University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 6

Solutions by P. Pebler

1 *Purcell 2.27 The electrostatic potential at a point on the edge of a disc of radius r and uniform charge density σ is $\phi = 4\sigma r$. Calculate the energy stored in the electric field of a charged disk of radius a.*

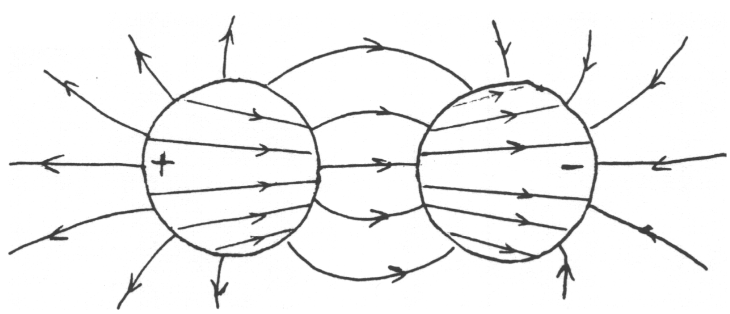
We calculate the total energy by bringing in each infinitesimal ring of charge from infinity and adding up the energy for each ring. We assume that we have already built up the disc to radius r. We now bring in a ring of width dr and stick it on the edge. Recall that the energy necessary to bring in a test charge from infinity to some point is just the potential at that point times the charge. (This is more or less the definition of the potential.) The potential just outside the disc where we are packing on the next ring is $4\sigma r$. The energy necessary is then

$$dU = \phi(r)dq = (4\sigma r)(2\pi r dr \sigma) = 8\pi\sigma^2 r^2\, dr.$$

To add up all the rings integrate from 0 to a.

$$U = 8\pi\sigma^2 \int_0^a r^2\, dr = \frac{8}{3}\pi a^3\sigma^2 = \frac{8}{3}\pi a^3\left(\frac{Q}{\pi a^2}\right)^2 = \frac{8Q^2}{3\pi a}$$

2 *Purcell 2.29 Two nonconducting spherical shells of radius a carry charges of Q and $-Q$ uniformly distributed over their surfaces. The spheres are brought together until they touch. What does the electric field look like, both outside and inside the shells? How much work is needed to move them far apart?*



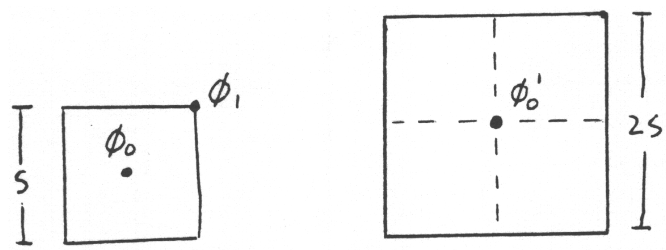
The field of a uniformly charged shell is zero inside the shell and that of a point charge outside. Outside both shells, we have the field of two point charges. Inside either shell, the field is that of a single point charge at the center of the other shell.

To find the energy we use the following argument. Consider instead a uniform shell of charge $-Q$ and a point charge Q a distance r from the center of the shell (but outside it). We know that outside the shell, the potential due to the shell is just $-Q/r$, so the energy needed to bring in the point charge is $-Q^2/r$ and the energy needed to move it out is Q^2/r. However, this must be the same energy as that required to move out the shell while keeping the point charge fixed. So we find that the energy needed to move a shell out to infinity in the field of a point charge is Q^2/r. But since the other shell creates the field of a point charge outside of it, this is also the energy needed to separate our two shells.

$$E = \frac{Q^2}{2a}$$

If you don't like this argument, you can integrate a shell distribution times the potential of a point charge which isn't too hard and find the same answer.

3 *Purcell 2.30* *Consider a cube with sides of length b and constant charge density ρ. Denote by ϕ_o the potential at the center of the cube and ϕ_1 the potential at a corner, with zero potential at inifinity. Determine the ratio ϕ_o/ϕ_1.*



We imagine another cube with the same charge density but with twice the side length. Let the potential at the center of this cube be ϕ_o'. The point at the center of this new cube lies at the corner of each of eight cubes of the original size. Because the potential is additive, we have

$$\phi_o' = 8\phi_1.$$

We can also use dimensional arguments to find ϕ_o'. We can write

$$\phi_o = f(Q, s),$$

where Q is the total charge, s is the side length and the functional form of f depends on the shape and nature of the distribution. We can now ask for what's called a scaling law which tells us what happens if we multiply the variables Q and s by numerical factors while keeping all other details of the distribution the same. Whatever the functional form of f is, we know it has units of charge per length, the units of the potential. Fortunately, the only parameters carrying units which enter into f are Q and s. The only way then to get the right units is if

$$f(Q, s) \propto \frac{Q}{s}.$$

The function f then satisfies the simple scaling

$$f(\alpha Q, \beta s) = \frac{\alpha}{\beta} f(Q, s).$$

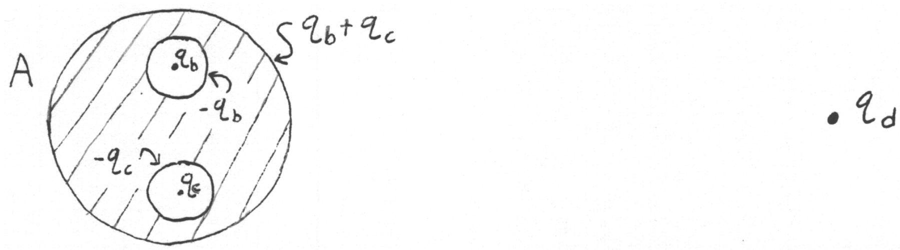
In our case $s' = 2s$ and because we are keeping the charge density constant, $Q' = \rho s'^3 = \rho(2s)^3 = 8Q$. Then

$$\phi_o' = f(8Q, 2s) = \frac{8}{2} f(Q, s) = 4\phi_o,$$

$$4\phi_o = 8\phi_1,$$

$$\frac{\phi_o}{\phi_1} = 2.$$

4 *Purcell 3.1* *A spherical conductor A contains two spherical cavities. The total charge on the conductor is zero. There are point charges q_b and q_c at the center of each cavity. A considerable distance r away is another charge q_d. What force acts on each of the four objects A, q_b, q_c, q_d? Which answers, if any, are only approximate, and depend on r being relatively large?*

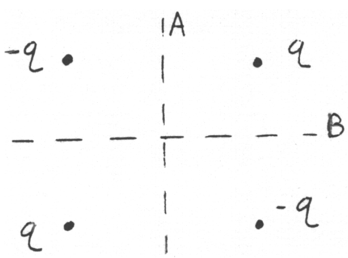


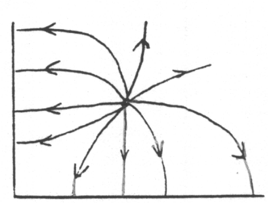
The force on q_b and q_c is zero. The field inside the spherical cavity is quite independent of anything outside. A charge $-q_b$ is uniformly distributed over the conducting surface to cancel the field from the point charge. The same happens with q_c. This leave an excess charge of $q_c + q_b$ on the outside surface of the conductor. If q_d were absent, the field outside A would be the symmetrical, radial field $E = |q_b + q_c|/r^2$, the same as a point charge because the excess charge would uniformly distribute itself over the spherical outer surface. The influence of q_d will slightly alter the distribution of the charge on A, but without affecting the total amount. Hence for large r, the force on q_d will be approximately

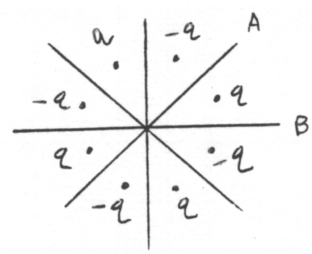
$$\mathbf{F}_d = \frac{q_d(q_b + q_c)}{r^2}\,\hat{\mathbf{r}}.$$

The force on A must be precisely equal and opposite to the force on q_d.

5 *Purcell 3.9* *Two charges q and two charges $-q$ lie at the corners of a square with like charges opposite one another. Show that there are two equipotential surfaces that are planes. Obtain and sketch qualitatively the field of a single point charge located symmetrically in the inside corner formed by bending a metal sheet through a right angle. Which configurations of conducting planes and point charges can be solved this way and which can't?*







The potential on each of the two lines A and B shown is zero because the contribution at each point on either line from any charge is cancelled by the opposite charge directly across from it. Therefore, the field of a point charge in the corner of a bent conductor is the same as the field from these four point charges. You should be able to see by looking at the first few cases that this strategy will work any time we divide the space into an even number of wedges. This allows the contributions to the potential to cancel pairwise. For example, in the picture at right the potential is zero on lines A and B because all the charges come in equal and opposite pairs. The applicable angles are $\theta_n = 2\pi/(2n) = \pi/n$, where n is an integer. This would not work for an angle of 120^o.

6 *Purcell 3.17* *A spherical vacuum capacitor has radius a for the outer sphere. What radius b should be chosen for the inner spherical conductor to store the greatest amount of electrical energy subject to the constraint that the electric field strength at the surface of the inner sphere may not exceed E_o? How much energy can be stored?*

We first need the capacitance of this capacitor. Assuming there is a charge Q on the inner shell and a charge $-Q$ on the outer shell, the field between the shells is

$$\mathbf{E} = \frac{Q}{r^2}\,\hat{\mathbf{r}}.$$

The potential difference is

$$V = -\int_a^b \frac{Q}{r^2}\,dr = Q\left(\frac{a-b}{ab}\right),$$

and the capacitance

$$C = Q/V = \frac{ab}{a-b}.$$

The energy stored by this capacitor is

$$U = \frac{1}{2C}Q^2 = \frac{1}{2}\frac{a-b}{ab}Q^2.$$

The energy in the capacitor will depend on how much charge is on it. If we were allowed to put arbitrary amounts on, the energy would have no maximum. However, for a given b, the maximum field near the inner sphere gives us the maximum allowed charge. This gives us the maximum stored energy for a given capacitor.

$$E_o = \frac{Q_{max}}{b^2}$$

$$U_{max} = \frac{1}{2}\frac{a-b}{ab}E_o^2 b^4 = \frac{1}{2}\frac{ab^3-b^4}{a}E_o^2$$

Now we want to choose a b to make this as large as possible.

$$\frac{\partial U_{max}}{\partial b}(b_{max}) = \frac{1}{2}\frac{3ab^2-4b^3}{a}E_o^2 = 0$$

$$3a - 4b_{max} = 0$$

$$b_{max} = \frac{3}{4}a$$

The energy is then

$$U_{max} = \frac{1}{2}\frac{a-\left(\frac{3}{4}a\right)}{a\left(\frac{3}{4}a\right)}E_o^2\left(\frac{3}{4}a\right)^4 = \frac{27}{512}E_o^2 a^3.$$

7 *Purcell 3.23* *Find the capacitance of a capacitor that consists of two coaxial cylinder of radii a and b and length L. Assume $L \gg b-a$ so that end corrections may be neglected. Check your result in the limmit $b-a \ll a$ with the formula for the parallel-plate capacitor.*

A cylinder of 2.00 *in outer diameter hangs with its axis vertical from one arm of a beam balance. The lower portion of the hanging cylinder is surrounded by a stationary cylinder with inner diameter* 3.00 *in. Calculate the magnitude of the force down when the potential difference between the two cylinders is* 5 *kV.*

The field between charged cylinders is

$$\mathbf{E} = \frac{2\lambda}{r}\,\hat{\mathbf{r}} = \frac{2Q}{rL}\,\hat{\mathbf{r}},$$

assuming we have Q on the inside and $-Q$ on the outside. The potential difference is

$$V = \int_a^b \frac{2Q}{L}\frac{dr}{r} = \frac{2Q}{L}\ln\frac{b}{a}.$$

Just arrange your signs so that the capacitance comes out positive.

$$C = \frac{L}{2\ln(b/a)}$$

Let us now consider the general case where the potential difference is being held constant by a battery while the capacitance is changing. Initially we have charge and energy

$$Q = CV \qquad U = \frac{1}{2}CV^2.$$

After a change in capacitance ΔC,

$$Q' = (C + \Delta C)V = Q + V\Delta C \qquad U' = \frac{1}{2}(C + \Delta C)V^2.$$

The battery has done work on this system by moving this extra charge across the potential difference.

$$W_b = (\Delta C)V^2$$

If the change in capacitance is caused by movement of the components, the electric field does work on the plates or plate.

$$W = F(\Delta L)$$

From conservation of energy we have

$$U + W_b = U' + W,$$

$$(\Delta C)V^2 = \frac{1}{2}(\Delta C)V^2 + W,$$

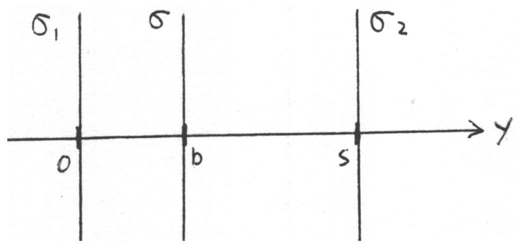
$$W = F(\Delta L) = \frac{1}{2}(\Delta C)V^2,$$

$$F = \frac{1}{2}V^2\frac{\partial C}{\partial L}.$$

In our case we have

$$F = \frac{1}{2}V^2\frac{1}{2\ln(b/a)} = \frac{1}{2}\frac{(16.7\ statvolts)^2}{2\ln(3/2)} = 172\ dynes.$$

8 *Purcell 3.24 Two parallel plates are connected by a wire. Let one plate coincide with the xz plane and the other with the plane $y = s$. The distance s is much smaller than the lateral dimensions of the plates. A point charge Q is located between the plates at $y = b$. What is the magnitude of the total surface charge on the inner surface of each plate?*



The total induced charge is $-Q$. We need to find the fraction of induced charge on either conductor. For this we may notice that the fraction of induced charge on both planes will be the same for any distribution located at $y = b$ because we may view it as the superposition of many little point charges. So we want to consider the simplest possible case which is a uniformly charged plane. (Once again, we are ignoring edge effects.) Using a Gaussian pillbox with its left face inside the left plate and its right face at y, where $0 < y < b$, the field in the left region is

$$E_l = 4\pi\sigma_1\hat{\mathbf{y}}.$$

Similarly, the field in the right region is

$$E_r = -4\pi\sigma_2\hat{\mathbf{y}}.$$

Since the two conductors are connected by a wire, they are at the same potential so the line integral from the middle to the left and right should be the same.

$$4\pi\sigma_1(-b) = -4\pi\sigma_2(s-b)$$

$$\frac{\sigma_2}{\sigma_1} = \frac{b}{s-b}$$

Now switch back to the original problem.

$$\frac{Q_2}{Q_1} = \frac{b}{s-b} \qquad Q_1 + Q_2 = -Q$$

$$Q_1 = -\frac{s-b}{s}Q \qquad Q_2 = -\frac{b}{s}Q$$