University of California, Berkeley
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SOLUTION TO PROBLEM SET 5

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1. A mass M rests on a table, and a mass m is supported on top of it by a massless spring connecting it to M.

(a.) We want to find the force F needed to push down on the spring so that the whole system will barely leave the table. We solve this by conservation of energy. The energy stored in a spring is given by $kx^2/2$ where x is the displacement from equilibrium of the spring. We can measure the gravitational potential of the small mass relative to the equilibrium point of the spring. Initially then, the spring is compressed with a force $F + mg$ which is just the weight of the small mass plus the added force. Hooke's law tells us that $x = -(F + mg)/k$. We can now write the initial energy of the system

$$E_i = -\frac{mg}{k}(F + mg) + \frac{(F + mg)^2}{2k}$$

The first term is the gravitational energy relative to the equilibrium point of the spring, and the second term is the energy stored in the spring. We want the spring to be able to lift the mass M off the table. To do this it must apply a force equal to Mg, its weight. When the spring is released, it will oscillate. At the top of the oscillation, there will be no kinetic energy. The displacement y of the spring must barely provide the force to lift the lower block: $ky = Mg$. The energy here is the following

$$E_f = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

Conservation of energy tells us that these are the same, so now we can solve for F.

$$-\frac{mg}{k}(F + mg) + \frac{(F + mg)^2}{2k} = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

Cancelling k and using the quadratic formula to solve for $F + mg$,

$$F + mg = \left(mg \pm \sqrt{m^2g^2 + 2Mmg^2 + M^2g^2}\right)$$

The expression under the square root is just $(M + m)g$, so the expression simplifies a lot:

$$F = \pm(M + m)g$$

We obviously want the plus sign.

(b.) This is a similar situation. The mass M is dangling and barely touching the table. The displacement of the spring just supports the weight of the block so $kx = Mg$. At the other end, the small mass is momentarily at rest at some distance $-y$ from equilibrium. The energies are

$$E_i = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

$$E_f = -mgy + \frac{ky^2}{2}$$

We equate these energies and solve for y.

$$-mgy + \frac{ky^2}{2} = \frac{Mmg^2}{k} + \frac{M^2g^2}{2k}$$

Using the quadratic formula again

$$ky = mg \pm \sqrt{m^2g^2 + 2Mmg^2 + M^2g^2}$$

Again the discriminant is a perfect square, and we want the positive value of y, so

$$y = \frac{(2m + M)g}{k}$$

The total distance that the mass falls is $x+y = d$.

$$d = 2\frac{(M + m)g}{k}$$

This is just twice the displacement caused by the force in part (a.), which makes sense, because the displacement upwards should be the same as the displacement downwards.

(c.) (1) When M is zero, the necessary applied force is mg. This is just the weight of the small mass. The spring will bounce back with the same force, so this is what is needed to lift the whole assembly. The distance fallen makes sense also, because the spring starts at its equilibrium length. The mass wants to sit at mg/k below this, to just support the weight. Thus it will oscillate down to $2mg/k$ below this point.

(2) When m is zero, the force needed to move the assembly is Mg; again this is the total weight. The distance traveled by the end of the spring in the second case is just $2Mg/k$. The end of the spring is Mg/k away from equilibrium when it begins, so the total distance traveled by the end is $2Mg/k$. While this seems to work out, it does not necessarily agree with common sense: a massless spring would not seem to be able to pull a massive block off the table by virtue of its own motion. However, we realize that, as the spring mass approaches zero in this idealization, its maximum velocity approaches infinity. This explains why the spring is still able to pull the block off the table, defying our intuition.

2.

(a.) The rocket is fired directly upward from the ground. The initial energy is just U, the energy in the fuel. After the fuel is spent, the fuel mass m is moving down at speed u and the remaining rocket mass M is moving upwards at speed v. Because momentum is conserved over this very short time, $mu = Mv$. The energy of the system is given by conservation of energy, and at launch, all of the energy is kinetic:

$$U = \frac{1}{2}Mv^2 + \frac{1}{2}mu^2 = \frac{Mv^2}{2}\left(1 + \frac{M}{m}\right)$$

Now we want to consider the motion of the mass M alone. Its kinetic energy is $Mv^2/2$, which we can find from the previous equation.

$$K_M = \frac{U}{1 + M/m}$$

Energy for the mass M is now conserved, so we can just set $K_M = Mgd$, where d is the maximum height achieved by the rocket. This gives the answer to part (a.):

$$d = \frac{U}{Mg}\frac{1}{1 + M/m}$$

(b.) This is a little more involved. The rocket has gone around part of an oval track and is now a distance h below where it started. The gravitational energy $(M + m)gh$ gets converted to kinetic energy, so we get the velocity v_0 of the rocket before the fuel is used:

$$(M + m)gh = \frac{1}{2}(M + m)v_0^2 \;\Rightarrow\; v_0 = \sqrt{2gh}$$

Ignoring for the moment the gravitational energy, the energy of the rocket at this point is

$$E_i = U + \frac{1}{2}(M + m)v_0^2 = U + (M + m)gh$$

The spring (fuel) imparts a change in velocity Δv to M and Δu to m. As in part (a.), instantaneous conservation of momentum gives $M\Delta v = m\Delta u$. After the spring is released, the energy corresponding to E_i is

$$E_f = \frac{1}{2}M(v_0 + \Delta v)^2 + \frac{1}{2}m(v_0 - \Delta u)^2$$

We know by conservation of energy that $E_i = E_f$, and we have $M\Delta v = m\Delta u$, so we can find Δv:

$$\Delta v = \sqrt{\frac{2U}{M(1 + M/m)}}$$

The total velocity of M is now $v = v_0 + \Delta v$:

$$v = \sqrt{2gh} + \sqrt{\frac{2U}{M(1 + M/m)}}$$

(c.) We can easily find the kinetic energy of the remaining rocket, and, using energy conservation, the maximum height H to which it rises above its current position:

$$K = \frac{1}{2}M(v_0 + \Delta v)^2 = MgH$$

Using $v_0 = \sqrt{2gh}$, we can solve for H:

$$H = \frac{2gh + \Delta v\sqrt{2gh} + \Delta v^2}{2g}$$

Plugging in the result for Δv, we arrive at the final answer

$$H = h + \frac{U}{Mg(1+M/m)} + \sqrt{\frac{hU}{Mg(1+M/m)}}$$

This is an interesting result. The first term just gets the rocket back to the height where it started in the first place. The second term gets it to the maximum height of the rocket in part (a.). The fact that the third term is positive means that the rocket actually flies higher in this case. The gain in height is just the third term

$$\Delta H = \sqrt{\frac{hU}{Mg(1+M/m)}}$$

(d.) This result does not conflict with energy conservation, which says only that the total energy of the system is conserved. We have been neglecting what happens to the mass m, which will take away a smaller amount of energy in the second case. If we looked at the total energy of both pieces, it would be conserved.

3. K&K problem 4.7. This problem is one in which both force and energy need to be considered. The forces on the ring are gravity, the tension in the thread T, and the normal forces due to the beads. The forces on the ring in the vertical direction are

$$F_{\text{ring}} = T - Mg - 2N(\theta)\cos\theta$$

where θ is the angle of the bead's position from the top, and $N(\theta)$ is taken to be positive outward. The two beads will move symmetrically. We now need to find the normal force $N(\theta)$. First we determine the velocity of the bead from conservation of energy. It yields the following:

$$E/2 = mgL(\cos\theta - 1) + \frac{1}{2}mv^2(\theta) = 0$$

This gives the velocity, and thus the centripetal acceleration a_c, as a function of θ:

$$v^2(\theta) = 2gL(1-\cos\theta) \;\Rightarrow\; a_c = 2g(1-\cos\theta)$$

The centripetal acceleration is provided by gravity and the normal force. Since a positive normal force is outward, at the top the normal force will be negative. The radial equation of motion will be

$$ma_c = N + mg\cos\theta \Rightarrow N(\theta) = mg(2 - 3\cos\theta)$$

Now we can go back to the force equation for the ring and use this result. The total force on the ring will be zero, but the ring will just start to move upwards when the thread is slack, which is when $T = 0$. Using both of these facts, we get an equation for θ

$$Mg = 2mg\cos\theta(3\cos\theta - 2)$$

This is just a quadratic equation in $\cos\theta$. Multiplying it out, we can apply the quadratic formula to get the answer

$$\cos\theta = \frac{1}{3} \pm \frac{1}{3}\sqrt{1 - \frac{3M}{2m}}$$

There is a small problem here in that the discriminant can be negative, making the cosine of the angle complex. This of course is unphysical. The problem is that for sufficiently small m, the motion of the small masses is never important enough to cause the tension in the rope to vanish, so our calculation is wrong from the start. Insisting that $\cos\theta$ be real, we obtain the condition

$$m > \frac{3}{2}M$$

Taking the positive root, the final answer is

$$\theta = \cos^{-1}\left(\frac{1}{3} + \frac{1}{3}\sqrt{1 - \frac{3M}{2m}}\right)$$

4. We assume that the moon is a uniform sphere of mass $M = 7.3 \times 10^{22}$ kg and radius $R = 1740$ km. A straight, frictionless tunnel connects two points on the surface. Given the mass and radius, the density is just $\rho = 3M/4\pi R^3$. We need to know the acceleration due to gravity at a distance r from the center of the moon. This is also straightforward. Recall that a spherical shell of mass exerts no force on objects inside it, so at a radius r, the only force we need to consider is due to the mass in the moon interior to radius r. This is just the density times the volume interior to r, or $\mathcal{M}(r) = Mr^3/R^3$. The acceleration due to gravity is then just $g(r) = -G\mathcal{M}(r)/r^2 = -GMr/R^3$. Thus the acceleration due to gravity increases linearly as one moves away from the center of a uniform solid sphere.

(a.) In a spherical polar coordinate system with its $\hat{\mathbf{z}}$ axis at the moon's north pole, assume that the tunnel lies in a straight line between $(r, \theta_0, \phi) = (R, \theta_0, 0)$ and (R, θ_0, π), *i.e.* between two points at the same north latitude $\pi/2 - \theta_0$ having the largest possible difference in longitude. This means that the distance along a great circle between the ends of the tunnel is $2\theta_0$, while the distance from the center of the moon to the center of the tunnel is $z_0 = R\cos\theta_0$. Now assume that the mass makes an angle ψ $(-\frac{\pi}{2} < \psi < \frac{\pi}{2})$ with a line connecting the center of the moon and the center of the tunnel, *i.e.* with the $\hat{\mathbf{z}}$ axis. The distance of the mass from the center of the tunnel is then $x = z_0 \tan\psi$, while its distance from the center of the moon is $r = z_0/\cos\psi$. We now need to know the component F_x of the gravitational force $-GMm\mathbf{r}/R^3$ which lies in the $(\hat{\mathbf{x}})$ direction of the tunnel, which makes an angle $\pi/2 - \psi$ with the radial direction. This is

$$\begin{aligned} F_x &= -\frac{GMmr}{R^3}\cos\left(\pi/2 - \psi\right) \\ &= -\frac{GMm}{R^3}\frac{z_0}{\cos\psi}\sin\psi \\ &= -\frac{GMm}{R^3}x \end{aligned}$$

This is like the force from a Hooke's law spring with effective spring constant $k_{\text{eff}} = GMm/R^3$, yielding simple harmonic oscillation with resonant angular frequency

$$\omega_0 = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{GM}{R^3}}$$

(b.) Plugging in values of M and R for the moon, and using $T = 2\pi/\omega_0$, we get for the period of oscillation

$$T = 6536 \text{ seconds } = \text{ 109 minutes}$$

(c.) A satellite traveling in a circular orbit must have centripetal acceleration provided by gravity, which means that

$$\frac{v^2}{R} = \frac{GM}{R^2} = \omega^2 R$$

From the last equality we see that the angular frequency ω of a circular orbit of radius R around the moon is the same as ω_0 above. Of course the period is the same as well.

5. K&K problem 4.23. Two balls of masses M and m are dropped from height h and collide elastically. The small ball is on top of the larger ball. Conservation of energy for the system gives its speed v right before the balls hit the ground:

$$(M+m)gh = \frac{1}{2}(M+m)v^2 \;\Rightarrow\; v = \sqrt{2gh}$$

The ball M collides with the ground first. In order to conserve energy, it must still have speed v instantaneously after it bounces from the ground. Now it immediately collides with the small ball. (Think of this problem as if there were a very small gap between the two balls so that the first ball to hit the ground has a chance to bounce before the second one hits it.) We consider the elastic collision between the two balls, each moving at speed v towards the other.

The easiest frame in which to study this collision is a comoving (inertial) frame that is instantaneously at rest with respect to the large ball M immediately after it has rebounded with velocity $v = \sqrt{2gh}$ from its elastic collision with the ground. In this frame, M is instantaneously at rest, and m has (upward) velocity $-2v$. When the collision occurs, if $m \ll M$ as stated in the problem, M seems to m like a "brick wall" from which it bounces back elastically with the same speed. Thus, in the comoving frame immediately after the collision, m has velocity $+2v$. Finally, transforming back to the lab frame, m acquires an extra velocity increment v, for a total of $3v$. Since the height that m reaches is proportional to the square of its velocity, this means that m reaches nine times the height from which it originally was dropped.

A less elegant approach considers the collision between M and m in the lab frame. Here is it essential not to apply the approximation $m \ll M$ until near the end, since cancellations occur which may make nonleading terms more important than would initially be suspected.

In the lab frame, conservation of momentum gives

$$Mv - mv = MV_M + mV_m$$

We also have conservation of energy through this collision. This condition gives

$$\frac{1}{2}(M+m)v^2 = \frac{1}{2}MV_M^2 + \frac{1}{2}mV_m^2$$

These are two equations in the two unknowns V_m and V_M, since we already know $v = \sqrt{2gh}$. We are interested in V_m, which yields the desired final height $V_m^2/2g$ of m, but we are not interested in V_M. So we plan to eliminate V_M by solving for it using the first equation and then substituting for it in the second.

Before proceeding with this algebra, it is convenient to substitute

$$\begin{aligned} \epsilon &= m/M \\ u &= V_m/v \\ U &= V_M/v \end{aligned}$$

so that all terms are dimensionless. The two equations above become

$$\begin{aligned} 1 - \epsilon &= U + \epsilon u \\ 1 + \epsilon &= U^2 + \epsilon u^2 \end{aligned}$$

Solving the first equation for U,

$$U = 1 - \epsilon - \epsilon u$$

Substituting this value for U in the second equation,

$$\begin{aligned} 1 + \epsilon &= 1 - 2\epsilon + \epsilon^2 - 2\epsilon u + 2\epsilon^2 u + \epsilon^2 u^2 + \epsilon u^2 \\ 0 &= \epsilon(1+\epsilon)u^2 - 2\epsilon(1-\epsilon)u - \epsilon(3-\epsilon) \\ 0 &= u^2 - 2\frac{1-\epsilon}{1+\epsilon}u - \frac{3-\epsilon}{1+\epsilon} \end{aligned}$$

Neglecting ϵ with respect to 3 or 1 in both quotients, the polynomial is

$$u^2 - 2u - 3 = (u-3)(u+1)$$

with the physical solution

$$\begin{aligned} u &= 3 \\ V_m &= 3v = \sqrt{18gh} \\ h' &= \frac{V_m^2}{2g} = 9h \end{aligned}$$

as before.

6. This is a collision problem that has different unknown quantities than those to which you are accustomed, but it is still solvable. We have two collisions to study, and the unknowns are the neutron mass and the initial and final speeds of the neutrons. The initial speeds are the same, so there are four unknowns in total. We have two collisions, each of which yields two equations (one for momentum conservation, one for energy conservation since the collisions are elastic). Therefore the system can be solved uniquely. The directions of the scattered neutrons relative to the incident directions do not represent additional unknowns, since the maximum recoil velocities of the target nucleii will occur when the collisions take place head-on, with the incoming neutrons bouncing straight back. Thus we can take this to be a one dimensional problem.

The equations are the following (the energy equations have been multiplied by 2):

$$m_n v = m_n v' + m_H v_H \quad m_n v^2 = m_n v'^2 + m_H v_H^2$$

$$m_n v = m_n v'' + m_N v_N \quad m_n v^2 = m_n v''^2 + m_N v_N^2$$

Solving these equations for m_n and v requires careful algebra. We square the first momentum equation to get a relation between v', m_n, and v

$$v'^2 = \frac{(m_n v - m_H v_H)^2}{m_n^2}$$

Now we plug this into the first energy equation

$$m_n v^2 = \frac{(m_n v - m_H v_H)^2}{m_n} + m_H v_H^2$$

Expanding,

$$m_H^2 v_H^2 - 2m_n m_H v v_H + m_n m_H v_H^2 = 0$$

Writing this as an equation for v, we get

$$v = \frac{1}{2}\left(1 + \frac{m_H}{m_n}\right) v_H$$

This is fairly simple result. If we perform the same manipulations on the nitrogen equations, we will get an analogous result

$$v = \frac{1}{2}\left(1 + \frac{m_N}{m_n}\right) v_N$$

We can now use these to solve for m_n and v. Equating the right hand sides, we get a single equation for the mass.

$$\begin{aligned} m_n v_H + m_H v_H &= m_n v_N + m_N v_N \\ m_n &= \frac{m_N v_N - m_H v_H}{v_H - v_N} \ . \end{aligned}$$

We can now use this to find the initial velocity of the neutrons:

$$\begin{aligned} v &= \frac{v_H}{2}\left(1 + \frac{m_H(v_H - v_N)}{m_N v_N - m_H v_H}\right) \\ &= \frac{v_H}{2}\left(\frac{m_N v_N - m_H v_N}{m_N v_N - m_H v_H}\right) \\ &= \frac{v_H v_N}{2}\left(\frac{m_N - m_H}{m_N v_N - m_H v_H}\right) \ . \end{aligned}$$

We want to know the mass of the neutron in amu, so we plug in $m_H = 1$ and $m_N = 14$ (greater accuracy is unnecessary, since the recoil velocities are measured only to 10%). We also look at both boundaries of the nitrogen velocity, calling these results $m_\pm$ and $v_\pm$. Plugging in numbers, the values of m_n are

$$\begin{aligned} m_n &= 1.159 \text{ amu} \\ m_+ &= 1.415 \text{ amu} \\ m_- &= 0.911 \text{ amu} \ . \end{aligned}$$

Chadwick's experimental work is seen to be reliable; today's accepted value for the neutron mass is 1.008665 amu, or 938.27231 $\pm$ 0.00028 MeV/c^2, well within his experimental range. The range of initial neutron velocity is given by

$$\begin{aligned} v &= 3.07 \times 10^7 \text{ m/sec} \\ v_+ &= 2.82 \times 10^7 \text{ m/sec} \\ v_- &= 4.13 \times 10^7 \text{ m/sec} \ . \end{aligned}$$

7. K&K problem 4.13. The Lennard-Jones potential is given by

$$U = \epsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6\right]$$

(a.) We find the minimum of this potential by differentiating it with respect to r and setting the results equal to zero:

$$\frac{dU}{dr} = -\frac{12\epsilon}{r}\left[\left(\frac{r_0}{r}\right)^{12} - \left(\frac{r_0}{r}\right)^6\right] = 0$$

This is easy to solve:

$$\left(\frac{r_0}{r}\right)^{12} = \left(\frac{r_0}{r}\right)^6 \Rightarrow r = r_0$$

The depth of the potential well is just $U(r_0) = -\epsilon$. Thus the potential well has a depth ϵ.

(b.) We find the frequency of small oscillations by making a Taylor expansion of the potential about $r = r_0$. Read section 4.10 in K&K for more information on this. We can write the potential as follows:

$$\begin{aligned} U(r) = U(r_0) &+ \left(\frac{dU}{dr}\right)_{r=r_0}(r - r_0) \\ &+ \frac{1}{2}\left(\frac{d^2U}{dr^2}\right)_{r=r_0}(r - r_0)^2 + \cdots \end{aligned}$$

We know that $dU/dr = 0$ at $r = r_0$, so we drop the middle term.

$$U(r) \approx -\epsilon + \frac{1}{2}\left(\frac{d^2U}{dr^2}\right)_{r=r_0}(r - r_0)^2$$

This is exactly the form of the potential of a mass on a spring. We only have to identify the spring constant. Remembering that $U_{\text{spring}} = kx^2/2$, we make the identification

$$k = \left(\frac{d^2U}{dr^2}\right)_{r=r_0}$$

For the Lennard-Jones potential, we already know the first derivative, so we need to differentiate once more.

$$\frac{d^2U}{dr^2} = \frac{12\epsilon}{r^2}\left[13\left(\frac{r_0}{r}\right)^{12} - \left(7\frac{r_0}{r}\right)^{6}\right]$$

Plugging in $r = r_0$, we find the effective spring constant for this potential

$$k = \frac{72\epsilon}{r_0^2}$$

We now consider two identical masses m on the ends of this "spring". Their (coupled) equations of motion are:

$$m\ddot{r_1} = k(r - r_0) \quad m\ddot{r_2} = -k(r - r_0)$$

where $r = r_2 - r_1$ is the distance between the masses. Subtracting these two equations, we get

$$m\ddot{r} = -2k(r - r_0)$$

The frequency of oscillation is then $\omega^2 = 2k/m$. (Note that we could have obtained the same result by considering the two-mass system to be a *single* mass of *reduced mass* $m_{\text{reduced}} = m_1 m_2/(m_1 + m_2)$). Plugging in the above value for the effective spring constant k,

$$\omega = 12\sqrt{\frac{\epsilon}{r_0^2 m}}$$