)$ has maximum probability to be found on the left-hand side, when

$$\frac{\Delta E \cdot t_0}{\hbar} = n\pi$$

$(n = 1, 3, 5...)$ the probability to be found on the right-hand side is a maximum. So

$$t_0 = \frac{n\pi\hbar}{\Delta E}.$$

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 10

1.
A particle of mass m in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega_0^2 x^2$ has possible definite-energy wavefunctions $u_n(x)$ with energies $E_n = \hbar\omega_0(n + \frac{1}{2})$, where n is zero or a positive integer. The particle is in thermal equilibrium with a bath at temperature T. Its probability of having total energy E (relative to the bottom of the well) is proportional to the Boltzmann factor $e^{-E/kT}$ where k is Boltzmann's constant.

Calculate the average energy $\bar{E}$ of the particle, true for any temperature. Then take the high temperature limit $kT >> \hbar\omega_0$ and show that $\bar{E}$ reduces to kT, the classical result.

2.
Consider the potential $\frac{1}{2}m\omega^2 x^2$ of the harmonic oscillator, where ω is a constant. Define the operator $A \equiv y + iq$, where $y \equiv x\sqrt{m\omega/2}$ and $q \equiv p/\sqrt{2m\omega}$. Use the fact that $\langle x\rangle$ and $\langle p\rangle$ are physical observables so that $x^\dagger = x$ and $p^\dagger = p$, where † denotes the Hermitian conjugate. Remember that $c^\dagger = c^*$ when c is a constant. $\mathcal{H} = \mathcal{H}^\dagger$ is the Hamiltonian. For this potential, prove that

$$\begin{aligned}
A^\dagger &= y - iq \\
[A, A^\dagger] &= \hbar \\
\mathcal{H} &= \omega(q^2 + y^2) \\
\mathcal{H} &= \omega(A^\dagger A + \hbar/2) \\
\mathcal{H} &= \omega(AA^\dagger - \hbar/2) \\
[\mathcal{H}, A^\dagger] &= \hbar\omega A^\dagger \\
[\mathcal{H}, A] &= -\hbar\omega A
\end{aligned}$$

(This is the formal basis for the assertion, proved in lecture, that a harmonic oscillator has energy levels $E_n = \hbar\omega(n+\frac{1}{2})$, where n is an integer ≥ 0.)

3.
Consider the angular momentum operator $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = (\hbar/i)\mathbf{r} \times \nabla$. For example, $L_x \equiv (\hbar/i)\big(y(\partial/\partial z) - z(\partial/\partial y)\big)$. Define $L_\pm \equiv L_x \pm iL_y$. Prove that

$$\begin{aligned}
[L_x, L_y] &= i\hbar L_z \\
[L_y, L_z] &= i\hbar L_x \\
[L_z, L_x] &= i\hbar L_y \\
[L_+, L_-] &= 2\hbar L_z \\
[L_-, L_z] &= \hbar L_- \\
[L_+, L_z] &= -\hbar L_+ \\
[L^2, L_z] &= 0 \\
[L^2, L_\pm] &= 0 \\
L^2 &= L_- L_+ + L_z^2 + \hbar L_z \\
L^2 &= L_+ L_- + L_z^2 - \hbar L_z
\end{aligned}$$

This is the formal basis for the assertion, proved in lecture, that angular momentum is quantized in integral or (for intrinsic [spin] angular momentum) *half-integral* units of $\hbar$.

4.
The spherical harmonic $Y_{lm}(\theta, \phi)$ is an eigenfunction of L^2 with eigenvalue $\hbar^2 l(l+1)$ and also of L_z with eigenvalue $\hbar m$. It is normalized so that

$$\int d\Omega\, Y_{lm}^*(\theta, \phi) Y_{lm}(\theta, \phi) = 1.$$

(a.)
Show formally that $L_+ Y_{ll} = 0$. (*Hint: evaluate* $\int d\Omega (L_+ Y_{ll})^*(L_+ Y_{ll})$.)
(b.)
In lecture it was proved that L_- is a lowering operator:

$$L_- Y_{lm} = C_-(l, m) Y_{l,m-1} \ ,$$

where $C_-(l, m)$ is a constant depending on l and m. Using the normality of the Y_{lm}'s, derive the value of $|C_-(l, m)|^2$.

5.
Consider the problem of an electron bound to an infinitely heavy nucleus, here using our modern understanding of orbital and spin angular momenta. Suppose that the nucleus is spinless, and

that the atom is in a state of definite *orbital* angular momentum $l = 2$. Moreover, the atom is in a state of definite projection $m_j = \frac{5}{2}$ of its *total* (spin + orbital) angular momentum on the z axis.

(a.)

What value(s) of total angular momentum quantum number j is (are) possible? Why?

(b.)

Define the expectation value of the cosine of the angle between the atom's orbital and spin angular momenta as follows:

$$\langle \cos\theta \rangle \equiv \frac{\langle \mathbf{L} \cdot \mathbf{S} \rangle}{\sqrt{\langle L^2 \rangle \langle S^2 \rangle}}.$$

Evaluate $\langle \cos\theta \rangle$ for this problem. (*Hint.* Consider $\langle J^2 \rangle$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$.)

6.

Rohlf 7.27.

7.

Rohlf 8.21.

8.

Rohlf 8.25.

"It's really quite straightforward, there's nothing mystical about it."

- Prof. Eugene D. Commins, U.C. Berkeley, on the subject of the Einstein-Podolsky-Rosen paradox.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

The average energy of a particle $\bar{E}$ in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega_0^2 x^2$ is given by:

$$\bar{E} = \frac{U}{N}, \tag{1}$$

where U is the total energy of N particles in identical potentials (this is the usual imaginary ensemble of identical quantum mechanical systems used to calculate expectation values, etc.). The total energy is given by:

$$U = \sum_{n=0}^{\infty} N_0 E_n e^{-E_n\beta} = \sum_{n=0}^{\infty} N_0\hbar\omega_0\left(n+\frac{1}{2}\right)e^{-n\beta\hbar\omega_0}, \tag{2}$$

where $\beta = 1/(kT)$ and the total number of particles is:

$$N = \sum_{n=0}^{\infty} N_0 e^{-n\beta\hbar\omega_0}. \tag{3}$$

The average energy is then given by the expression:

$$\frac{U}{N} = \frac{\sum_{n=0}^{\infty} N_0\hbar\omega_0 n e^{-n\beta\hbar\omega_0}}{\sum_{n=0}^{\infty} N_0 e^{-n\beta\hbar\omega_0}} + \frac{1}{2}\hbar\omega_0. \tag{4}$$

Cancelling out common factors of N_0 and noting that

$$(n\hbar\omega_0)e^{-\beta n\hbar\omega_0} = -\frac{\partial}{\partial\beta}e^{-\beta n\hbar\omega_0},$$

we can simplify Eq. (4) to:

$$\frac{U}{N} = \frac{-\frac{\partial}{\partial\beta}\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_0}}{\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_0}} + \frac{1}{2}\hbar\omega_0. \tag{5}$$

Next we use the fact that for a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Thus for the average energy, we have:

$$\bar{E} = \frac{-\frac{\partial}{\partial\beta}\left(\frac{1}{1-e^{-\beta\hbar\omega_0}}\right)}{\frac{1}{1-e^{-\beta\hbar\omega_0}}} + \frac{1}{2}\hbar\omega_0. \tag{6}$$

From which we find:

$$\bar{E} = \frac{\hbar\omega_0 e^{-\beta\hbar\omega_0}}{1-e^{-\beta\hbar\omega_0}} + \frac{1}{2}\hbar\omega_0 = \frac{\hbar\omega_0}{e^{\hbar\omega_0/kT}-1} + \frac{1}{2}\hbar\omega_0. \tag{7}$$

This is an expression for the average energy of the particle at any temperature. If we take the high temperature limit ($kT \gg \hbar\omega_0$), $e^{\beta\hbar\omega_0} \approx 1 + \beta\hbar\omega_0$, so

$$\boxed{\bar{E} \approx \frac{1}{\beta} = kT.}$$

Problem 2

Here, we consider the potential $\frac{1}{2}m\omega^2x^2$ of the harmonic oscillator, where ω is a constant. We define the operator $A \equiv y + iq$, where $y \equiv x\sqrt{m\omega/2}$ and $q \equiv p/\sqrt{2m\omega}$. Since $\langle x\rangle$ and $\langle p\rangle$ are physical observables, $x^\dagger = x$ and $p^\dagger = p$ and subsequently $y^\dagger = y$ and $q^\dagger = q$.

(a)

$$A^\dagger = (y+iq)^\dagger = y^\dagger - iq^\dagger = y - iq$$

(b)

$$\left[A, A^\dagger\right] = (y+iq)(y-iq) - (y-iq)(y+iq) = [y,y] + [y,-iq] + [iq,y] + [iq,-iq]$$

We know that y and q commute with themselves, and that

$$[y,q] = -[q,y].$$

Employing these results, we find that

$$\left[A, A^\dagger\right] = -2i[y,q].$$

From the definition of y and q,

$$[y,q] = \left[x\sqrt{\frac{m\omega}{2}}, \frac{p}{\sqrt{2m\omega}}\right] = \frac{1}{2}[x,p] = i\hbar/2.$$

From which we conclude:

$$\left[A, A^\dagger\right] = -2i[y,q] = \hbar.$$

(c)

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2$$

Since $p^2 = 2m\omega q^2$ and $x^2 = 2y^2/(m\omega)$, we find

$$\mathcal{H} = \omega(q^2 + y^2).$$

(d),(e) Note that from the definitions of A and $A^\dagger$, we have

$$y = \frac{A + A^\dagger}{2}$$

and

$$q = \frac{A - A^\dagger}{2i}.$$

Thus we find that

$$y^2 = \frac{1}{4}\left(A^2 + A^{\dagger 2} + AA^\dagger + A^\dagger A\right)$$

and

$$q^2 = \frac{1}{4}\left(-A^2 - A^{\dagger 2} + AA^\dagger + A^\dagger A\right)$$

From (c) and the above considerations we have that

$$\mathcal{H} = \omega(q^2 + y^2) = \frac{\omega}{2}\left(AA^\dagger + A^\dagger A\right).$$

By adding and subtracting $AA^\dagger$ or $A^\dagger A$ where appropriate,

$$AA^\dagger = AA^\dagger - A^\dagger A + A^\dagger A = \left[A, A^\dagger\right] + A^\dagger A = \hbar + A^\dagger A$$

and also

$$A^\dagger A = A^\dagger A - AA^\dagger + AA^\dagger = \left[A^\dagger, A\right] + AA^\dagger = -\hbar + AA^\dagger.$$

These expressions can be used in our above expression for $\mathcal{H}$, and from them we find

$$\mathcal{H} = \frac{\omega}{2}\left(AA^\dagger + A^\dagger A\right) = \omega\left(A^\dagger A + \hbar/2\right) = \omega\left(AA^\dagger - \hbar/2\right).$$

(f) Since constants commute with anything $\left[\hbar/2, A^\dagger\right] = 0$. Thus we get:

$$\left[\mathcal{H}, A^\dagger\right] = \left[\omega AA^\dagger, A^\dagger\right] = \omega\left(AA^\dagger A^\dagger - A^\dagger AA^\dagger\right) = \omega\left[A, A^\dagger\right]A^\dagger = \hbar\omega A^\dagger.$$

(g) Similarly,

$$[\mathcal{H}, A] = \left[\omega AA^\dagger, A\right] = \omega\left(AA^\dagger A - AAA^\dagger\right) = \omega A\left[A^\dagger, A\right] = -\hbar\omega A.$$

Problem 3

Consider the angular momentum operator $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = (\hbar/i)\mathbf{r} \times \nabla$. For example, $L_x \equiv (\hbar/i)\big(y(\partial/\partial z) - z(\partial/\partial y)\big)$. Define $L_\pm \equiv L_x \pm iL_y$.

(a)

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] + [zp_y, xp_z] - [zp_y, zp_x] - [yp_z, xp_z] \tag{8}$$

Recall that in figuring out these commutation relations, it often helps to think of the commutators as operators acting on functions. This is especially helpful in dealing with commutators involving derivatives. We'll look at each of the terms in the above expression individually:

$$[yp_z, zp_x] = -\hbar^2 y\frac{\partial}{\partial z}z\frac{\partial}{\partial x} + \hbar^2 z\frac{\partial}{\partial x}y\frac{\partial}{\partial z}$$

$$[yp_z, zp_x] = -\hbar^2 yz\frac{\partial^2}{\partial z\partial x} - \hbar^2 y\frac{\partial}{\partial x} + \hbar^2 yz\frac{\partial^2}{\partial x\partial z} = -\hbar^2 y\frac{\partial}{\partial x}$$

$$\boxed{[yp_z, zp_x] = -\hbar^2 y\frac{\partial}{\partial x}}$$

$$[zp_y, xp_z] = -\hbar^2 z\frac{\partial}{\partial y}x\frac{\partial}{\partial z} + \hbar^2 x\frac{\partial}{\partial z}z\frac{\partial}{\partial y}$$

$$[zp_y, xp_z] = -\hbar^2 zx\frac{\partial^2}{\partial y\partial z} + \hbar^2 x\frac{\partial}{\partial y} + \hbar^2 xz\frac{\partial^2}{\partial z\partial y}$$

$$\boxed{[zp_y, xp_z] = \hbar^2 x\frac{\partial}{\partial y}}$$

$$[zp_y, zp_x] = -\hbar^2 z\frac{\partial}{\partial y}z\frac{\partial}{\partial x} + \hbar^2 z\frac{\partial}{\partial x}z\frac{\partial}{\partial y}$$

$$[zp_y, zp_x] = -\hbar^2 z^2\frac{\partial^2}{\partial y\partial x} + \hbar^2 z^2\frac{\partial^2}{\partial x\partial y}$$

$$\boxed{[zp_y, zp_x] = 0}$$

$$[yp_z, xp_z] = -\hbar^2 y\frac{\partial}{\partial z}x\frac{\partial}{\partial z} + \hbar^2 x\frac{\partial}{\partial z}y\frac{\partial}{\partial z}$$

$$[yp_z, xp_z] = -\hbar^2 yx\frac{\partial^2}{\partial z^2} + \hbar^2 xy\frac{\partial^2}{\partial z^2}$$

$$\boxed{[yp_z, xp_z] = 0}$$

Now we can put these simplified expressions into Eq. (8), and we find:

$$[L_x, L_y] = \hbar^2\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = i\hbar\left(x\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - y\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\right)$$

$$[L_x, L_y] = i\hbar(xp_y - yp_x)$$

$$\boxed{[L_x, L_y] = i\hbar L_z}$$

(b),(c) The arguments used in (a) can be basically repeated, just changing the identities of some of the variables. Or you can argue that since space is rotationally invariant, if we rotate our coordinate system in such a way that $x \to y$, $y \to z$ and $z \to x$, the same commutation relation holds with the appropriate change of coordinate names. The basic principle is that for any (i, j, k) which are a cyclic permutation of (x, y, z), we have:

$$[L_i, L_j] = i\hbar L_k.$$

We know that in general if we interchange two operators in a commutator, the result of the commutator acquires a negative sign. In other words, for any two operators A, B:

$$[A, B] = -[B, A].$$

Thus if (i, j, k) are an anti-cyclic permutation of (x, y, z) (e.g., (y, x, z)), we have:

$$[L_i, L_j] = -i\hbar L_k.$$

(d)

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] = [L_x, L_x] + i[L_y, L_x] - i[L_x, L_y] + [L_y, L_y]$$

An operator always commutes with itself, so

$$[L_x, L_x] = 0, \quad [L_y, L_y] = 0.$$

Thus we have

$$[L_+, L_-] = -2i[L_x, L_y] = -2i(i\hbar L_z)$$

$$\boxed{[L_+, L_-] = 2\hbar L_z}$$

(e)

$$[L_-, L_z] = [L_x - iL_y, L_z] = [L_x, L_z] - i[L_y, L_z]$$

$$[L_-, L_z] = -i\hbar L_y - i(i\hbar L_x) = \hbar(L_x - iL_y)$$

$$\boxed{[L_-, L_z] = \hbar L_-}$$

(f)

$$[L_+, L_z] = [L_x + iL_y, L_z] = [L_x, L_z] + i[L_y, L_z]$$

$$[L_+, L_z] = -i\hbar L_y + i(i\hbar L_x) = -\hbar(L_x + iL_y)$$

$$\boxed{[L_+, L_z] = -\hbar L_+}$$

(g)

$$\left[L^2, L_z\right] = \left[L_x^2, L_z\right] + \left[L_y^2, L_z\right] + \left[L_z^2, L_z\right] = \left[L_x^2, L_z\right] + \left[L_y^2, L_z\right]$$

$$\left[L^2, L_z\right] = L_xL_xL_z - L_zL_xL_x + L_yL_yL_z - L_zL_yL_y$$

Now we employ a common trick in calculating commutation relations. We add and subtract terms which allow us to substitute commutators we know into our expression.

$$\left[L^2, L_z\right] = L_xL_xL_z - L_xL_zL_x + L_xL_zL_x - L_zL_xL_x + L_yL_yL_z - L_yL_zL_y + L_yL_zL_y - L_zL_yL_y$$

$$\left[L^2, L_z\right] = L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$

Maybe, if you are like me, when you first see this trick it seems quite clever. This trick is used very often because of the potential non-commutativity of operators. Suppose we have two operators A, B which don't commute. If we have AB and want to get BA for some reason, we can use

$$AB = AB - BA + BA = [A, B] + BA.$$

This is a very useful operator identity! Anyhow, continuing on with the math by substituting in results from (a),(b) and (c) of this problem:

$$\left[L^2, L_z\right] = -i\hbar L_xL_y - i\hbar L_yL_x + i\hbar L_yL_x + i\hbar L_xL_y$$

$$\boxed{\left[L^2, L_z\right] = 0}$$

This relation proves it is possible to find simultaneous eigenfunctions of both operators. This is an important result, so it probably won't hurt to see why this is the case once more. Suppose we have operators A, B where $[A, B] = 0$. Consider an eigenfunction of B, ψ_b, with eigenvalue λ_b. Is $A\psi_b$ an eigenfunction of B? It is, since

$$BA\psi_b = AB\psi_b = A(\lambda_b\psi_b) = \lambda_b(A\psi_b).$$

As you can see, this result relies on the fact that A and B commute. Thus measurement of one observable does not affect measurement of the other observable. This shows that there is no fundamental quantum uncertainty in measurement of observables which correspond to commuting operators. So we can find a function $\psi_a b$ which is is an eigenfunction of both A and B with eigenvalues λ_a and λ_b, as we do for L^2 and L_z.

(h) We can use arguments analagous to those above to show that:

$$\left[L^2, L_x\right] = 0, \quad \left[L^2, L_y\right] = 0.$$

If you want to avoid the math, you can just use the isotropy of space to claim that L^2 should not preferentially commute with a particular direction in space. It immediately follows that

$$\boxed{\left[L^2, L_\pm\right] = 0}$$

(i),(j) First, let's consider L_+L_- and L_-L_+:

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + iL_yL_x - iL_xL_y + L_y^2$$

$$L_+L_- = L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 + \hbar L_z$$

To go from L_+L_- to L_-L_+ we use the identity

$$L_+L_- = [L_+, L_-] + L_-L_+ = L_-L_+ + 2\hbar L_z.$$

Since

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

and

$$L_+L_- = L_x^2 + L_y^2 + \hbar L_z, \quad L_-L_+ = L_x^2 + L_y^2 - \hbar L_z$$

we have:

$$\boxed{L^2 = L_-L_+ + L_z^2 + \hbar L_z}$$

$$\boxed{L^2 = L_+L_- + L_z^2 - \hbar L_z}$$

Problem 4

The spherical harmonic $Y_{lm}(\theta, \phi)$ is an eigenfunction of L^2 with eigenvalue $\hbar^2 l(l+1)$ and also of L_z with eigenvalue $\hbar m$. It is normalized so that

$$\int d\Omega\, Y_{lm}^*(\theta, \phi) Y_{lm}(\theta, \phi) = 1.$$

(a)

Consider the integral:

$$\int d\Omega (L_+Y_{ll})^*(L_+Y_{ll}) = \int d\Omega Y_{ll}^*(L_-L_+)Y_{ll}$$

where we make use of the fact that:

$$(L_+Y_{ll})^* = Y_{ll}^* L_+^\dagger = Y_{ll}^* L_-$$

From problem 3 parts (i),(j) we have:

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z.$$

Conveniently, Y_{ll} is an eigenfunction of L^2 and L_z with eigenvalues $\hbar^2 l(l+1)$ and $\hbar l$ respectively. Thus we obtain

$$\int d\Omega (L_+Y_{ll})^*(L_+Y_{ll}) = -\int d\Omega Y_{ll}^* \hbar^2 \big(l(l+1) - l^2 - l\big) Y_{ll} = 0.$$

Therefore, it must be that

$$\boxed{L_+Y_{ll} = 0}$$

(b)

We can use quite similar methods to find $C_-(l, m)$. Consider the integral:

$$\int d\Omega (L_-Y_{lm})^*(L_-Y_{lm}) = \int d\Omega Y_{lm}^*(L_+L_-)Y_{lm}$$

From problem 3 parts (i),(j) we have:

$$L_+L_- = L^2 - L_z^2 + \hbar L_z.$$

Y_{lm} is an eigenfunction of L^2 and L_z with eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$ respectively. Thus we have the integral

$$\int d\Omega (L_-Y_{lm})^*(L_-Y_{lm}) = \int d\Omega Y_{lm}^* \hbar^2 \big(l(l+1) - m^2 + m\big) Y_{lm}$$

Because the Y_{lm}'s are orthonormal, we have that

$$\boxed{|C_-(l,m)|^2 = \hbar^2\big(l(l+1) - m^2 + m\big)}$$

Problem 5

(a)

Since the atom is in a state of definite projection $m_j = 5/2$ of its *total* (spin + orbital) angular momentum on the z axis, $j \geq 5/2$. This follows from the fact that possible values for m_j range between $+j$ and $-j$. What values of j are possible for a one electron atom with orbital angular momentum $l = 2$? The values of j range between $l + s$ and $|l - s|$ where s is the electron spin. Therefore, in general $j = 5/2, 3/2$ are possible, but since the atom is in the $m_j = 5/2$ state we know $j = 5/2$.

(b)

Consider

$$J^2 = (\mathbf{L} + \mathbf{S})^2 = L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S}.$$

Since the atom is in an eigenstate of J^2, L^2 and S^2, we have that:

$$\mathbf{L}\cdot\mathbf{S} = \frac{\hbar^2}{2}(j(j+1) - l(l+1) - s(s+1)).$$

With $j = 5/2$, $l = 2$, and $s = 1/2$, we obtain $\mathbf{L}\cdot\mathbf{S} = \hbar^2$. Also,

$$\sqrt{\langle L^2\rangle\langle S^2\rangle} = \hbar^2\sqrt{l(l+1)\cdot s(s+1)} = \hbar^2\frac{3}{\sqrt{2}}.$$

So we find,

$$\boxed{\langle\cos\theta\rangle = \frac{\sqrt{2}}{3}}$$

Problem 6 *Rohlf 7.27*

The energies for the particle in the 3D box are given by:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2mL^2}\big(n_1^2 + n_2^2 + 4n_3^2\big).$$

The following table shows the first 5 unique energies (in units of $\frac{\hbar^2\pi^2}{2mL^2}$) and the quantum numbers of the states that possess them.

$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{n_3}$	**Energy**
1	1	1	6
1	2	1	9
2	1	1	9
2	2	1	12
1	3	1	14
3	1	1	14
2	3	1	17
3	2	1	17

Problem 7 *Rohlf 8.21*

(a)

Of course, first we specify that we know l and s. The possible values for j are $j = l+s, l+s-1, ..., |l-s|$. We choose one particular value of j. Then there are $2j+1$ states with total angular momentum j. There are many ways to show this result. You could start with a "stretched" state ($m_j = \pm j$) and use the raising or lowering operator, for example. But if you simply note that m_j can take on the possible values

$$m_j = j, j-1, ..., -j+1, -j$$

and count these states, we see immediately that the number of states is $2j+1$.

(b)

The quantum numbers of the $n = 2$ states of hydrogen in terms n, l, m_l and m_s are shown in the following table:

n	**l**	$\mathbf{m_l}$	$\mathbf{m_s}$
2	1	1	1/2
2	1	1	-1/2
2	1	0	1/2
2	1	0	-1/2
2	1	-1	1/2
2	1	-1	-1/2
2	0	0	1/2
2	0	0	-1/2

Note there are 8 states in total in the table.
The quantum numbers of the $n = 2$ states of hydrogen in terms n, l, j and m_j are shown in the following table:

n	**l**	**j**	$\mathbf{m_j}$
2	1	3/2	3/2
2	1	3/2	1/2
2	1	3/2	-1/2
2	1	3/2	-3/2
2	1	1/2	1/2
2	1	1/2	-1/2
2	0	1/2	1/2
2	0	1/2	-1/2

Here there are also 8 states. The system we are considering is described by an 8D Hilbert space, so any complete, orthonormal set of eigenfunctions which span the space must consist of 8 states.

(c)

Since $J_z = L_z + S_z$, for the state $m_j = 3/2$, $l = 1$, $m_l = 1$ and $m_s = 1/2$. This is the "stretched" state, and we can readily convert from the n, l, j and m_j basis to the n, l, m_l and m_s basis by making the correspondence between the stretched states and employing the raising and lowering operators. Note also that $J_\pm = L_\pm + S_\pm$. Such transformations are used quite often and are tabulated (these are the famed Clebsch-Gordan coefficients).

(d)

If $m_j = 1/2$, then we can have

$$(l, m_l, m_s) = (0, 0, 1/2)$$

$$(l, m_l, m_s) = (1, 0, 1/2)$$

$$(l, m_l, m_s) = (1, 1, -1/2).$$

Problem 8 *Rohlf 8.25*

(a)

The magnetic dipole moment $\vec{\mu}$ of a hydrogen atom, in the limit of a strong $\vec{B}$-field, is given by:

$$\vec{\mu} = -\frac{e}{2m}(\mathbf{L} + 2\mathbf{S})$$

and the energy shift ΔE due to the external field is given by

$$\Delta E = -\vec{\mu} \cdot \vec{B}.$$

So we get energy shifts proportional to $m_l + 2m_s$.
For hydrogen in the n=3 state with a strong $\vec{B}$-field, we have the following possible values for the angular momentum quantum numbers and energy shifts in units of $e\hbar B/(2m)$:

$\mathbf{m_l}$	$\mathbf{m_s}$	$\mathbf{\Delta E}$
2	1/2	3
2	-1/2	1
1	1/2	2
1	-1/2	0
0	1/2	1
0	-1/2	-1
-1	1/2	0
-1	-1/2	-2
-2	1/2	-1
-2	-1/2	-3

(b)

In the absence of a magnetic field the energy separation ΔE_0 of the $3p$ and $1s$ states is

$$\Delta E_0 = 13.6 \text{ eV} - \frac{13.6}{9} \text{ eV} = 12.1 \text{ eV}.$$

The electric dipole transition selection rules demand that the difference in the projection of the orbital angular momentum on the z-axis bewteen the initial and final states of an atomic transition must obey

$$\Delta m_l = 1, 0, -1.$$

So the energies of the photons E_γ can be

$$E_\gamma = 12.1 \text{ eV} \pm \frac{e\hbar B}{2m},\ 12.1 \text{ eV}.$$

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 11

1.
Rohlf 8.41.

2.
Rohlf 9.3.

3.
Rohlf 9.10.

4.
Rohlf 9.25.

5.
Rohlf 9.31.

For those of you itching to do more, here is a preview of the first three problems in Problem Set 12:

1.
Rohlf 12.5.

2.
N electrons each of mass m are confined within a (formerly) cubic infinite potential well that has been "squashed" almost flat: $V = 0$ for ($0 < x < L$ and $0 < y < L$ and $0 < z < \epsilon L$), $V = \infty$ otherwise. Here $\epsilon \ll 1$ (cube is "squashed" in the z direction) and $N \gg 1$. The electrons do not interact with each other and are at very low temperature so that they fill up the available states in order of increasing energy. Take $\epsilon N \ll 1$, so that the z part of each electron's wavefunction may be assumed to be the same (lowest possible k_z). Thus the problem is *reduced to two dimensions*. Calculate the difference Δ between the energy of the most energetic electron (Fermi energy) and the energy of a ground state electron, using the approximation $N \gg 1$. Δ should depend on m, N, and L, but not ϵ.

3.
Write an integral equation for the fraction $\mathcal{F}$ of nonrelativistic fermions in a gas at *finite* temperature T which have energy *above* the Fermi energy E_F. The density of states is proportional to $E^{1/2}$ and the probability that a state is occupied is

$$\frac{1}{\exp\big(\beta(E - E_F)\big) + 1}$$

where $\beta = (kT)^{-1}$. You don't need to perform the integration, but you should set up the integral so that doing it would yield the correct answer without any additional physical reasoning.

"Civilization as we know it is based on ten ideas. These are Newton's three laws, the three laws of thermodynamics and Maxwell's equations. Everything that you see around you which differentiates modern times from the past is based on these concepts. Soon, Schrodinger's equation will join these ten ideas. People will tell you that civilization is about art, or literature, or architecture... but these components of civilization have more or less been the same for thousands of years. The only real difference between today and thousands of years ago is physics. If civilization collapsed tomorrow, we could rebuild it in the same fashion armed with these ideas."

- Prof. Seamus C. Davis, U.C. Berkeley.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1 *Rohlf 8.41*

Here we wish to show that the average value of $1/r$ is independent of the orbital angular momentum for the hydrogen atom. This result is straightforward if we introduce the virial theorem (see, e.g., B.H. Bransden and C.J. Joachain, **Introduction to Quantum Mechanics**, pgs. 227-228), which states (in one particular form) that for a spherically symmetric potential $V(r) \propto r^n$ one has for a stationary state:

$$2\langle T\rangle = n\langle V(r)\rangle,$$

where T is the kinetic energy. This is analagous to a classical result of the same name. The average potential energy for hydrogen is proportional to $\langle 1/r\rangle$, as is the kinetic energy and hence the total energy from the virial theorem. We know that the total energy for the hydrogen atom (from the Bohr model) is independent of l and depends only on the principal quantum number n. Consequently, $\langle 1/r\rangle$ is also independent of the orbital angular momentum.

Problem 2 *Rohlf 9.3*

We seek a totally antisymmetric wavefunction Ψ for 3 electrons in terms of $\psi_a(\mathbf{r}_1)$, $\psi_b(\mathbf{r}_2)$ and $\psi_c(\mathbf{r}_3)$. The wavefunction must be totally antisymmetric because we have three identical fermions. Such a wavefunction is given below. It is easily verified that under particle interchange it flips sign.

$$\begin{aligned}\Psi = \frac{1}{\sqrt{3!}}[&\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)\psi_c(\mathbf{r}_3) - \psi_a(\mathbf{r}_1)\psi_c(\mathbf{r}_2)\psi_b(\mathbf{r}_3) + \psi_b(\mathbf{r}_1)\psi_c(\mathbf{r}_2)\psi_a(\mathbf{r}_3)\\ &-\psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\psi_c(\mathbf{r}_3) + \psi_c(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_3) - \psi_c(\mathbf{r}_1)\psi_b(\mathbf{r}_2)\psi_a(\mathbf{r}_3)]\end{aligned} \tag{1}$$

Problem 3 *Rohlf 9.10*

An atom has two electrons in the d-subshell. What are the possible values of total z-angular momentum?

Well, we know that the total orbital angular momentum $l_{\rm tot}$ can range from l_1+l_2 to $|l_1-l_2|$. Thus $l_{\rm tot} = 4,3,2,1,0$. If $l_{\rm tot}$ is even, then the spatial part of the wavefunction $\Psi_{\rm spatial}$ is symmetric, and if $l_{\rm tot}$ is odd, then $\Psi_{\rm spatial}$ is antisymmetric.

With two electrons, the total spin can be $s_{\rm tot} = s_1+s_2 = 1,0$. If $s_{\rm tot}=1$ then the spin function $\Xi_{\rm spin}$ is symmetric. The total wavefunction $\Phi_{\rm total} = \Psi_{\rm spatial}\cdot\Xi_{\rm spin}$ must be antisymmetric since we are dealing with identical fermions. Therefore if $s_{\rm tot}=1$ then $\Psi_{\rm spatial}$ must be antisymmetric, meaning that $l_{\rm tot}$ is odd (1 or 3). In this case the total angular momentum j can take on the values $j=4,3,2,1,0$.

If $s_{\rm tot}=0$ then the spin function $\Xi_{\rm spin}$ is antisymmetric, and $\Psi_{\rm spatial}$ must be symmetric. In this case $l_{\rm tot}$ is even (0, 2 or 4). The possible values of j are 0, 2 or 4.

The largest j value possible is 4, so the possible m_j values are:

$$\boxed{m_j = -4,-3,-2,-1,0,1,2,3,4}$$

Problem 4 *Rohlf 9.25*

(a)

Sodium atoms are placed in a magnetic field of 1.5 T. The Zeeman splitting of the ground state ($n=0$, $l=0$) is given by the shift of energy due to the different spin states the single valence electron can have. Energy shifts are

$$E_\pm = \pm\mu_B B$$

where μ_B is the Bohr magneton. The numerical value for the splitting is given by

$$\Delta E = 2\mu_B B = 2(6\times10^{-5}\ \text{ev/T})(1.5\ \text{T}) = 1.8\times10^{-4}\ \text{eV}.$$

(b)

If 1/3 of the sodium atoms are in the higher energy state, then the Boltzmann factor

$$e^{-\Delta E/(kT)} = \frac{1/3}{2/3} = 1/2.$$

From which we calculate

$$kT = \frac{\Delta E}{\ln 2} = 2.5 \times 10^{-4} \text{ eV}.$$

Hence,

$$\boxed{T \approx 2.9 \text{ K}}$$

(c)

Same as above:

$$e^{-\Delta E/(kT)} = 49/51.$$

From which we calculate

$$kT = 4.4 \times 10^{-3} \text{ eV}$$

and

$$\boxed{T \approx 51 \text{ K}}$$

Problem 5 *Rohlf 9.31*

A sample of Na atoms are placed in a 1.0 T magnetic field. We calculate the energy shifts for the $3s_{1/2}$, $3p_{1/2}$ and $3p_{3/2}$ states. Note that in this case $\mu_B B = 6 \times 10^{-5}$ eV/T.

For the $3s_{1/2}$ state ($s = 1/2$, $l = 0$, and $j = 1/2$), the Lande factor is given by:

$$g_L = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1) = 2}.$$

Therefore the energy shifts, given by

$$\Delta E = \mu_z B_z = g_L \mu_B B m_j$$

are

$$\Delta E = \pm \mu_B B.$$

Similarly, for the $3p_{1/2}$ state, $g_L = 2/3$ so

$$\Delta E = \pm \frac{1}{3} \mu_B B.$$

For the $3p_{3/2}$ states $g_L = 4/3$ so

$$\Delta E = \pm \frac{2}{3} \mu_B B, \quad \pm 2\mu_B B.$$

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 12

1.
Rohlf 12.5.

2.
N electrons each of mass m are confined within a (formerly) cubic infinite potential well that has been "squashed" almost flat: $V = 0$ for ($0 < x < L$ and $0 < y < L$ and $0 < z < \epsilon L$), $V = \infty$ otherwise. Here $\epsilon \ll 1$ (cube is "squashed" in the z direction) and $N \gg 1$. The electrons do not interact with each other and are at very low temperature so that they fill up the available states in order of increasing energy. Take $\epsilon N \ll 1$, so that the z part of each electron's wavefunction may be assumed to be the same (lowest possible k_z). Thus the problem is *reduced to two dimensions*. Calculate the difference Δ between the energy of the most energetic electron (Fermi energy) and the energy of a ground state electron, using the approximation $N \gg 1$. Δ should depend on m, N, and L, but not ϵ.

3.
Write an integral equation for the fraction $\mathcal{F}$ of nonrelativistic fermions in a gas at *finite* temperature T which have energy *above* the Fermi energy E_F. The density of states is proportional to $E^{1/2}$ and the probability that a state is occupied is

$$\frac{1}{\exp\big(\beta(E - E_F)\big) + 1}$$

where $\beta = (kT)^{-1}$. You don't need to perform the integration, but you should set up the integral so that doing it would yield the correct answer without any additional physical reasoning.

4.
Rohlf 12.16.

5.
Rohlf 17.27.

6.
Rohlf 18.11.

7.
Rohlf 19.18.

8.
Rohlf 19.29.

"Entropy in the universe is always increasing. At some point, the universe will reach its maximum state of entropy and then no work can be done. The universe will become a cold, lifeless place. This is known as the heat death of the universe. Get ready, it's coming..."

- Prof. Seamus C. Davis, U.C. Berkeley.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1 *Rohlf 12.5*

Consider a system of 6 spin-1/2 fermions having a total energy of 10 units. The fermions are in a quantum mechanical system where the ground state has 0 energy units, the first excited state has 1 energy unit, the second excited state has 2 energy units, etc. Determine the energy distribution function df/dE. Make an estimate of the Fermi energy.

We make a table of the possible distributions and the spin degeneracy of each state (i.e., if there is an isolated electron, it can be either spin up or spin down). We then total up number of times an electron is found in a state, and from this total we arrive at our distribution function f, which describes the probability to find an electron in a particular energy level. The total number of times we find an electron in an energy level for a distinct distribution (including spin degeneracy) is shown on the bottom line.

Degen.	**Energy**						
	0	**1**	**2**	**3**	**4**	**5**	**6**
4	2	2	1	0	0	0	1
4	2	2	0	1	0	1	0
1	2	2	0	0	2	0	0
16	2	1	1	1	1	0	0
4	2	1	2	0	0	1	0
1	2	0	2	2	0	0	0
4	1	2	1	2	0	0	0
4	1	2	2	0	1	0	0
TOTAL	68	54	42	30	22	8	4

f, which in this case is discrete, is just the ratio of the various total number of times a particle is found in an energy level in one of the distributions to the ground state ($E = 0$) number. So $f(E)$ is a discrete function described by the following relations:

$$f(0) = 1$$

$$f(1) = 0.79$$

$$f(2) = 0.62$$

$$f(3) = 0.44$$

$$f(4) = 0.32$$

$$f(5) = 0.12$$

$$f(6) = 0.6$$

$$f(E > 6) = 0$$

The Fermi energy E_F is where this distribution has $f(E_F) \approx 1/2$. This is seen from the Fermi-Dirac distribution:

$$f_{FD}(E) = \frac{1}{e^{(E-E_F)/(kT)} + 1}$$

$$f_{FD}(E_F) = 1/2.$$

This is around $E = 3$ for this system. It doesn't work out exactly because this is a discrete system with a small number of possible distributions. If the number of particles was greatly increased, the distribution would become increasingly well-described by f_{FD}, which is derived in the large N limit.

Problem 2

The energy of the 3D infinite square well in this case is given by:

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + \frac{n_z^2}{\epsilon^2}\right).$$

ϵ is small, so since the number of particles is chosen to be small ($\epsilon N \ll 1$), the temperature must be large in order to excite states with $n_z \neq 1$. We consider the case of low temperature, where always $n_z = 1$. In this case, the problem is effectively 2D, with the energies:

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + \frac{1}{\epsilon^2}\right).$$

We seek the difference Δ between the ground state energy and the Fermi energy, both of which have a term

$$\frac{\hbar^2\pi^2}{2mL^2\epsilon^2}$$

which will cancel out.

First, let's calculate the Fermi energy. The Fermi-Dirac distribution at $T = 0$ is given by

$$f_{FD} = 1 \quad E < E_F$$
$$f_{FD} = 0 \quad E > E_F.$$

The electrons will try to achieve the lowest energy possible, filling up states in accordance with the Pauli exclusion principle. The total number of particles is given by:

$$N = \int_0^\infty 2\frac{dN}{dE} f_{FD}(E)dE$$

where the 2 is for spin degeneracy of the electrons. At $T = 0$, this integral becomes:

$$N = \int_0^{E_F} 2\frac{dN}{dE}dE.$$

Now we must determine the density of states for 2D. The energy of a state (neglecting the common factor $\frac{\hbar^2\pi^2}{2mL^2\epsilon^2}$) is $\propto N^2$ where $N^2 = n_x^2 + n_y^2$ and is the square of the total number of available states. The derivative of N^2 with respect to energy is

$$\frac{d}{dE}N^2 = \frac{1}{4}2\pi N\frac{dN}{dE}$$

where the factor of $1/4$ arises because we consider only positive n_x, n_y and the 2π is from the integration about a ring of thickness dN in n-space. From this we can explicitly solve for $\frac{dN}{dE}$:

$$\frac{dN}{dE} = \frac{4mL^2}{\hbar^2\pi^3}\frac{1}{N} = \frac{2\sqrt{2m}L}{\hbar\pi^2}E^{-1/2}.$$

Now we employ this expression for $\frac{dN}{dE}$ in our integral for the total number of particles:

$$N = \int_0^{E_F} 2\frac{2\sqrt{2m}L}{\hbar\pi^2}E^{-1/2}dE = 8\frac{\sqrt{2m}L}{\hbar\pi^2}E_F^{1/2}.$$

Solving for E_F yields

$$\boxed{E_F = \frac{\hbar^2\pi^4N^2}{128mL^2}}$$

Problem 3

We start with the knowledge that the density of states is proportional to $E^{1/2}$ and the probability of occupation is

$$P(E) = \frac{1}{e^{\beta(E-E_F)} + 1}.$$

Let

$$\frac{dN}{dE} = cE^{1/2}.$$

where c is a constant. The fraction of nonrelativistic fermions in a gas of finite temperature T above the Fermi energy is given by the integral:

$$\mathcal{F} = \frac{\int_{E_F}^\infty P(E)\frac{dN}{dE}dE}{\int_0^\infty P(E)\frac{dN}{dE}dE}.$$

The denominator of the above equation is just the total number of particles, which is easiest to evaluate at $T = 0$ where, since $P(E)$ is just the Fermi-Dirac distribution function, we have:

$$P(E) = 1 \quad E < E_F$$
$$P(E) = 0 \quad E > E_F.$$

So the denominator is simply

$$\int_0^{E_F}\frac{dN}{dE}dE = \int_0^{E_F} cE^{1/2}dE = \frac{2}{3}cE_F^{3/2}.$$

So employing all of our information, the simplest expression we can get for $\mathcal{F}$ is:

$$\boxed{\mathcal{F} = \frac{3}{2}E_F^{-3/2}\int_{E_F}^\infty \frac{E^{1/2}dE}{e^{\beta(E-E_F)} + 1}}$$

Problem 4 *Rohlf 12.16*

We want to deduce the expression for density of states of a relativistic electron gas. We first get the density of states with respect to k (where $\hat{k}$ is the electron wave vector) in the usual manner for a 3D particle in a box problem. The volume of a shell of thickness dk in k-space (considering only positive k_x, k_y, and k_z) is $(4\pi k^2dk)/8$. From the boundary conditions $k_i = (\pi/L)n_i$ (where $i = x, y, z$ and the box is length L on a side), we know the number of states per unit volume of k-space is $(L/\pi)^3$. So

$$\frac{dN}{dk} = \frac{k^2V}{2\pi^2}$$

where V is the volume of the box. Dividing by volume, to get the density of states per unit volume, and converting to momentum p using the deBroglie relation $p = \hbar k$, we have:

$$\frac{dn}{dp} = \frac{4\pi p^2}{h^3}.$$

To convert this expression into density of states per unit energy $\rho(E)$, we use:

$$\rho(E) = \frac{dn}{dE} = \frac{dn}{dp}\frac{dp}{dE}.$$

The relativistic expression for momentum in terms of energy is:

$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4}.$$

Therefore

$$\frac{dp}{dE} = \frac{E}{c^2p}.$$

We now solve for $\rho(E)$:

$$\rho(E) = 2\frac{4\pi pE}{c^2h^3}$$

where the factor of 2 is for spin degeneracy.
The relativistic momentum is $p = \gamma mv$ and $v \approx c$. The relativistic energy is $E = \gamma mc^2$. Substituting these expressions in,

$$\boxed{\rho(E) = 2\left(\frac{4\pi m^2c}{h^3}\right)\gamma^2}$$

Problem 5 *Rohlf 17.27*

(a)

There are six states of charmonium with $n = 2$. Charmonium is a bound state of a charm and anti-charm quark. We don't have to worry about symmetrization of the wavefunction because these are *not* identical fermions. The total spin s of the system can be:

$$s = s_1 + s_2 = 1, 0$$

where $s_1 = 1/2$ and $s_2 = 1/2$ are the spins of the quarks. The orbital angular momentum l of charmonium can be $l = 1, 0$ in the $n = 2$ state. The total (spin + orbital) angular momentum is given by

$$J = L + S = l + s,\ \ldots, |l - s|.$$

So we have the following possible states:

$$\begin{aligned} j &= 2,1,0 && \text{for } s = 1, l = 1 \\ j &= 1 && \text{for } s = 0, l = 1 \\ j &= 1 && \text{for } s = 1, l = 0 \\ j &= 0 && \text{for } s = 0, l = 0 \end{aligned}$$

which total six.

(b)

The unobserved state has $s = 0$, $l = 1$ and $j = 1$ by inspection.

(c)

By analogy to similar states in the chart of charmonium, the energy difference between states with aligned vs. anti-aligned spins of the quarks is ~ 100 MeV (the energy difference of the $\psi(2s)$ and $\eta_c(2s)$ states and $\psi(1s)$ and $\eta_c(1s)$ states, see Rohlf pg. 494). One could imagine that the energy splitting arises because of some spin-spin interaction between the quarks, so the $s = 0$, $l = 1$, $j = 1$ state of charmonium (called the ☺ or smiley particle) should be split in energy from the $\chi_{c1}(2p)$ state by ~ 100 MeV. So the mass of ☺ should be roughly 3400 MeV.

(d)

The ☺ particle is not observed because the method by which all the states of charmonium were observed involved creation of a $\psi(2s)$ particle and subsequent electromagnetic decay. The energy of emitted photons was measured and the spectrum of charmonium particles was established. Note that decays of $\psi(2s) \to$ ☺ change both s and l by 1. Such a decay would involve interaction with both the electric (to change l) and magnetic (to change s) components of the photon and is highly suppressed.

Problem 6 *Rohlf 18.11*

(a)

Consider this problem in the context of the theory of weak interactions as it existed before Glashow, Weinberg, Salam discovered how to unify it with the theory of electromagnetic interactions. In this context (Rohlf p. 509), the coupling constant for weak interactions is the Fermi constant G_F. According to Rohlf's Eq. (18.28), G_F has dimensions GeV fm^3. In a system of "natural" units in which $\hbar = c = 1$, we can transform a length (fm) into an inverse energy (inverse GeV) using the fact that $\hbar c \approx 0.2$ GeV fm. In natural units, G_F therefore has dimensions GeV^{-2}.

Since G_F is a coupling constant, like the fine structure constant α, it describes the strength of a quantum mechanical amplitude. The *rate* is proportional to the square of the modulus of this amplitude. Thus the decay rate W (which is inversely proportional to the muon lifetime τ_μ) is proportional to the square of G_F.

Now we have a dilemma. If $|G_F|^2$ were the only dimensionful component of W, W would have units of GeV^{-4}. However, using the fact that $\hbar = 6.6 \times 10^{-25}$ GeV sec, in natural units we know that W must have units of sec^{-1} or GeV. So far we are off by five powers of energy!

The solution is to bring in the only other relevant dimensionful quantity around, the muon mass m_μ. Remembering that mc^2 is the same as m in natural units, we find that we need five powers of m_μ in the numerator of W to make its units correct. Therefore its inverse, the muon lifetime, must have five powers of m_μ in its denominator.

Alternatively, we can approach this problem a bit more formally. Fermi's Golden rule says that the decay rate W is given by:

$$W = \frac{2\pi}{\hbar}|\mathcal{M}|^2 \times (\text{phase space})$$

where $\mathcal{M}$ is a transition amplitude obtained from perturbation theory (you'll learn all about this in 137B). In this case, all we need to know is that $\mathcal{M} \propto 1/M_W^2$. By dimensionality, we need to cancel the mass of the W boson with something, the best guess is the mass of the muon. The phase space available to the decay products in this case is proportional to the available energy, in other words the muon mass again. So

$$|\mathcal{M}|^2 \times (\text{phase space}) \propto \mathrm{m}_\mu^5$$

which again implies $\tau_\mu \propto m_\mu^{-5}$.

(b)

The tau particle has five times as many decay channels as the muon, so the phase space is increased by a factor of 5. So the tau lifetime is given, from the above arguments, by:

$$\tau_\tau = \tau_\mu \left(\frac{1}{5}\right)\left(\frac{105.7\text{ MeV}}{1777\text{ MeV}}\right)^5 = 0.3\text{ ps}.$$

Problem 7 *Rohlf 19.18*

We have redshift parameter $z = 2$. We can employ the formula (19.15) on pg. 539 of Rohlf:

$$(1+z)^2 = \frac{1+\beta}{1-\beta}$$

where $\beta = v/c$ as usual. We can solve for β:

$$\beta = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = 0.8$$

and then use β in Hubble's law to determine the distance to the galaxy. Hubble's law is

$$d = \frac{\beta c}{H_0}$$

where H_0 is the Hubble constant. We find:

$$d = \frac{\beta c}{H_0} = \frac{0.8 \times (3 \times 10^8\text{ m/s})}{7 \times 10^4\ m/s \cdot \text{Mpc}^-1} = 3400\text{ Mpc}.$$

Problem 8 *Rohlf 19.29*

As a rough estimate, we simply set the thermal energy of particles in the early universe $\sim kT$ equal to the mass of 2 bottom quarks (actually a bottom and anti-bottom, which have the same mass). The bottom quarks must be produced in pairs so that "beauty" is conserved, since the b and $\bar{b}$ have equal and opposite beauties. You begin to wonder where this stuff comes from. Anyhow, the mass of 2 bottom quarks is 10 GeV, which implies $T = 10^{14}$ K. The characteristic expansion time t_{exp} comes from

$$t_{\text{exp}} = \frac{1}{H(t)} = \left(\frac{2.7\text{ K}}{T}\right)^2 \sqrt{\frac{3c^2}{8\pi\rho G}} = 5 \times 10^{19}\text{ s}\left(\frac{2.7\text{ K}}{T}\right)^2.$$

You should probably check out the discussion on pp. 558-559 of Rohlf. So we can estimate:

$$t_{\mathrm{exp}} \approx \frac{4 \times 10^{20}\ \mathrm{s} \cdot \mathrm{K}^2}{T^2} = 4 \times 10^{-8}\ \mathrm{s}.$$

And that's all folks!
Good luck on your finals! Merry winter break!

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

EXAMINATION 1

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (58 points)
In a free-electron laser, a beam of relativistic electrons is subjected to a transverse magnetic field that varies sinusoidally with lab coordinate z, the (average) beam direction:

$$\mathbf{B} = \hat{\mathbf{x}} B_0 \cos \frac{2\pi z}{\lambda_0}$$

where B_0 and λ_0 are constants. In the lab, the z component of the electrons' velocity is

$$v_z = \beta_0 c$$

where β_0 is a constant.

a. (8 points) Consider a Lorentz frame $\mathcal{S}'$ moving with velocity

$$\boldsymbol{\beta}_0 c = \hat{\mathbf{z}} \beta_0 c$$

with respect to the lab. The Lorentz transformation for electromagnetic fields is

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma_0(\mathbf{E}_{\perp} + \boldsymbol{\beta}_0 \times c\mathbf{B}_{\perp}) \\ c\mathbf{B}'_{\perp} &= \gamma_0(c\mathbf{B}_{\perp} - \boldsymbol{\beta}_0 \times \mathbf{E}_{\perp})\ , \end{aligned}$$

where $\gamma_0 \equiv (1-\beta_0^2)^{-1/2}$. Calculate the electric field $\mathbf{E}'$ seen in $\mathcal{S}'$; continue to express it in terms of $2\pi z/\lambda_0$.

b. (8 points) Defining

$$2\pi z/\lambda_0 \equiv \omega'_0 t'\ ,$$

where t' is the time as observed at the origin of $\mathcal{S}'$, compute ω'_0 in terms of the constants previously given.

c. (8 points) Consider an electron of charge $-e$ and mass m whose *average* position is

$$\langle x', y', z' \rangle = (0, 0, 0)$$

as observed in $\mathcal{S}'$. In this frame, its velocity is so small that you may ignore $\mathbf{v}' \times \mathbf{B}'$ with respect to $\mathbf{E}'$. In frame $\mathcal{S}'$, making this approximation, compute the electron's motion $y'(t')$. (In case you didn't get part **b.** exactly right, leave your answer in terms of ω'_0.)

d. (8 points) The electric dipole moment $\mathbf{p}$ of a distribution of N point charges q_i at positions $\mathbf{r_i}$ is defined as

$$\mathbf{p} = \sum_{i=1}^{N} \mathbf{r}_i q_i\ .$$

The power $P(t)$ radiated by a charge distribution with time-varying dipole moment $\mathbf{p}(t)$ is

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{(d^2\mathbf{p}/dt^2)^2}{c^3}\ .$$

As seen in $\mathcal{S}'$, calculate $\langle P' \rangle$, the *time-averaged* power radiated by a single electron in the free-electron laser.

e. (8 points) Energy and time both transform as the 0^{th} component of a four-vector. Calculate $\langle P \rangle$, the time-averaged power radiated by a single electron as observed in the lab. You may leave your answer in terms of $\langle P' \rangle$.

f. (10 points) For light, the relativistic Doppler shift is

$$\omega = \frac{\omega'}{\gamma_0(1 - \beta_0 \cos\theta)} \; .$$

Calculate the ratio λ_0/λ, where λ_0, as before, is the characteristic length describing the spatial variation in the lab of the free-electron laser's magnetic field, and λ is the wavelength of the light that its electrons radiate in the forward direction, as observed in the lab. Express this ratio in terms of γ_0, in the limit $\beta_0 \to 1$.

g. (8 points) What is the state of polarization of the free-electron laser's light? Explain.

2. (42 points)
Semi-infinite regions $y > L$ and $y < -L$ are filled by perfect conductor, while the intervening slab $-L < y < L$ is filled by dielectric with constant dielectric constant ϵ and permeability μ.

a. (8 points) In SI units, write Maxwell's equations for $\mathbf{E}$ and $\mathbf{H}$ inside the dielectric. Do not write any terms involving free charges or free currents, which both vanish there.

b. (8 points) Prove that E_x and E_z both must vanish at $y = \pm L$.

For parts **c.** and **d.** only, assume, for $-L < y < L$, that the fields are given by

$$\mathbf{E}_{\text{physical}} = \Re\big(\hat{\mathbf{y}} E_2 \exp\big(i(kz - \omega t)\big)\big)$$
$$\mathbf{H}_{\text{physical}} = \Re\big((\hat{\mathbf{x}} H_1 + \hat{\mathbf{z}} H_3) \exp\big(i(kz - \omega t)\big)\big) \; ,$$

where E_2, H_1, and H_3 are unknown complex constants, and k and ω are unknown real constants.

c. (8 points) Prove that $H_3 = 0$.

d. (10 points) Calculate the ratio H_1/E_2 in terms of known quantities.

e. (8 points) Can a linear combination of a right-hand and a left-hand circularly polarized plane wave in the region $-L < y < L$ propagate in the z direction? If not, why not? If so, what combination(s) would be possible? Explain fully.

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

SOLUTION TO EXAMINATION 1

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

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where B_0 and λ_0 are constants. In the lab, the z component of the electrons' velocity is

$$v_z = \beta_0 c$$

where β_0 is a constant.

a. (8 points) Consider a Lorentz frame $\mathcal{S}'$ moving with velocity

$$\boldsymbol{\beta}_0 c = \hat{\mathbf{z}} \beta_0 c$$

with respect to the lab. The Lorentz transformation for electromagnetic fields is

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma_0(\mathbf{E}_{\perp} + \boldsymbol{\beta}_0 \times c\mathbf{B}_{\perp}) \\ c\mathbf{B}'_{\perp} &= \gamma_0(c\mathbf{B}_{\perp} - \boldsymbol{\beta}_0 \times \mathbf{E}_{\perp}) \ , \end{aligned}$$

where $\gamma_0 \equiv (1-\beta_0^2)^{-1/2}$. Calculate the electric field $\mathbf{E}'$ seen in $\mathcal{S}'$; continue to express it in terms of $2\pi z/\lambda_0$.

Solution. There are no electric or parallel magnetic fields in the lab frame so the third equation gives us the total electric field seen in $\mathcal{S}'$:

$$\begin{aligned} \mathbf{E}' &= \boldsymbol{\beta}_0 \times c\mathbf{B}_{\perp} \\ &= \gamma_0 \beta_0 c B_0 \cos\Big(\frac{2\pi z}{\lambda_0}\Big)(\hat{z} \times \hat{x}) \\ &= \gamma_0 \beta_0 c B_0 \cos\Big(\frac{2\pi z}{\lambda_0}\Big)\hat{y} \ . \end{aligned}$$

b. (8 points) Defining

$$2\pi z/\lambda_0 \equiv \omega'_0 t' \ ,$$

where t' is the time as observed at the origin of $\mathcal{S}'$, compute ω'_0 in terms of the constants previously given.

Solution. We need a Lorentz transformation to relate (ct, z) to (ct', z'). Taking advantage of the fact that $z' = 0$, we minimize algebra by choosing the inverse transformation:

$$\begin{aligned} z &= \gamma_0 z' + \gamma_0 \beta_0 ct' \\ &= 0 + \gamma_0 \beta_0 ct' \\ &= \gamma_0 \beta_0 ct' \\ \frac{2\pi}{\lambda_0} z &= \frac{2\pi}{\lambda_0} \gamma_0 \beta_0 ct' \\ &\equiv \omega'_0 t' \\ \omega'_0 &= \frac{2\pi \gamma_0 \beta_0 c}{\lambda_0} \ . \end{aligned}$$

c. (8 points) Consider an electron of charge $-e$ and mass m whose *average* position is

$$\langle x', y', z' \rangle = (0, 0, 0)$$

as observed in $\mathcal{S}'$. In this frame, its velocity is so small that you may ignore $\mathbf{v}' \times \mathbf{B}'$

with respect to $\mathbf{E}'$. In frame $\mathcal{S}'$, making this approximation, compute the electron's motion $y'(t')$. (In case you didn't get part **b.** exactly right, leave your answer in terms of ω_0'.)

Solution. The Lorentz force on the electron in $\mathcal{S}'$ is given by:

$$\mathbf{F}' = -e\mathbf{E}' - e\mathbf{v}' \times \mathbf{B}' ,$$

but we can ignore the $\mathbf{v}' \times \mathbf{B}'$ term since $\mathbf{v}'$ is always small. So then we have an equation for the acceleration:

$$m_e \mathbf{a}' = -e\mathbf{E}'.$$

Plugging in $\mathbf{E}'$ from (**a.**) and integrating twice with respect to t',

$$y'(t') = \frac{e}{m}\gamma_0\beta_0 c B_0 \frac{\cos\omega_0' t'}{(\omega_0')^2} .$$

d. (8 points) The electric dipole moment $\mathbf{p}$ of a distribution of N point charges q_i at positions $\mathbf{r_i}$ is defined as

$$\mathbf{p} = \sum_{i=1}^{N} \mathbf{r}_i q_i .$$

The power $P(t)$ radiated by a charge distribution with time-varying dipole moment $\mathbf{p}(t)$ is

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{(d^2\mathbf{p}/dt^2)^2}{c^3} .$$

As seen in $\mathcal{S}'$, calculate $\langle P' \rangle$, the *time-averaged* power radiated by a single electron in the free-electron laser.

Solution. The second derivative of the time-varying electric dipole moment for a single electron in the free electron laser is given by:

$$\frac{d^2\mathbf{p}'}{dt^2} = -e\mathbf{a}' ,$$

where $\mathbf{a}'$ is the acceleration found in (**c.**):

$$\mathbf{a}' = -\frac{e}{m}\gamma_0\beta_0 c B_0 \cos\omega_0' t'.$$

Plugging this result into the formula given above for radiated power P',

$$P' = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^4\gamma_0^2\beta_0^2 c^2 B_0^2}{m^2c^3}\cos^2\omega_0' t' .$$

Since $\langle \cos^2\omega_0' t' \rangle = 1/2$, the time average of P' is given by:

$$\langle P' \rangle = \frac{1}{4\pi\epsilon_0}\frac{1}{3}\frac{e^4\gamma_0^2\beta_0^2 B_0^2}{m^2c} .$$

e. (8 points) Energy and time both transform as the 0^{th} component of a four-vector. Calculate $\langle P \rangle$, the time-averaged power radiated by a single electron as observed in the lab. You may leave your answer in terms of $\langle P' \rangle$.

Solution. Since both energy and time transform as the 0^{th} component of a four-vector, if we have measured a change in energy of the electron $\Delta E'$ and a change in time $\Delta t'$ in the electron's rest frame $\mathcal{S}'$, then the same quantities in the lab frame are given by $\Delta E = \gamma_0 \Delta E'$ and $\Delta t = \gamma_0 \Delta t'$. So

$$\begin{aligned}\langle P' \rangle &= \Delta E'/\Delta t' \\ &= \Delta E/\Delta t \\ &= \langle P \rangle .\end{aligned}$$

This was also solved in problem set 3, problem 3!

f. (10 points) For light, the relativistic Doppler shift is

$$\omega = \frac{\omega'}{\gamma_0(1-\beta_0\cos\theta)} .$$

Calculate the ratio λ_0/λ, where λ_0, as before, is the characteristic length describing the spatial variation in the lab of the free-electron laser's magnetic field, and λ is the wavelength of the light that its electrons radiate in the forward direction, as observed in the lab. Express this ratio in terms of γ_0, in the limit $\beta_0 \to 1$.

Solution. In $\mathcal{S}'$, we know from (**c.**) that the electron is oscillating with angular frequency ω_0'.

Then, in $\mathcal{S}'$, the EM radiation produced by that electron has the same angular frequency. A forward observer in $\mathcal{S}$, upon whom the beam impinges with $\theta = 0$, sees this radiation with a Doppler shifted angular frequency ω_0 that, by the above Doppler formula, is equal to

$$\omega_0 = \frac{\omega_0'}{\gamma_0(1-\beta_0)} \ .$$

Substituting $\omega_0 = 2\pi c/\lambda$, and plugging in the value of ω_0' from (**b.**),

$$\begin{aligned}
\frac{2\pi c}{\lambda} &= \frac{2\pi\gamma_0\beta_0 c}{\lambda_0}\frac{1}{\gamma_0(1-\beta_0)} \\
\frac{\lambda_0}{\lambda} &= \frac{\beta_0}{1-\beta_0} \\
&= \frac{1+\beta_0}{1+\beta_0}\frac{\beta_0}{1-\beta_0} \\
&= \gamma_0^2\beta_0(1+\beta_0) \ .
\end{aligned}$$

This reduces to $2\gamma_0^2$ in the relativistic limit, a famous (and simple) result. If, say, λ_0 is 0.1 m and the electron energy is 500 MeV ($\gamma^2 \approx 10^6$), the FEL or wiggler can be made to radiate in the far UV ($\lambda \approx 0.05\ \mu$m), where no conventional laser is available.

g. (8 points) What is the state of polarization of the free-electron laser's light? Explain.

Solution. From (**c.**) we know that the electron oscillates in the $\hat{y}$ direction. The on-axis radiation from the dipole is propagating in the $\hat{z}$ direction, so light is polarized orthogonal to $\hat{z}$. From the formula derived in class describing dipole radiation, we know the radiation is polarized in the $\hat{\theta}$ direction where the vertical axis is defined by the direction the dipole oscillates in. Hence we can conclude that the light is linearly polarized in the $\hat{y}$ direction.

2. (42 points)
Semi-infinite regions $y > L$ and $y < -L$ are filled by perfect conductor, while the intervening slab $-L < y < L$ is filled by dielectric with constant dielectric constant ϵ and permeability μ.

a. (8 points) In SI units, write Maxwell's equations for $\mathbf{E}$ and $\mathbf{H}$ inside the dielectric. Do not write any terms involving free charges or free currents, which both vanish there.

Solution. Within the dielectric, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, where ϵ and μ are constants. The source-free equations are

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= 0 \\
\mathbf{B} &= \mu\mathbf{H} \\
\Rightarrow \nabla\cdot\mathbf{H} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\
\mathbf{B} &= \mu\mathbf{H} \\
\Rightarrow \nabla\times\mathbf{E} &= -\mu\frac{\partial\mathbf{H}}{\partial t}
\end{aligned}$$

The useful source-dependent equations are the variety that depend on free rather than total charges and currents, because free charges and currents are zero in the dielectric. These are

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho_{\text{free}} = 0 \\
\mathbf{D} &= \epsilon\mathbf{E} \\
\Rightarrow \nabla\cdot\mathbf{E} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\nabla\times\mathbf{H} &= \mathbf{J}_{\text{free}} + \frac{\partial\mathbf{D}}{\partial t} = \frac{\partial\mathbf{D}}{\partial t} \\
\mathbf{D} &= \epsilon\mathbf{E} \\
\Rightarrow \nabla\times\mathbf{H} &= \epsilon\frac{\partial\mathbf{E}}{\partial t}
\end{aligned}$$

b. (8 points) Prove that E_x and E_z both must vanish at $y = \pm L$.

Solution. Electric fields vanish in perfect conductors, because the infinitely mobile free charges instantaneously rearrange themselves to shield out any externally applied electric field. Consider a rectangular loop with long side S and short side s. One long side lies in the conductor, parallel to the plane $y = L$; the other long side lies in the dielectric. In the limit $s \to 0$, the right-hand side of Faraday's law,

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\oint \mathbf{B}\cdot d\mathbf{a} \ ,$$

vanishes because the area vanishes, and the contributions of the short parts to the rectangular loop on the left-hand side also vanish. The only nonvanishing contribution to the left-hand side is $\mathbf{E}_{\|} \cdot \mathbf{S}$ in the dielectric. This proves that $\mathbf{E}_{\|}$ in the dielectric must vanish at $|y| = L$. This includes $\mathbf{E}_{\|}$ in either the $\hat{x}$ or $\hat{z}$ directions, which both are parallel to the interface.

For parts **c.** and **d.** only, assume, for $-L < y < L$, that the fields are given by

$$\mathbf{E}_{\text{physical}} = \Re\big(\hat{\mathbf{y}} E_2 \exp\big(i(kz - \omega t)\big)\big)$$
$$\mathbf{H}_{\text{physical}} = \Re\big((\hat{\mathbf{x}} H_1 + \hat{\mathbf{z}} H_3) \exp\big(i(kz - \omega t)\big)\big) ,$$

where E_2, H_1, and H_3 are unknown complex constants, and k and ω are unknown real constants.

c. (8 points) Prove that $H_3 = 0$.

Solution. As usual we require the complex electromagnetic fields to satisfy Maxwell's equations (not just their (physical) real part). When their dependence on $\mathbf{r}$ and t is of the form $\exp\big(i(\mathbf{k} \cdot \mathbf{r} - \omega t)\big)$, the operators $\nabla\cdot$ and $\nabla\times$ reduce to $i\mathbf{k}\cdot$ and $i\mathbf{k}\times$, respectively, while the operator $\partial/\partial t$ reduces to $-i\omega$. Using the first Maxwell equation in (**a.**),

$$\begin{aligned} \nabla \cdot \mathbf{H} &= 0 \\ i\mathbf{k} \cdot (\hat{x} H_1 + \hat{z} H_3) &= 0 \\ \mathbf{k} &= \hat{z} k \\ \Rightarrow ikH_3 &= 0 \, . \end{aligned}$$

d. (10 points) Calculate the ratio H_1/E_2 in terms of known quantities.

Solution. Using the methods of (**c.**), the second Maxwell equation in (**a.**) requires

$$\begin{aligned} ik(\hat{z} \times \hat{y}) E_2 &= +i\omega\mu \hat{x} H_1 \\ -ik\hat{x} E_2 &= +i\omega\mu \hat{x} H_1 \\ -\frac{k}{\mu\omega} &= \frac{H_1}{E_2} \end{aligned}$$

while the fourth Maxwell equation in (**a.**) requires

$$\begin{aligned} ik(\hat{z} \times \hat{x}) H_1 &= -i\omega\epsilon \hat{y} E_2 \\ ik\hat{y} H_1 &= -i\omega\epsilon \hat{y} E_2 \\ \frac{H_1}{E_2} &= -\frac{\epsilon\omega}{k} \, . \end{aligned}$$

Setting the two values of H_1/E_2 equal,

$$\begin{aligned} -\frac{k}{\mu\omega} &= -\frac{\epsilon\omega}{k} \\ \frac{k}{\omega} &= \sqrt{\epsilon\mu} \, . \end{aligned}$$

Plugging this value for k/ω into either of the equations for H_1/E_2,

$$\frac{H_1}{E_2} = -\sqrt{\frac{\epsilon}{\mu}} \, .$$

e. (8 points) Can a linear combination of a right-hand and a left-hand circularly polarized plane wave in the region $-L < y < L$ propagate in the z direction? If not, why not? If so, what combination(s) would be possible? Explain fully.

Solution. From (**b.**) we know that $E_x = 0$ at $|y| = L$. In this part (only!) we are asked to assume that a plane wave is propagating in the $\hat{z}$ direction within the dielectric. This means that $\mathbf{E}$ must be $\perp$ to $\hat{z}$ and it cannot depend on x and y. So, if E_x vanishes at the boundaries $|y| = L$, it must vanish throughout the dielectric. Thus the only nonzero component of $\mathbf{E}$ lies in the $\hat{y}$ direction. We then ask, what combination of RH polarization ($\propto \hat{x} - i\hat{y}$) and LH polarization ($\propto \hat{x} + i\hat{y}$) add to pure $\hat{y}$ polarization? Evidently, the RH and LH waves must have equal amplitude and opposite sign.

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

EXAMINATION 2

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1. (25 points)
The basis of scalar diffraction theory is the *Fresnel-Kirchoff integral formula*. In Fowles' notation (Eq. 5.11), this formula states

$$U_{\rm p} = -\frac{ikU_0 \exp{(-i\omega t)}}{4\pi} \times \\ \times \int\!\!\int \frac{\exp\big(ik(r+r')\big)}{rr'} \left[\hat{\mathbf{n}}\cdot\hat{\mathbf{r}} - \hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'\right] d\mathcal{A}$$

where $\mathbf{r}'$ is a vector from the (point) source to a point on the aperture, $\mathbf{r}$ is a vector from the observer to the same point on the aperture,

$$U_0 \frac{\exp\big(i(kr' - \omega t)\big)}{r'}$$

is the optical disturbance at a point on the aperture, $U_{\rm p}$ is the optical disturbance at the observer, $\omega = ck = 2\pi c/\lambda$ is the angular frequency of the light, $d\mathcal{A}$ is an element of aperture area, and $\hat{\mathbf{n}}$ is the normal to $d\mathcal{A}$. [Note that, in a typical geometry (source on the left, aperture in the middle, observer on the right, and $\hat{\mathbf{n}}$ pointing to the left), $\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}$ is positive while $\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'$ is negative, so that both terms in the square bracket are positive.]

Consider this simple geometry: Let z be the axis pointing from left to right. Place the source at $(x, y, z) = (0, 0, -D)$, the observer at (X, Y, D), and the aperture in the plane $z = 0$. The aperture is characterized by an *aperture function* $g(x, y)$ such that $g = 1$ where the aperture is open, and $g = 0$ where the aperture is opaque.

a. (10 points) Let δ be the maximum value of $\sqrt{x^2 + y^2}$ on the aperture plane for which the aperture is *not* opaque. Thus, for this part of the problem, there are three characteristic lengths: λ, δ, and D. By moving around in the plane $z = D$, restricting her own coordinates X, Y such that $\sqrt{X^2 + Y^2} \ll D$, the observer finds that the optical disturbance there is proportional to the *Fourier transform* of $g(x, y)$. As someone who understands the physics of diffraction, you realize that this information implies that a single strong condition must be satisfied which relates λ, δ, and D. *Write down this condition.* (You needn't prove it, and you may omit factors of order unity.)

b. (15 points) For this part of the problem, take the aperture function to be

$$g(x, y) = 0,\ \ x < 0$$
$$g(x, y) = 1,\ \ x > 0\ .$$

This describes a "knife edge" at $x = 0$ extending from $y = -\infty$ to $y = \infty$. Therefore, in this part of the problem, $\delta = \infty$: the strong condition of part **a.** cannot be satisfied. In this part of the problem, the observer is fixed at $(0, 0, D)$, *i.e.* at $X = Y = 0$. With this aperture in place, the observer records an *irradiance* $I_{\rm a}$. With the aperture completely removed ($g \equiv 1$), the observer records an irradiance I_0. Give the ratio $I_{\rm a}/I_0$. To receive credit you must *explain why this ratio is correct.*

2. (25 points)
James Rainwater was awarded the Nobel Prize in the 1980's for experiments done at the Nevis (Columbia) cyclotron in the 1950's. He measured the sizes of nuclei using their interactions with muons (heavy electrons) which were in orbit about them.

In the following, use the Bohr picture to describe the muon orbit. For ease of numerical computation, you may take the natural length unit $\hbar/m_e c$ to be 400 fm; the ratio m_μ/m_e of muon to electron masses to be 200; and the fine structure constant α to be 1/150. You may neglect the difference between the muon's actual and reduced mass.

A muon in $n = 1$ Bohr orbit reacts with (is "captured" by) a $Z = 50$ nucleus before it decays:

$$\mu^- + (A, Z) \to (A, Z - 1) + \nu_\mu,$$

where the neutrino ν_μ has negligible rest mass. Assuming that the initial and final nuclei have the same infinitely large rest mass and therefore a negligible kinetic energy, what is the neutrino energy expressed in units of $m_e c^2$? (1% accuracy is sufficient.)

3. (25 points)
Consider the elastic scattering of a photon from an infinitely massive, perfectly reflective, spherical target of finite radius R (like a bowling ball polished to a mirror finish). The bowling ball is centered on the origin. The photon is incident along the $\hat{\mathbf{z}}$ direction and scatters (reflects) into the direction (θ, ϕ), where θ and ϕ are the usual spherical polar angles. Note that $\theta = 0$ means that the photon remains undeflected. For this problem, ignore diffraction and any other effects which arise from the wavelike properties of the photon.

- **a.** (10 points) What is the total scattering cross section σ_T, corresponding to any deflection of the photon? (You don't need a calculation here, just a correct answer and a convincing explanation for it.)
- **b.** (15 points) Calculate the differential cross section
$$\frac{d\sigma}{d\Omega},$$
where $d\Omega = \sin\theta \, d\theta \, d\phi$ is an element of solid angle. (When you integrate your result over the full solid angle, do you confirm your answer to **a.**?)

4. (25 points)
A nonrelativistic particle of mass m is confined to a one-dimensional box extending from $x = 0$ to $x = L$. Here a "box" is a square potential well with infinite sides.

- **a.** (10 points) In terms of n and other constants, write down the energies E_n, $1 \le n < \infty$, measured with respect to the bottom of the potential well, that the particle is allowed by Schrödinger's equation to have.
- **b.** (15 points) Define $N(E)$ to be the total number of allowed states with energy $\le E$. Taking $n \gg 1$, so that the distribution of E is approximately continuous, calculate the density of states
$$\rho(E) \equiv \frac{dN}{dE}.$$

University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

SOLUTION TO EXAMINATION 2

Directions. Do all four problems (weights are indicated). This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Calculators are allowed but not essential – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (25 points)
The basis of scalar diffraction theory is the *Fresnel-Kirchoff integral formula*. In Fowles' notation (Eq. 5.11), this formula states

$$U_{\mathrm{p}} = -\frac{ikU_0 \exp(-i\omega t)}{4\pi} \times \\ \times \int\int \frac{\exp\big(ik(r+r')\big)}{rr'} \left[\hat{\mathbf{n}}\cdot\hat{\mathbf{r}} - \hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'\right] d\mathcal{A}$$

where $\mathbf{r}'$ is a vector from the (point) source to a point on the aperture, $\mathbf{r}$ is a vector from the observer to the same point on the aperture,

$$U_0 \frac{\exp\big(i(kr' - \omega t)\big)}{r'}$$

is the optical disturbance at a point on the aperture, U_{p} is the optical disturbance at the observer, $\omega = ck = 2\pi c/\lambda$ is the angular frequency of the light, $d\mathcal{A}$ is an element of aperture area, and $\hat{\mathbf{n}}$ is the normal to $d\mathcal{A}$. [Note that, in a typical geometry (source on the left, aperture in the middle, observer on the right, and $\hat{\mathbf{n}}$ pointing to the left), $\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}$ is positive while $\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'$ is negative, so that both terms in the square bracket are positive.]

Consider this simple geometry: Let z be the axis pointing from left to right. Place the source at $(x, y, z) = (0, 0, -D)$, the observer at (X, Y, D), and the aperture in the plane $z = 0$. The aperture is characterized by an *aperture function* $g(x, y)$ such that $g = 1$ where the aperture is open, and $g = 0$ where the aperture is opaque.

a. (10 points) Let δ be the maximum value of $\sqrt{x^2+y^2}$ on the aperture plane for which the aperture is *not* opaque. Thus, for this part of the problem, there are three characteristic lengths: λ, δ, and D. By moving around in the plane $z = D$, restricting her own coordinates X, Y such that $\sqrt{X^2+Y^2} \ll D$, the observer finds that the optical disturbance there is proportional to the *Fourier transform* of $g(x, y)$. As someone who understands the physics of diffraction, you realize that this information implies that a single strong condition must be satisfied which relates λ, δ, and D. *Write down this condition.* (You needn't prove it, and you may omit factors of order unity.)

Solution. In order for the optical disturbance $U_p(X, Y)$ to be the Fourier transform of $g(x, y)$, our system must satisfy the Fraunhofer condition (see discussion in Fowles Section 5.6). The basic idea of this condition is that the spherical curvature of the wavefront at the aperture must be small compared to the wavelength of the light, allowing us to treat the light at the aperture as a plane wave. Omitting factors of order unity, the Fraunhofer condition is

$$\delta^2 \ll D\lambda\ .$$

This condition implies that the obliquity factor $\left[\hat{\mathbf{n}}\cdot\hat{\mathbf{r}} - \hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'\right]$ is constant over the aperture, the quantity $e^{ikr'}/r'$ is nearly constant, and the quantity $e^{ikr}/r \approx e^{ikr}$. With these approximations, the optical disturbance

$$U_p(X, Y) \propto \int\int e^{ikr} d\mathcal{A}.$$

b. (15 points) For this part of the problem, take the aperture function to be

$$g(x,y) = 0, \quad x < 0$$
$$g(x,y) = 1, \quad x > 0 \,.$$

This describes a "knife edge" at $x = 0$ extending from $y = -\infty$ to $y = \infty$. Therefore, in this part of the problem, $\delta = \infty$: the strong condition of part **a.** cannot be satisfied. In this part of the problem, the observer is fixed at $(0, 0, D)$, *i.e.* at $X = Y = 0$. With this aperture in place, the observer records an *irradiance* I_a. With the aperture completely removed ($g \equiv 1$), the observer records an irradiance I_0. Give the ratio I_a/I_0. To receive credit you must *explain why this ratio is correct.*

Solution. According to the Fresnel-Kirchoff integral, the optical disturbance U_p arises from a superposition of secondary waves which originate at the $z = 0$ plane. When the semi-infinite screen is in place, due to the symmetry of the system exactly half the secondary waves are blocked. Thus

$$U_p = \frac{1}{2}U_0 \,,$$

or, in terms of intensity $I \propto |U|^2$,

$$\frac{I}{I_0} = \frac{1}{4} \,.$$

2. (25 points)
James Rainwater was awarded the Nobel Prize in the 1980's for experiments done at the Nevis (Columbia) cyclotron in the 1950's. He measured the sizes of nuclei using their interactions with muons (heavy electrons) which were in orbit about them.

In the following, use the Bohr picture to describe the muon orbit. For ease of numerical computation, you may take the natural length unit $\hbar/m_e c$ to be 400 fm; the ratio m_μ/m_e of muon to electron masses to be 200; and the fine structure constant α to be 1/150. You may neglect the difference between the muon's actual and reduced mass.

A muon in $n = 1$ Bohr orbit reacts with (is "captured" by) a $Z = 50$ nucleus before it decays:

$$\mu^- + (A, Z) \rightarrow (A, Z-1) + \nu_\mu,$$

where the neutrino ν_μ has negligible rest mass. Assuming that the initial and final nuclei have the same infinitely large rest mass and therefore a negligible kinetic energy, what is the neutrino energy expressed in units of $m_e c^2$? (1% accuracy is sufficient.)

Solution. The binding energy of the muon in the Bohr model is given by:

$$BE = \frac{1}{2}m_\mu c^2 (Z\alpha)^2$$

and in our case $Z\alpha \approx 1/3$ and $m_\mu c^2 \approx 200 m_e c^2$. So

$$BE \approx 11 m_e c^2 \,.$$

Since the rest energies of the initial and final nuclei are taken to be the same, the kinetic energy of the neutrino must be equal to the rest mass of the muon minus the binding energy, or

$$KE(\nu_\mu) \approx 200 m_e c^2 - 11 m_e c^2 \approx 189 m_e c^2 \,.$$

3. (25 points)
Consider the elastic scattering of a photon from an infinitely massive, perfectly reflective, spherical target of finite radius R (like a bowling ball polished to a mirror finish). The bowling ball is centered on the origin. The photon is incident along the $\hat{\mathbf{z}}$ direction and scatters (reflects) into the direction (θ, ϕ), where θ and ϕ are the usual spherical polar angles. Note that $\theta = 0$ means that the photon remains undeflected. For this problem, ignore diffraction and any other effects which arise from the wavelike properties of the photon.

a. (10 points) What is the total scattering cross section σ_T, corresponding to any deflection of the photon? (You don't need a calculation here, just a correct answer and a convincing explanation for it.)

Solution. The total cross section σ_T is the cross sectional area of the photon beam that suffers *any* deflection as a result of interaction with the target. Neglecting diffractive effects, the only photons scattered are those which intercept the area of a hemisphere of radius R, projected into the $z = 0$ plane. This is a circle of area πR^2. Thus

$$\sigma_T = \pi R^2 \ .$$

b. (15 points) Calculate the differential cross section

$$\frac{d\sigma}{d\Omega} \ ,$$

where $d\Omega = \sin\theta \, d\theta \, d\phi$ is an element of solid angle. (When you integrate your result over the full solid angle, do you confirm your answer to **a.**?)

Solution. A photon with impact parameter $b = \sqrt{x^2 + y^2}$ intercepts the sphere at a point on the sphere described by

$$\theta_s = \pi - \arcsin\frac{b}{R} \equiv \pi - \psi \ .$$

Just before it hits the sphere, it is travelling in the direction

$$\theta_0 = 0 \ .$$

Before impact, the angle that the photon makes with the normal to the sphere is ψ. Since the angle of incidence is equal to the angle of reflection, its direction changes by

$$\Delta\theta = \pi - 2\psi \ .$$

Therefore the final angle θ of the photon is

$$\begin{aligned}
\theta &= \theta_0 + \Delta\theta \\
&= 0 + \pi - 2\psi \\
&= \pi - 2\arcsin\frac{b}{R} \ .
\end{aligned}$$

Rearranging and differentiating,

$$\begin{aligned}
\arcsin\frac{b}{R} &= \frac{\pi}{2} - \frac{\theta}{2} \\
b &= R\,\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\
&= R\,\cos\frac{\theta}{2} \\
db &= -\frac{R}{2}\,\sin\frac{\theta}{2}\,d\theta \ .
\end{aligned}$$

An element $d\sigma$ of beam cross section is equal to $|b\,db\,d\phi|$. Substituting from above,

$$\begin{aligned}
d\sigma &= |b\,db\,d\phi| \\
&= \frac{R^2}{2}\,\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}\,d\theta\,d\phi \\
&= \frac{R^2}{4}\,\sin\theta\,d\theta\,d\phi \\
&= \frac{R^2}{4}\,d\Omega \\
\frac{d\sigma}{d\Omega} &= \frac{R^2}{4} \ .
\end{aligned}$$

This is an isotropic (constant) differential cross section. Integrated over $\Delta\Omega = 4\pi$, it yields $\sigma_T = \pi R^2$ as in (**a.**). Note that the isotropy of the differential cross section doesn't follow automatically from the spherical symmetry of the potential (an infinite wall at $r = R$). A different spherically symmetric potential, for example the Coulomb potential, yields the dramatically different Rutherford result

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\frac{\theta}{2}} \ .$$

4. (25 points)
A nonrelativistic particle of mass m is confined to a one-dimensional box extending from $x = 0$ to $x = L$. Here a "box" is a square potential well with infinite sides.

a. (10 points) In terms of n and other constants, write down the energies E_n, $1 \le n < \infty$, measured with respect to the bottom of the potential well, that the particle is allowed by Schrödinger's equation to have.

Solution. We measure the energy E of the particle with respect to the bottom of the well, where $V \equiv 0$. We seek solutions of the time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)u_E(x) = E u_E(x) \ ,$$

with the boundary condition (because of the infinite potential wall)

$$u_E(0) = u_E(L) = 0 \ .$$

The solutions are of the form

$$u_E(x) \propto \sin k_n x$$

with $k_n = n\pi/L$. Therefore, from the time-independent Schrödinger equation,

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2} \ ,$$

with $1 \le n \le \infty$. [As posed, the problem doesn't require a proof like the above; you just need to write down the correct values of E_n.]

b. (15 points) Define $N(E)$ to be the total number of allowed states with energy $\le E$. Taking $n \gg 1$, so that the distribution of E is approximately continuous, calculate the density of states

$$\rho(E) \equiv \frac{dN}{dE} \ .$$

Solution. The difference in energy between two adjacent states is

$$\begin{aligned}\Delta E &\equiv E_n - E_{n-1} \\ &= \left(n^2 - (n-1)^2\right)\frac{\pi^2\hbar^2}{2mL^2} \\ &= (2n-1)\frac{\pi^2\hbar^2}{2mL^2} \ .\end{aligned}$$

So when we increase the number of states by $\Delta N = 1$ we increase the maximum energy by ΔE. The density of states is just the ratio:

$$\begin{aligned}\rho(E) &\equiv \frac{dN}{dE} \\ &\approx \frac{\Delta N}{\Delta E} \text{ as } n \to \infty \\ &= \frac{2mL^2}{(2n-1)\pi^2\hbar^2} \\ &\approx \frac{mL^2}{n\pi^2\hbar^2} \text{ as } n \to \infty \ .\end{aligned}$$

Expressing $\rho(E)$ in terms of E and other constants, we substitute

$$\begin{aligned}n^2 &= \frac{2mL^2E}{\pi^2\hbar^2} \\ \rho(E) &= \frac{mL^2}{\pi^2\hbar^2}\sqrt{\frac{\pi^2\hbar^2}{2mL^2E}} \\ &= \frac{L}{\pi\hbar}\sqrt{\frac{m}{2E}} \ .\end{aligned}$$

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FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Calculators are allowed but not essential – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (35 points)
One circularly polarized photon is trapped between two parallel perfectly conducting plates, separated by a distance L, which are parallel to the photon's electric and magnetic fields.

- **a.** (10 points) If the photon has the smallest definite energy possible under these circumstances, at a point halfway between the plates describe a possible motion of its electric field vector.
- **b.** (10 points) Same for its magnetic field vector.
- **c.** (15 points) For this plate configuration, making no restriction on the photon energy, evaluate the density of photon states

$$\frac{d^2N}{dL\,d\lambda}$$

where N is the number of states and λ is the photon wavelength $\ll L$. Take into account the possible states of circular polarization.

2. (35 points)
A linearly ($\hat{\mathbf{x}}$) polarized plane EM wave travelling along $\hat{\mathbf{z}}$ is incident on an opaque baffle located in the plane $z = 0$. The baffle has two slits cut in it, which are of infinite extent in the $\hat{\mathbf{y}}$ direction. In the $\hat{\mathbf{x}}$ direction, the slit widths are each a and their center-to-center distance is d. (Obviously $d > a$, but you may *not* assume that $d \gg a$.) The top and bottom slits are each an equal distance from $x = 0$.

The diffracted image is viewed on a screen located in the plane $z = L$, where $L \gg d$; also $\lambda L \gg d^2$, where λ is the EM wavelength.

Quarter-wave plates are placed in each slit. They are identical, except that the top plate's "slow" (high-index) axis is along $(\hat{\mathbf{x}}+\hat{\mathbf{y}})/\sqrt{2}$ ($+45°$ with respect to the $\hat{\mathbf{x}}$ axis), while the bottom plate's slow axis is along $(\hat{\mathbf{x}}-\hat{\mathbf{y}})/\sqrt{2}$ ($-45°$ with respect to the $\hat{\mathbf{x}}$ axis).

- **a.** (15 points) What is the state of polarization of the diffracted light that hits the center of the screen, at $x = y = 0$? Explain.
- **b.** (20 points) At what diffracted angle θ_x does the first minimum of the irradiance occur?

3. (35 points)
A lens has an f-number (ratio of focal length to diameter) equal to F. The lens is used to concentrate sunlight on a ball whose diameter is equal to the diameter of the sun's image. The ball is convectively and conductively insulated, but it freely radiates energy outward so that its temperature can approach an equilibrium value T_b.

- **a.** (10 points) The sun subtends a half-angle of ≈ 0.005 radians. Is the size of its image "diffraction-limited", *i.e.* determined largely by the effects of diffraction? Make an order-of-magnitude argument assuming that the lens is a typical camera lens, with a radius of order 10^{-2} m.
- **b.** (25 points) Assuming the sun to be a blackbody of temperature T, calculate the ball's temperature T_b. Neglect reflection by the lens. (*Hint:* your answer should depend

only on T and F.)

4. (30 points)
You are given a Hamiltonian

$$\mathcal{H} = \frac{1}{2}(LR + RL) ,$$

where R and L are two operators such that

$$[L, R] = E_0$$

with E_0 a constant. You are also given an eigenfunction $u_E(x)$ of $\mathcal{H}$, such that

$$\mathcal{H}u_E = Eu_E ,$$

where E, another constant, is the energy eigenvalue.

Prove that

$$\mathcal{H}(Ru_E) = (E + E_0)(Ru_E) ,$$

i.e. R is a *raising operator*.

5. (35 points).
Consider a harmonic oscillator potential

$$V(x) = \frac{1}{2}m\omega_0^2 x^2$$

in one dimension. An *even* number N of particles of mass m are placed in this potential. There are no special interactions between the particles – no significant mutual electrostatic repulsion, gravitational attraction, *etc.*, compared to the strength of their interaction with the harmonic potential itself.

You may use what you already know about the levels of a harmonic oscillator.

The system is in its ground state, *i.e.* $T = 0$ Kelvin.

Calculate the total energy E of the N-particle system, relative to the bottom of the well, for the cases

a. (10 points) The N particles are *distinguishable.*

b. (10 points) The N particles are identical *bosons.*

c. (15 points) The N particles are identical spin $\frac{1}{2}$ *fermions.*

6. (30 points)
In the rest frame $\mathcal{S}'$ of a star, ignoring the gravitational redshift, some of the photons emitted by the star arise from a particular atomic transition with an unshifted wavelength λ'. When these photons are observed on earth, they are shifted to longer wavelength $\lambda = \lambda' + \Delta\lambda$ because the star is receding from the earth with velocity $\beta_0 c$ due to the Hubble expansion of the universe. Astronomers measure this redshift by means of the parameter z, defined by

$$z \equiv \frac{\Delta\lambda}{\lambda'} .$$

For light, the relativistic Doppler shift is

$$\omega = \frac{\omega'}{\gamma_0(1 - \beta_0 \cos\theta)} .$$

a. (10 points) In the "neighboring star" limit $\beta_0 \ll 1$, show that β_0 is approximately equal to the measured z.

b. (10 points) In the "distant star" limit $\gamma_0 \gg 1$, derive an expression for γ_0 in terms of the measured z.

c. (10 points) The observation of Supernova 1987A marked the dawn of a new astronomy, in which humans are able to detect fermions (neutrinos) as well as bosons (photons) from (spatially or temporally) resolved sources outside the solar system. About a dozen such neutrinos were detected in each of two huge underground water tanks. The photons from Supernova 1987A were redshifted by

$$z \approx 10^{-5} .$$

Taking the Hubble constant to be

$$H_0 \approx 0.7 \times 10^{-10}\text{yr}^{-1} ,$$

for how many years did the neutrinos from SN1987A travel before humans observed them?

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SOLUTION TO FINAL EXAMINATION

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1. (35 points)
One circularly polarized photon is trapped between two parallel perfectly conducting plates, separated by a distance L, which are parallel to the photon's electric and magnetic fields.

a. (10 points) If the photon has the smallest definite energy possible under these circumstances, at a point halfway between the plates describe a possible motion of its electric field vector.

Solution. The electric field of the light $\vec{E}$ is parallel to the conducting plates, so $\vec{E}$ must vanish at the plates. Therefore we have

$$\vec{E}(0,t) = \vec{E}(L,t) = 0 \ .$$

Of course, $\vec{E}(z,t)$ is a solution to the wave equation:

$$\Big(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\vec{E}(z,t) = 0 \ .$$

In order to satisfy both the wave equation and the boundary conditions, we choose

$$\vec{E}(z,t) = E_0 \sin(kz) \exp(-i\omega t)\big(\hat{x} \pm i\hat{y}\big) \ ,$$

where

$$k = \frac{n\pi}{L} \quad \text{where } n = 1, 2, \ldots$$

The smallest definite energy is achieved when $k = \pi/L$ (consequently $\omega = \pi c/L$), in which case we have for the electric field:

$$\vec{E}(z,t) = E_0 \sin\big(\frac{\pi z}{L}\big) \exp\big(-i\frac{\pi c t}{L}\big)\big(\hat{x} \pm i\hat{y}\big) \ .$$

Halfway between the plates ($z = \frac{L}{2}$), we find that

$$\vec{E}(\frac{L}{2},t) = E_0 \exp\big(-i\frac{\pi c t}{L}\big)\big(\hat{x} \pm i\hat{y}\big) \ .$$

So $\vec{E}$ moves in a circle with angular frequency $\omega = \pi c/L$.

b. (10 points) Same for its magnetic field vector.

Solution. From Maxwell's equations,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ .$$

Therefore we have

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$
$$\frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} \ .$$

From the above relations, it follows that the spatial dependence of $\vec{B}(z,t)$ is given by $\cos(\pi z/L)$. Halfway between the plates $\vec{B} = 0$.

c. (15 points) For this plate configuration, making no restriction on the photon energy, evaluate the density of photon states

$$\frac{d^2N}{dL\, d\lambda}$$

where N is the number of states and λ is the photon wavelength $\ll L$. Take into account the possible states of circular polarization.

Solution. We have from part (**a.**) that $k = n\pi/L$, so in terms of wavelength λ (recalling $k = 2\pi/\lambda$):

$$n = \frac{2L}{\lambda} .$$

Since we have 2 possible polarization states, the total number of states N as a function of wavelength is

$$N = \frac{4L}{\lambda} .$$

The derivative with respect to wavelength is

$$\frac{dN}{d|\lambda|} = \frac{4L}{\lambda^2}$$

and so we obtain for the density of photon states

$$\frac{d^2N}{d|\lambda|dL} = \frac{4}{\lambda^2} .$$

2. (35 points)
A linearly ($\hat{\mathbf{x}}$) polarized plane EM wave travelling along $\hat{\mathbf{z}}$ is incident on an opaque baffle located in the plane $z = 0$. The baffle has two slits cut in it, which are of infinite extent in the $\hat{\mathbf{y}}$ direction. In the $\hat{\mathbf{x}}$ direction, the slit widths are each a and their center-to-center distance is d. (Obviously $d > a$, but you may *not* assume that $d \gg a$.) The top and bottom slits are each an equal distance from $x = 0$.

The diffracted image is viewed on a screen located in the plane $z = L$, where $L \gg d$; also $\lambda L \gg d^2$, where λ is the EM wavelength.

Quarter-wave plates are placed in each slit. They are identical, except that the top plate's "slow" (high-index) axis is along $(\hat{\mathbf{x}}+\hat{\mathbf{y}})/\sqrt{2}$ ($+45°$ with respect to the $\hat{\mathbf{x}}$ axis), while the bottom plate's slow axis is along $(\hat{\mathbf{x}}-\hat{\mathbf{y}})/\sqrt{2}$ ($-45°$ with respect to the $\hat{\mathbf{x}}$ axis).

a. (15 points) What is the state of polarization of the diffracted light that hits the center of the screen, at $x = y = 0$? Explain.

Solution. Light that exits the top slit (slow axis at $+45°$) is in a state

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} ,$$

or left-hand circular polarization; light that exits the bottom slit (slow axis at $-45°$) is in a state of polarization

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} ,$$

or right-hand circular polarization. At the center of the screen, light from each slit contributes equally; the state of polarization is proportional to

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} ;$$

it is $\hat{\mathbf{x}}$ polarized like the incident beam. (See Fowles page 34 and Table 2.1 for the Jones vectors and matrices.)

b. (20 points) At what diffracted angle θ_x does the first minimum of the irradiance occur?

Solution. Right- and left-hand polarized states are orthogonal; they do not interfere. To see this formally (though this is not required as part of the solution), consult Fowles Eq. 3.11; the interference term there is proportional to

$$\mathbf{E}_2^* \cdot \mathbf{E}_1 \propto \begin{pmatrix} 1 & -i^* \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 1 + i^2 = 0 .$$

Since there is no interference between the light from the top and bottom slit, the resulting irradiance is just twice that expected from a single slit of width a. According to Fowles Eq. 5.18, this pattern is proportional to

$$\left(\frac{\sin\beta}{\beta}\right)^2 ,$$

where in this problem's notation

$$\beta = \frac{1}{2}ka\sin\theta_x$$

and $k = 2\pi/\lambda$. The first minimum occurs at $\beta = \pi$, or

$$\sin\theta_x = \frac{\lambda}{a} .$$

3. (35 points)
A lens has an f-number (ratio of focal length to diameter) equal to F. The lens is used to concentrate sunlight on a ball whose diameter is equal to the diameter of the sun's image. The ball is convectively and conductively insulated, but it freely radiates energy outward so that its temperature can approach an equilibrium value T_b.

a. (10 points) The sun subtends a half-angle of ≈ 0.005 radians. Is the size of its image "diffraction-limited", *i.e.* determined largely by the effects of diffraction? Make an order-of-magnitude argument assuming that the lens is a typical camera lens, with a radius of order 10^{-2} m.

Solution. The image of a distant point source formed at the focal plane of a lens is actually a Fraunhofer diffraction pattern where the aperture is the lens opening. The image becomes diffraction limited when the size of the image is near the size of an Airy disk. From this condition, we have the *Rayleigh criterion*:

$$2\theta > \frac{1.22\lambda}{D} ,$$

where θ is the half-angle subtended by the sun (≈ 0.005 rad), λ is the wavelength of the light, and D is the diameter of the lens. We can assume $\lambda \approx 600$ nm for the sun, $D = 2 \times 10^{-2}$ m, so the image is not diffraction limited if

$$\theta > 2.5 \times 10^{-5} \text{ rad} ,$$

which is clearly satisfied in this problem. Thus the image is not diffraction limited.

(Note that this question requires only an order-of-magnitude analysis. So you don't need to know anything about the details of Rayleigh's criterion, or Airy disks, to get full credit; all that you need to say is that the diffraction angle is of order λ/D, which here is much smaller than the sun's angular width.)

b. (25 points) Assuming the sun to be a blackbody of temperature T, calculate the ball's temperature T_b. Neglect reflection by the lens. (*Hint:* your answer should depend only on T and F.)

Solution. The sun radiates total power

$$P_S = \sigma T_S^4 \cdot 4\pi R_S^2 ,$$

where T_S is the sun's surface temperature and R_S is the sun's radius. The lens collects a fraction of this light power given by

$$\frac{\Delta\Omega}{\Omega} = \frac{\pi(D/2)^2}{4\pi R_{ES}^2} ,$$

where R_{ES} is the distance from the earth to the sun. The entirety of this light is focused on the ball. The ball re-radiates power

$$P_b = \sigma T_b^4 \cdot 4\pi R_b^2 .$$

In equilibrium we have the light power absorbed by the ball equal to the light power radiated by the ball. Setting the two equal, we obtain

$$T_S^4 \frac{D^2}{4} \frac{R_S^2}{R_{ES}^2} = T_b^2 r_b^2 .$$

To relate this result to F, we note that $2r_b/f = 1/F$ where f is the focal length. Also we have $r_b/f \approx R_S/R_{ES} \approx \theta$, where θ is the half-angle subtended by the sun. Employing the above relations in the equation relating the sun's temperature to the ball's temperature:

$$T_b^4 = T_S^4 \frac{d^2\theta^2}{4r_b^2} = \frac{1}{16F^2} T_S^4 .$$

So we find that

$$T_b = \frac{T_S}{2}\sqrt{\frac{1}{F}} .$$

4. (30 points)
You are given a Hamiltonian

$$\mathcal{H} = \frac{1}{2}(LR + RL) ,$$

where R and L are two operators such that

$$[L, R] = E_0$$

with E_0 a constant. You are also given an eigenfunction $u_E(x)$ of $\mathcal{H}$, such that

$$\mathcal{H}u_E = Eu_E \ ,$$

where E, another constant, is the energy eigenvalue.

Prove that

$$\mathcal{H}(Ru_E) = (E + E_0)(Ru_E) \ ,$$

i.e. R is a *raising operator*.

Solution.

$$\begin{aligned}
H(Ru) &= \frac{1}{2}\big(LRR + RLR\big)u \\
&= \frac{1}{2}\big(LRR - RLR + 2RLR\big)u \\
&= \frac{1}{2}\big([L,R]R + 2RLR\big)u \\
&= \frac{1}{2}\big(E_0R + 2RLR\big)u \\
&= \frac{1}{2}\big(E_0R + RLR - RRL + RRL + RLR\big)u \\
&= \frac{1}{2}\big(E_0R + R[L,R] + RRL + RLR\big)u \\
&= \frac{1}{2}\big(2E_0R + R(RL + LR)\big)u \\
&= \frac{1}{2}\big(2E_0R + R(2H)\big)u \\
&= (E + E_0)Ru \ .
\end{aligned}$$

5. (35 points).
Consider a harmonic oscillator potential

$$V(x) = \frac{1}{2}m\omega_0^2x^2$$

in one dimension. An *even* number N of particles of mass m are placed in this potential. There are no special interactions between the particles – no significant mutual electrostatic repulsion, gravitational attraction, *etc.*, compared to the strength of their interaction with the harmonic potential itself.

You may use what you already know about the levels of a harmonic oscillator.

The system is in its ground state, *i.e.* $T = 0$ Kelvin.

Calculate the total energy E of the N-particle system, relative to the bottom of the well, for the cases

a. (10 points) The N particles are *distinguishable*.

Solution. The energy levels of a harmonic oscillator are

$$E_n = \Big(n + \frac{1}{2}\Big)\hbar\omega_0 \ ,$$

where $n = 0, 1, 2 \ldots$. Nothing prevents mutually noninteracting distinguishable particles from occupying the same spatial wavefunction. At $T = 0$ this will be the ground state $n = 0$. Then

$$E = \frac{N}{2}\hbar\omega_0 \ .$$

b. (10 points) The N particles are identical *bosons*.

Solution. Same as (**a.**). All the identical bosons are in the ground state at $T = 0$.

c. (15 points) The N particles are identical spin $\frac{1}{2}$ *fermions*.

Solution. Each spatial wavefunction can accommodate two identical spin $\frac{1}{2}$ fermions, one with spin up, one with spin down. At $T = 0$ the lowest occupied state is the ground state, with energy

$$E_0 = \frac{1}{2}\hbar\omega_0 \ ,$$

while the highest-energy occupied state has (Fermi) energy equal to

$$E_F = E_0 + \Big(\frac{N}{2} - 1\Big)\hbar\omega_0 \ .$$

The total energy is N times the average energy, which is the mean of E_0 and E_F:

$$\begin{aligned}
E &= N\frac{E_0 + E_F}{2} \\
&= \frac{N}{2}\Big(2E_0 + \Big(\frac{N}{2} - 1\Big)\hbar\omega_0\Big) \\
&= \frac{N^2}{4}\hbar\omega_0 \ .
\end{aligned}$$

6. (30 points)
In the rest frame $\mathcal{S}'$ of a star, ignoring the gravitational redshift, some of the photons emitted by the star arise from a particular atomic transition with an unshifted wavelength λ'. When these photons are observed on earth, they are shifted to longer wavelength $\lambda = \lambda' + \Delta\lambda$ because the star is receding from the earth with velocity $\beta_0 c$ due to the Hubble expansion of the universe. Astronomers measure this redshift by means of the parameter z, defined by

$$z \equiv \frac{\Delta\lambda}{\lambda'} .$$

For light, the relativistic Doppler shift is

$$\omega = \frac{\omega'}{\gamma_0(1-\beta_0\cos\theta)} .$$

a. (10 points) In the "neighboring star" limit $\beta_0 \ll 1$, show that β_0 is approximately equal to the measured z.

Solution.

$$\begin{aligned}
\omega &= \frac{\omega'}{\gamma_0(1-\beta_0\cos\theta)} \\
&= \frac{2\pi c}{\lambda} \\
1/\lambda &= \frac{1/\lambda'}{\gamma_0(1-\beta_0\cos\theta)} \\
\frac{\lambda}{\lambda'} &= \gamma_0(1-\beta_0\cos\theta) \\
\frac{\Delta\lambda}{\lambda'} &= \gamma_0(1-\beta_0\cos\theta) - 1 \\
z &= \gamma_0(1-\beta_0\cos\theta) - 1 .
\end{aligned}$$

For a receding star, $\cos\theta = -1$, and so

$$z = \gamma_0(1+\beta_0) - 1 .$$

When $\beta_0 \ll 1$, $\gamma_0 \approx 1$ to *second* order in β_0. Then

$$z \approx 1 + \beta_0 - 1 = \beta_0 .$$

b. (10 points) In the "distant star" limit $\gamma_0 \gg 1$, derive an expression for γ_0 in terms of the measured z.

Solution. When $\gamma_0 \gg 1$, $\beta_0 \approx 1$. Then, from the solution to (**a.**),

$$\begin{aligned}
z &\approx 2\gamma_0 - 1 \\
\gamma_0 &\approx \frac{1+z}{2} .
\end{aligned}$$

Full credit is given with or without the "1" term.

c. (10 points) The observation of Supernova 1987A marked the dawn of a new astronomy, in which humans are able to detect fermions (neutrinos) as well as bosons (photons) from (spatially or temporally) resolved sources outside the solar system. About a dozen such neutrinos were detected in each of two huge underground water tanks. The photons from Supernova 1987A were redshifted by

$$z \approx 10^{-5} .$$

Taking the Hubble constant to be

$$H_0 \approx 0.7 \times 10^{-10}\mathrm{yr}^{-1} ,$$

for how many years did the neutrinos from SN1987A travel before humans observed them?

Solution. From Rohlf Eq. (19.17) (necessary for solving assigned problem 19.18), the velocity v with which SN1987A is receding from Earth is

$$v = H_0 d ,$$

where d is its present distance from Earth. Neutrinos are nearly massless and travel essentially at the speed of light c. Using the result of part (**a.**), the travel time T of the neutrinos from SN1987A was

$$\begin{aligned}
T &= \frac{d}{c} \\
&= \frac{v}{H_0 c} \\
&= \frac{\beta_0}{H_0} \\
&\approx \frac{z}{H_0} \\
&\approx \frac{1\times 10^{-5}}{0.7\times 10^{-10}\ \mathrm{yr}^{-1}} \\
&\approx 1.4 \times 10^5\ \mathrm{yr} .
\end{aligned}$$