

**SOLUTION TO PROBLEM SET 9**

*Solutions by P. Pebler*

**1** *Purcell 5.3* A beam of 9.5 MeV electrons ( $\gamma = 20$ ) amounting as current to  $0.05 \mu\text{A}$ , is traveling through vacuum. The transverse dimensions of the beam are less than 1 mm, and there are no positive charges in or near it. In the lab frame, what is approximately the electric field strength 1 cm away from the beam, and what is the average distance between the electrons, measured parallel to the beam? Answer the same questions for the electron rest frame.

With  $\gamma = 20$ , the speed of the electrons will be essentially  $c$ . The number of electrons passing per second is

$$\nu_e = \frac{0.05 \times 10^{-6} \text{ A}}{1.6 \times 10^{-19} \text{ C}} = 3.1 \times 10^{11} \text{ 1/s} .$$

The mean distance between electrons is

$$d = \frac{c}{\nu_e} = 1 \text{ mm} ,$$

and the charge per unit length is

$$\lambda = -\frac{4.8 \times 10^{-10} \text{ esu}}{0.1 \text{ cm}} = -4.8 \times 10^{-9} \text{ esu/cm} .$$

If we think of this as a continuous line charge, the field strength a distance 1 cm away is

$$E = -\frac{2\lambda}{r} = 9.6 \times 10^{-9} \text{ statvolt/cm} .$$

Since the electrons are moving, the distance between them will be contracted. Therefore, in the electron rest frame, they will be more spread out so that

$$d' = \gamma d = 2 \text{ cm} .$$

We might find a new field strength by

$$E' = \frac{E}{\gamma} = 4.8 \times 10^{-10} \text{ statvolt/cm} ,$$

however, this will be only the average field strength along a line parallel to the beam. Since the electrons are so far apart in this frame, there will be big variation in the field.

**2** *Purcell 5.7* A moving proton has  $\gamma = 10^{10}$ . How far away from such a proton would the field rise to 1 V/m as it passes?

The electric field strength of a moving point charge is (in SI units)

$$E' = \frac{Q}{4\pi\epsilon_0 r'^2} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta')^{3/2}} .$$

The maximum field strength is directly beside the particle where  $\theta' = \pi/2$ . Then

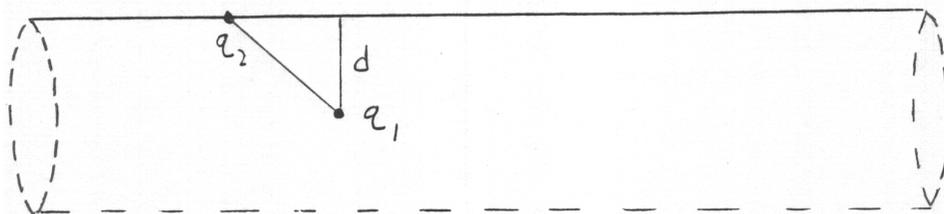
$$E' = \frac{Q}{4\pi\epsilon_0 r'^2} \frac{1}{\sqrt{1 - \beta^2}} = \gamma \frac{Q}{4\pi\epsilon_0 r'^2} .$$

We want the distance where the field is  $1 \text{ V/m}$ .

$$\frac{10^{10}(1.6 \times 10^{-19} \text{ C})}{4\pi(8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2)r'^2} = 1 \text{ V/m}$$

$$r' = 3.8 \text{ m}$$

**3 Purcell 5.10** In the rest frame of a particle with charge  $q_1$  another particle with charge  $q_2$  is approaching, moving with velocity  $v$  not small compared with  $c$ . If it continues to move in a straight line, it will pass a distance  $d$  from the position of the first particle. It is so massive that its displacement from the straight path during the encounter is small compared with  $d$ . Likewise, the first particle is so massive that its displacement from its initial position is small compared with  $d$ . Show that the increment in momentum acquired by each particle as a result of the encounter is perpendicular to  $\mathbf{v}$  and has magnitude  $2q_1q_2/vd$ . Expressed in terms of other quantities, how large must the masses of the particles be to justify our assumptions?



In the first approximation, we consider the trajectory to be a straight line. For this trajectory, the net impulse in the  $x$  direction will be zero. Using cylindrical coordinates,

$$\Delta p_r = \int F_r dt = \frac{1}{v} \int F_r dx = \frac{q_2}{2\pi dv} \int E_r 2\pi d dx \quad .$$

The purpose of transforming this integral in this way is so that it becomes the electric flux through an infinite cylinder of radius  $d$ . We can evaluate it easily by using Gauss's law.

$$\Delta p_r = \frac{q_2}{2\pi dv} \int \mathbf{E} \cdot d\mathbf{a} = \frac{q_2}{2\pi dv} 4\pi q_1 = \frac{2q_1q_2}{dv}$$

In the rest frame of  $q_2$ , the situation is reversed, and  $q_1$  acquires the same  $y$  momentum in the opposite direction. This will be the same momentum in the rest frame of  $q_1$  because the perpendicular component of the momentum is unchanged by a Lorentz transformation.

For this approximation to be good, the acquired  $y$  momentum must be much smaller than the  $x$  momentum, so that

$$\gamma mv \gg \frac{2q_1q_2}{dv} \quad ,$$

where  $m$  can represent either mass.

**4 Purcell 5.17** Two protons are moving parallel to one another a distance  $r$  apart, with the same velocity  $\beta c$  in the lab frame. At the instantaneous position of one of the protons the electric field strength caused by the other is  $\gamma e/r^2$ . But the force on the proton measured in the lab frame is not  $\gamma e^2/r^2$ . Verify that by finding the force in the proton rest frame and transforming that force back to the lab frame. Show that the discrepancy can be accounted for if there is a magnetic field  $\beta$  times as strong as the electric field, accompanying this proton as it travels through the lab frame.

In the rest frame of the protons, the force is  $e^2/r^2$ . In the lab frame this force is

$$F_y = \frac{1}{\gamma} \frac{e^2}{r^2} = \gamma \frac{e^2}{r^2} + F_y^{(b)} \quad ,$$

where the first term is the electric force in the lab frame. The extra term is

$$F_y^{(b)} = \gamma \frac{e^2}{r^2} \left( \frac{1}{\gamma^2} - 1 \right) = -\gamma \beta^2 \frac{e^2}{r^2} = - \left( \beta \gamma \frac{e}{r^2} \right) e \frac{v}{c} \quad .$$

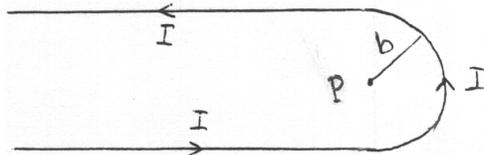
From the Lorentz force law

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{B} \quad ,$$

we can account for this extra force with a magnetic field out of the page

$$\mathbf{B} = \beta E \hat{\mathbf{z}} \quad .$$

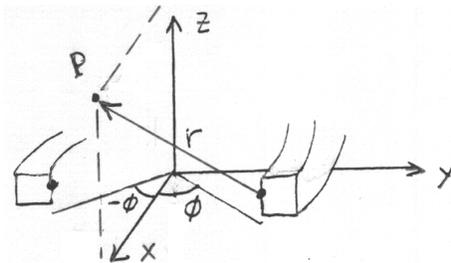
**5 Purcell 6.4** A long wire is bent into the hairpin like shape shown. find an exact expression for the magnetic field at the point  $P$  which lies at the center of the half-circle.



In the integral for the magnetic field of an infinite wire, each half of the wire contributes the same amount in the same direction, so for each of the half wires, we can take half of the formula for an infinite wire. Also, the half circle will contribute half of the ring formula. All contributions point out of the page so

$$B_z = \frac{1}{2} \frac{2\pi I}{cb} + \frac{1}{2} \frac{2I}{cb} + \frac{1}{2} \frac{2I}{cb} = \frac{(2 + \pi)I}{bc} \quad .$$

**6 Purcell 6.14** A coil is wound evenly on a torus of rectangular cross section. There are  $N$  turns of wire in all. Assume that the current on the surface of the torus flows exactly radially on the annular end faces, and exactly longitudinally on the inner and outer cylindrical surfaces. Show that the magnetic field everywhere would be circumferential. Second, prove that the field is zero at all points outside the torus, including the interior of the central hole. Third, find the magnitude of the field inside the torus as a function of radius.



We set a coordinate system so the point  $P$  under consideration is in the  $z - x$  plane with coordinates  $(x_o, 0, z_o)$ . Consider a current loop at an angle  $\phi$  from the  $x$  axis. One small section of the loop has coordinates  $(r \cos \phi, r \sin \phi, z)$  and

$$\mathbf{r} = (x_o - r \cos \phi, -r \sin \phi, z_o - z) \quad .$$

The direction of this little current is arbitrary within the plane so the current direction vector can be written

$$\mathbf{I} = I_r \hat{\mathbf{r}} + I_z \hat{\mathbf{z}} = (I_r \cos \phi, I_r \sin \phi, I_z) \quad .$$

Then the contribution to the magnetic field will be in the direction

$$\mathbf{I} \times \mathbf{r} = [\sin \phi (I_r (z_o - z) + r I_z)] \hat{\mathbf{x}} + [I_z (x_o - r \cos \phi) - I_r \cos \phi (z_o - z)] \hat{\mathbf{y}} - I_r x_o \sin \phi \hat{\mathbf{z}} \quad .$$

However, for each section of current, there is a similar section that is identical except that  $\phi \rightarrow -\phi$ . We see that the  $x$  and  $z$  components change sign. Therefore, these components cancel out and the net field at the point  $P$  is in the  $y$  direction. For our coordinate system, this is in the circumferential direction.

Once we have this information, it is easy to use Ampere's law.

$$\int \mathbf{B} \cdot d\mathbf{l} = 2\pi r B = \frac{4\pi}{c} NI$$

$$B = \frac{2NI}{cr} \quad a < r < b$$

If we are outside the solenoid, the enclosed current will be zero and so the field is zero.

**7 Purcell 6.18** *Two long coaxial aluminum cylinders are charged to a potential difference of 50 statvolts. The inner cylinder has an outer diameter of 6 cm, the outer cylinder an inner diameter of 8 cm. With the outer cylinder stationary the inner cylinder is rotated around its axis at a constant frequency of 30 Hz. Describe the magnetic field this produces and determine its intensity in gauss. What if both cylinders are rotated in the same direction at 30 Hz?*

The capacitance for two coaxial cylinders is

$$C = \frac{L}{2 \ln(b/a)} \quad .$$

Assuming we have equal amounts of positive and negative charge on the inside and outside respectively, we can find the charge per unit length.

$$\frac{Q}{L} = \frac{CV}{L} = \frac{V}{2 \ln(b/a)} = 87 \text{ esu/cm}$$

The spinning charged cylinders are essentially perfect solenoids. The field of a long solenoid is constant inside and practically zero outside. The solenoid formula is

$$B = \frac{4\pi I n}{c} \quad ,$$

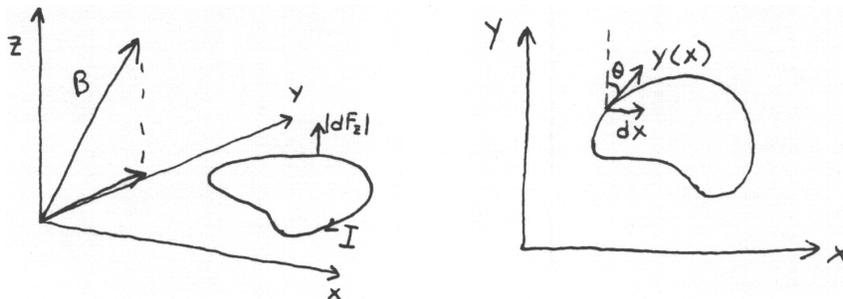
where  $n$  is the number of turns per unit length. The quantity  $nI$  can be thought of as the charge per unit time per unit length passing through a line running parallel to the cylinder. You should be able to convince yourself that this quantity in our case is  $Q\nu/L$ , where  $\nu$  is the frequency of revolution. With just the inner cylinder rotating with  $\nu = 30 \text{ Hz}$ ,

$$B = \frac{4\pi Q\nu}{cL} = \frac{4\pi(87 \text{ esu/cm})(30 \text{ Hz})}{3 \times 10^{10} \text{ cm/s}} = 1.1 \times 10^{-6} \text{ gauss} \quad r < a \quad ,$$

and the field elsewhere is zero. With a clockwise rotation as shown, the field is into the page.

If the outer cylinder rotates in the same direction, it will produce a field with the same magnitude but opposite direction. These two fields will cancel for  $r < a$ , but for  $a < r < b$ , the field is  $1.1 \times 10^{-6}$  gauss out of the page. It is zero again for  $r > b$ .

**8 Purcell 6.22** A constant  $\mathbf{B}$  field lies in the  $y-z$  plane. An arbitrary current loop lies in the  $x-y$  plane. Show, by calculating the torque about the  $x$  axis, that the torque on the current loop can be written  $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ , where the magnetic moment  $\mathbf{m}$  of the loop is defined as a vector of magnitude  $IA/c$  where  $I$  is the current in esu/s and  $a$  is the area of the loop in  $\text{cm}^2$ , and the direction of the vector is normal to the loop with a right-hand relation to the current. What about the net force on the loop?



The force on a small section of loop is

$$d\mathbf{F} = \frac{1}{c} I d\mathbf{l} \times \mathbf{B} .$$

Only the part of the force in the  $z$  direction will contribute to the torque about the  $x$  axis. We split the magnetic field into two parts in the  $z$  and  $y$  directions.  $d\mathbf{l} \times (B_z \hat{\mathbf{z}})$  is in the  $x-y$  plane, so we only need the  $y$  part of  $\mathbf{B}$ .

$$dF_z = \frac{1}{c} I B_y \sin \theta dl$$

From the figure we see that  $dl \sin \theta = dx$  so that  $dF_z = \frac{1}{c} I B_y dx$  and the torque is

$$N_x = \int y dF_z = \frac{1}{c} I B_y \int_{loop} y(x) dx .$$

This integral if we go forward along the top of the loop and back along the bottom is the area of the loop and

$$N_x = \frac{1}{c} I B_y a .$$

With  $m = Ia/c$  and pointing in the  $-z$  direction,

$$\mathbf{m} \times \mathbf{B} = \mathbf{m} \times (B_y \hat{\mathbf{y}}) = m B_y \hat{\mathbf{x}} = \frac{1}{c} I B_y a \hat{\mathbf{x}} .$$

For a constant field there is no net force because

$$\mathbf{F} = \int d\mathbf{F} = \int \frac{1}{c} I d\mathbf{l} \times \mathbf{B} = -\frac{1}{c} I \mathbf{B} \times \int d\mathbf{l}$$

and  $\int d\mathbf{l}$  around the loop is zero.