University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 4

1. RHK problem 26.36
Solution: Let
n = no. of moles of ideal monatomic gas = 1.00
R = universal gas constant = 8.314 J/mole·K
Then
(a)

$$\begin{aligned} W_{abc} &= W_{ab} + W_{bc} \\ &= -\int_a^b p\,dV + 0 \\ &= -p_0(4V_0 - V_0) \\ &= -3p_0V_0 \ . \end{aligned}$$

(b)

$$\begin{aligned} \Delta E_{\rm int}(b \to c) &= \frac{3}{2}nR(T_c - T_b) \\ &= \frac{3}{2}(p_cV_c - p_bV_b) \\ &= \frac{3}{2}(8p_0V_0 - 4p_0V_0) \\ &= 6p_0V_0 \ . \\ \Delta S_{bc} &= \int_b^c \frac{\delta Q}{T} \\ &= \int_b^c \frac{dE_{\rm int}}{T} - W_{bc} \\ &= \int_b^c \frac{dE_{\rm int}}{T} - 0 \\ E_{\rm int} &= \frac{3}{2}nRT \\ \Delta S_{bc} &= \int_b^c \frac{3}{2}nR\frac{dT}{T} \\ &= \frac{3}{2}nR\ln\left(\frac{T_c}{T_b}\right) \\ &= \frac{3}{2}nR\ln 2 \\ &= 8.644 \text{ J/K} \ . \end{aligned}$$

Note that the last result does not depend on p_0 or V_0, even though the problem asks us to express it in terms of p_0 and V_0.

(c)

$$\oint dE_{\rm int} \equiv 0 \ \ \text{(state function)} \ .$$

$$\oint dS \equiv 0 \ \ \text{(state function)} \ .$$

2. RHK problem 26.40
Solution: Let
n = no. of moles of ideal diatomic gas = 1.00
R = universal gas constant = 8.314 J/mole·K
Then
(a)

$$\begin{aligned} pV &= \text{constant} \ \ \text{(isotherm)} \\ p_2 &= p_1\frac{V_1}{V_2} \\ &= \frac{p_1}{3} \ . \\ pV^\gamma &= \text{constant} \ \ \text{(adiabat)} \\ p_3 &= p_1\left(\frac{V_1}{V_3}\right)^\gamma \\ \gamma &= \frac{7}{5} \ \ \text{(diatomic)} \\ p_3 &= p_1\left(\frac{1}{3}\right)^{7/5} \\ &= 0.215\,p_1 \ . \\ TV^{\gamma-1} &= \text{constant} \ \ \text{(adiabat)} \\ T_3 &= T_1\left(\frac{V_1}{V_3}\right)^{\gamma-1} \\ &= T_1\left(\frac{1}{3}\right)^{2/5} \\ &= 0.644\,T_1 \ . \end{aligned}$$

(b)

$$
\begin{aligned}
E_{\text{int}} &= \frac{5}{2}nRT \quad \text{(diatomic)} \\
\Delta E_{\text{int}}(1 \to 2) &= \frac{5}{2}nR(T_2 - T_1) \\
&= 0 \ . \\
\Delta E_{\text{int}}(2 \to 3) &= \frac{5}{2}nR(T_3 - T_2) \\
&= \frac{5}{2}nRT_1\left(\left(\frac{1}{3}\right)^{2/5} - 1\right) \\
&= -\frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) \\
&= -0.889\, p_1V_1 \ . \\
\oint dE_{\text{int}} &\equiv 0 \quad \text{(state function)} \\
\Delta E_{\text{int}}(3 \to 1) &= -\Delta E_{\text{int}}(1 \to 2) \\
&\quad - \Delta E_{\text{int}}(2 \to 3) \\
&= -0 + \frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) \\
&= 0.889\, p_1V_1 \ .
\end{aligned}
$$

$$
\begin{aligned}
W_{12} &= -\int_1^2 p\, dV \\
&= -\int_1^2 nRT_1 \frac{dV}{V} \\
&= -nRT_1 \ln \frac{V_2}{V_1} \\
&= -p_1V_1 \ln 3 \\
&= -1.099\, p_1V_1 \ . \\
W_{23} &= -\int_2^3 p\, dV \\
&= 0 \ . \\
W_{31} &= \Delta E_{\text{int}}(3 \to 1) - Q_{31} \\
&= \frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) - 0 \\
&= 0.889\, p_1V_1 \ .
\end{aligned}
$$

$$
\begin{aligned}
Q_{12} &= \Delta E_{\text{int}}(1 \to 2) - W_{12} \\
&= 0 + p_1V_1 \ln 3 \\
&= 1.099\, p_1V_1 \ . \\
Q_{23} &= \Delta E_{\text{int}}(2 \to 3) - W_{23} \\
&= -\frac{5}{2}p_1V_1\left(1 - \left(\frac{1}{3}\right)^{2/5}\right) - 0 \\
&= -0.889\, p_1V_1 \ . \\
Q_{31} &\equiv 0 \quad \text{(adiabat)} \ .
\end{aligned}
$$

$$
\begin{aligned}
\Delta S_{12} &= \int_1^2 \frac{\delta Q}{T} \\
T &= T_1 \quad \text{(isotherm)} \\
\Delta S_{12} &= \frac{Q_{12}}{T_1} \\
&= \frac{p_1V_1}{T_1} \ln 3 \\
&= nR \ln 3 \\
&= 9.134 \text{ J/K} \ . \\
\Delta S_{31} &\equiv 0 \quad \text{(adiabat)} \ . \\
\oint dS &\equiv 0 \quad \text{(state function)} \\
\Delta S_{23} &= -\Delta S_{12} - \Delta S_{31} \\
&= -nR \ln 3 - 0 \\
&= -9.134 \text{ J/K} \ .
\end{aligned}
$$

3. RHK problem 26.43

Solution: In Physics H7B, all problems involving numbers should be solved completely in terms of algebraic symbols before any numbers are plugged in (otherwise it is much more difficult to give part credit). Let

m_1 = initial amount of water = 1.780 kg
(initial amount of ice is 0.262 kg)
m_2 = final amount of water = $(1.780 + 0.262)/2$ = 1.021 kg
L_f = latent heat of fusion of water = 333000 J/kg
T_0 = temperature of melting ice = 273 K
Then

(a)

$$
\begin{aligned}
Q(m_1 \to m_2) &= L_f(m_2 - m_1) \\
\Delta S(m_1 \to m_2) &= \int_1^2 \frac{\delta Q}{T} \\
T &\equiv T_0 \\
\Delta S(m_1 \to m_2) &= \frac{Q(m_1 \to m_2)}{T_0} \\
&= \frac{L_f(m_2 - m_1)}{T_0} \\
&= -\frac{L_f(m_1 - m_2)}{T_0} \\
&= -925.8 \text{ J/K} .
\end{aligned}
$$

(b)

$$
\begin{aligned}
\oint dS &\equiv 0 \quad \text{(state function)} \\
\Delta S(m_2 \to m_1) &= -\Delta S(m_1 \to m_2) \\
&= \frac{L_f(m_1 - m_2)}{T_0} \\
&= 925.8 \text{ J/K} .
\end{aligned}
$$

(c) Here the change of entropy of the environment in this cycle is calculated assuming that the heat to melt the ice is supplied at a temperature $T_{>0}$ which is greater than T_0, for example by a Bunsen burner. Nevertheless, using the fact that the entropy of the environment is a state variable, we calculate its change by making use of a hypothetical reversible process, $\Delta S = Q/T$:

$$
\begin{aligned}
\oint dS_{\text{icewater}} &\equiv 0 \\
\oint dS_{\text{environ}} &= -\frac{Q(m_1 \to m_2)}{T_0} - \frac{Q(m_2 \to m_1)}{T_{>0}} \\
&= -\Big(-\frac{L_f(m_1 - m_2)}{T_0}\Big) \\
&\quad - \frac{L_f(m_1 - m_2)}{T_{>0}} \\
&= L_f(m_1 - m_2)\Big(\frac{1}{T_0} - \frac{1}{T_{>0}}\Big) \\
&> 0 \\
\oint dS_{\text{universe}} &> 0 .
\end{aligned}
$$

4. Purcell problem 1.5
Solution: Consider an element of charge $dQ = \lambda R\, d\phi$, where $d\phi$ is an element of azimuth around the semicircle ($0 < \phi < \pi$), and $\lambda = Q/\pi R$ is the charge per unit length (in esu/cm) around the semicircle.

Construct a Cartesian coordinate system with its origin at the center of the semicircle; choose $x = R\cos\phi$ and $y = R\sin\phi$. Then the symmetry about $x = 0$ and $z = 0$ requires the electric field at the origin from the full semicircle not to have any component in the x or z directions. So the net electric field must be parallel to the y axis; it points toward $-y$ if the charge Q is positive.

At the origin, Coulomb's law requires the above mentioned charge element dQ to create an element of electric field $d\mathbf{E}$ which has a magnitude equal to dQ/R^2. However, only a fraction $\sin\phi$ of that field magnitude points in the $-y$ direction. Therefore

$$
\begin{aligned}
dE_y &= -\frac{dQ}{R^2}\sin\phi \\
&= -\frac{\lambda R\, d\phi}{R^2}\sin\phi \\
&= -\frac{\frac{Q}{\pi R} R\, d\phi}{R^2}\sin\phi \\
&= -\frac{Q}{\pi R^2}\sin\phi\, d\phi \\
E_y &= -\frac{Q}{\pi R^2}\int_0^\pi \sin\phi\, d\phi \\
&= -\frac{2}{\pi}\frac{Q}{R^2} \\
\mathbf{E} &= \Big(0, -\frac{2}{\pi}\frac{Q}{R^2}, 0\Big) .
\end{aligned}
$$

5. Purcell problem 1.8
Solution: Let a be the ionic spacing of the one-dimensional crystal. Place the first positive ion at $x = 0$, two negative ions at $x = \pm a$, two more positive ions at $x = \pm 2a$, etc. Consider Purcell's Eq. 1.9:

$$
U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{q_j q_k}{r_{jk}} .
$$

This is a double sum. As the number N of ions approaches ∞, the sum of the terms of the double sum which involve any particular ion will be the same as the sum of the terms involving any other particular ion (see the argument at the bottom of Purcell's page 14). Thus the double sum reduces to a single sum:

$$U = \frac{1}{2} N \sum_{k=2}^{N} \frac{q_1 q_k}{r_{1k}} \ ,$$

where we have chosen to sum only the terms involving ion 1. Furthermore, since the string of ions is symmetric about $x = 0$, we may consider in the single sum only the ions with $x > 0$, at the expense of multiplying the result by an extra factor of 2:

$$U = \frac{1}{2} 2N \sum_{k=2;\ x>0}^{N} \frac{q_1 q_k}{r_{1k}} \ .$$

Here we evaluate $r_{1k} = a(k-1)$, and we use the fact that the sign of $q_1 q_k$ is equal to $(-1)^{k-1}$:

$$\begin{aligned} U &= \frac{1}{2} 2N \sum_{k=2}^{N} \frac{q_1 q_k}{a(k-1)} \\ &= \frac{Ne^2}{a} \sum_{k=2}^{N} \frac{(-1)^{k-1}}{(k-1)} \\ &= \frac{Ne^2}{a} \sum_{j=1}^{N-1} \frac{(-1)^j}{j} \ . \end{aligned}$$

Taking $N \to \infty$ in the limit of the sum,

$$\begin{aligned} U &= \frac{Ne^2}{a} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \\ &= -\frac{Ne^2}{a} \ln(1+1) \\ \frac{U}{N} &= -\frac{e^2}{a} \ln 2 \ , \end{aligned}$$

where, following the hint, we have evaluated the sum by using the Taylor series expansion

$$\ln(1+b) = \sum_{j=1}^{\infty} \frac{(-b)^{j-1}}{j} \ .$$

6. Purcell problem 1.14
Solution: This is similar to Purcell's problem 1.5, discussed above, and we will use similar notation. Consider an element of charge $dQ = \lambda b\, d\phi$, where $d\phi$ is an element of azimuth around the circle ($0 < \phi < 2\pi$), and $\lambda = Q/2\pi b$ is the charge per unit length (in esu/cm) around the circle.

Construct a Cartesian coordinate system with its origin at the center of the circle; choose z as the coordinate along the axis normal to plane of the circle. Consider a line drawn from dQ to a point $(0, 0, z)$ on this axis. Define ψ to be the angle that this line makes with the plane of the circle. With these definitions, $\tan\psi = z/b$ and $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$; the distance from dQ to $(0, 0, z)$ is $b\sec\psi$. Because the configuration is symmetric about $x = 0$ and $y = 0$, on the z axis the electric field must point in the z direction, away from the plane of the ring if its charge Q is positive.

At the point $(0, 0, z)$, Coulomb's law requires the above mentioned charge element dQ to create an element of electric field $d\mathbf{E}$ which has a magnitude equal to $dQ/(b\sec\psi)^2$. However, only a fraction $\sin\psi$ of that field magnitude points in the z direction. Therefore

$$\begin{aligned} dE_z &= \frac{dQ}{b^2\sec^2\psi} \sin\psi \\ &= \frac{\lambda b\, d\phi}{b^2\sec^2\psi} \sin\psi \\ &= \frac{\frac{Q}{2\pi b} b\, d\phi}{b^2\sec^2\psi} \sin\psi \\ &= \frac{Q\cos^2\psi\sin\psi}{2\pi b^2} d\phi \\ E_z &= \frac{Q\cos^2\psi\sin\psi}{2\pi b^2} \int_0^{2\pi} d\phi \\ &= \frac{Q\cos^2\psi\sin\psi}{b^2} \ . \end{aligned}$$

The problem thus reduces to finding the value of

ψ which maximizes the product $\cos^2\psi\sin\psi$:

$$\begin{aligned} u &\equiv \sin\psi \\ 0 &= \frac{d}{du}\big(u(1-u^2)\big) \\ &= 1-3u^2 \\ u &= \sqrt{\frac{1}{3}} \\ \psi &= \arcsin\sqrt{\frac{1}{3}} \\ z &= b\tan\left(\arcsin\sqrt{\frac{1}{3}}\right) \\ &= b\sqrt{\frac{1}{2}}\ . \end{aligned}$$

7. Purcell problem 1.26
Solution: Place the origin of a Cartesian coordinate system at the center of the semicircle, with both parallel rods lying in the xy plane. Orient the y coordinate so that the rods extend to $y=-\infty$.

At point C, the origin of this coordinate system, any electric field can point only in along the $\pm y$ direction, owing to the symmetry of the problem about $x=0$ and $z=0$. Purcell's figure refers us to two elements of charge. The element at point A has a value $dQ=\lambda b\,d\theta$ and generates an electric field at the origin of magnitude $\lambda b\,d\theta/b^2$. Only a fraction $\sin\theta$ of this field points in the $-y$ direction; thus

$$\begin{aligned} dE_{A,y} &= -\frac{\lambda b\,d\theta}{b^2}\sin\theta \\ &= -\frac{\lambda}{b}\sin\theta\,d\theta\ . \end{aligned}$$

The field from the element of charge at point B is slightly more complicated. This charge element has value $dQ=\lambda\,d|y|$, where $d|y|$ is an element of length along the straight rod, and $|y|=b\tan\theta$. Therefore $dQ=\lambda b\,d\tan\theta=\lambda b\sec^2\theta\,d\theta$. This element of charge lies a distance $b\sec\theta$ away from the origin. Again, only a fraction $\sin\theta$ of the field generated by this charge element points in the $+y$ direction. Putting it all together,

$$\begin{aligned} dE_{B,y} &= +\frac{\lambda b\sec^2\theta\,d\theta}{b^2\sec^2\theta}\sin\theta \\ &= +\frac{\lambda}{b}\sin\theta\,d\theta\ . \end{aligned}$$

We have seen that $dE_{A,y}$ exactly cancels $dE_{B,y}$ for any choice of θ; therefore $\mathbf{E}_C$ vanishes.