University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 2

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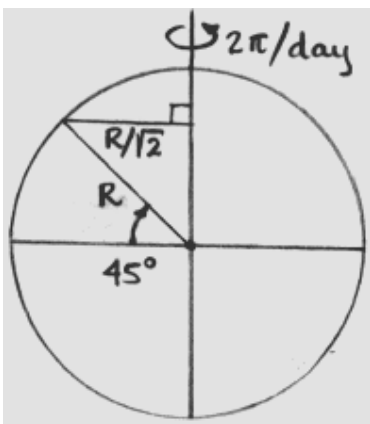
1. This problem is a simple application of the formula for centripetal acceleration. An object moving at speed v in a circular path of radius r has a centripetal acceleration directed inward:

$$\mathbf{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{\mathbf{r}}$$

If we know instead the angular velocity ω of the object, we can use it instead. Remember that an object with angular velocity ω going on a circular path of radius r has speed $v = \omega r$. Plugging into the above formula, we get

$$\mathbf{a}_{\text{centripetal}} = -\omega^2 r\hat{\mathbf{r}}$$



(a.) Standing at 45° latitude, the distance to the axis of the earth's rotation is just $r_{\text{earth}}/\sqrt{2}$. This is the radius to be used. The speed is just the angular velocity times this radius. The angular velocity is easy to guess, it's 2π/day. Using $r_{\text{earth}} =$ 6370 km, the distance to the axis is 4504 km.

Plugging in these numbers, the acceleration is 0.0238 m/sec^2. This is $2.4 \times 10^{-3}g$, a very small acceleration compared to the acceleration of gravity.

(b.) This is the same sort of calculation, except that the angular velocity is now $2\pi/28$ days. The mean radius of the moon's orbit is 3.84×10^8m. The centripetal acceleration is 0.0026 m/sec^2, which is $2.6 \times 10^{-4}g$.

(c.) This time you are given the speed, not the angular velocity, so we use the first formula. The acceleration is 8.0×10^{22} m/sec^2. This is $8.2 \times 10^{21}g$!

(d.) A point on the rim of a wheel is moving at the same speed that the wheel is rolling. The radius is 13 inches, and the velocity is 25 mph, so we need to convert these units. The official definition of the inch is 1 inch = 2.54 cm. This gives 1610 meters per mile. The acceleration is then 379 m/sec^2, which is $38.6g$.

2. K&K problem 1.17

In plane polar coordinates the velocity and acceleration are given by

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

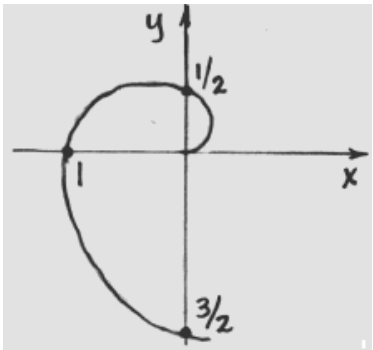
We know all of these things, so we can get a formula for the magnitudes of $\mathbf{v}$ and $\mathbf{a}$, given that both the radial and angular velocities are constant ($\ddot{r} = \ddot{\theta} = 0$).

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$
$$a = \sqrt{r^2\dot{\theta}^4 + 4\dot{r}^2\dot{\theta}^2}$$

Plugging into these, we find that $v = \sqrt{52}$ m/sec and $a = 20$ m/sec^2.

3. K&K problem 1.20

The motion of a particle is given by $r = A\theta$, $\theta = \alpha t^2/2$ and $A = (1/\pi)$ m/rad. The sketch of this motion should look something like this:



(b.) We plug the expression for θ into r to get $r = A\alpha t^2/2$. From these we can get all of the necessary derivatives. $\dot{r} = A\alpha t$, $\ddot{r} = A\alpha$, $\dot{\theta} = \alpha t$, and $\ddot{\theta} = \alpha$.

Using the equation for radial acceleration, we find an expression for points where the radial acceleration is zero.

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 \;\Rightarrow\; \ddot{r} = r\dot{\theta}^2$$

Plugging in the expressions for the r and θ variables, we find that $A\alpha = A\alpha(\alpha t^2)^2/2$. Using the fact that $\theta = \alpha t^2/2$, we arrive at the result $\theta = 1/\sqrt{2}$.

(c.) For this part, we set the magnitudes of the radial and tangential accelerations equal to each other and solve for the angle. Plugging into the formulas for the two accelerations, we get

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Plugging into the various terms, we get

$$\left|A\alpha - \frac{1}{2}A\alpha^3 t^4\right| = \left|\frac{1}{2}A\alpha^2 t^2 + 2A\alpha^2 t^2\right|$$

Plugging in the expression for θ where we can, we get

$$\left|1 - 2\theta^2\right| = |5\theta|$$

When the dust settles, there will be separate results depending on the value of θ. The final results are

$$\theta < \frac{1}{\sqrt{2}} : \quad \theta = \frac{\sqrt{33} - 5}{4}$$
$$\theta > \frac{1}{\sqrt{2}} : \quad \theta = \frac{\sqrt{33} + 5}{4}$$

4. Two objects are dropped from a building at times $t = 0$ and $t = t_0$. The distance that the first has fallen as a function of time is just $d_1 = gt^2/2$. The distance that the second has fallen is $d_2 = g(t - t_0)^2/2$. When $t > t_0$, the separation between them is $l = d_1 - d_2$. Thus we get

$$l = \frac{1}{2}gt^2 - \frac{1}{2}g(t^2 - 2tt_0 + t_0^2) = gtt_0 - \frac{1}{2}gt_0^2$$

Solving for t as a function of l, we get

$$t = \frac{l}{gt_0} + \frac{t_0}{2}$$

When $l < gt_0^2/2$ this time is negative. This does have a sensible interpretation, unlike the negative time in the airplane problem of the previous problem set. Think of the problem as if we wanted to throw both objects at the same time, but still have the initial conditions given. At $t = 0$, when we drop the first object, from where do we have to throw the other object? The answer is we want to throw upwards from below in such a way that at time $t = t_0$, the ball has reached the peak of its path and is momentarily at rest at the point where the first ball was dropped. During the time between $t = 0$ and $t = t_0$, the separation between the objects can be negative, meaning that the second one is below the first.

(b.) In this part we want to calculate the optimal value of t_0 so that the separation reaches some value l_0 at the earliest time possible. In other words, we want to minimize the function $t(t_0, l_0)$. First we take the derivative and set it to zero to find local extrema.

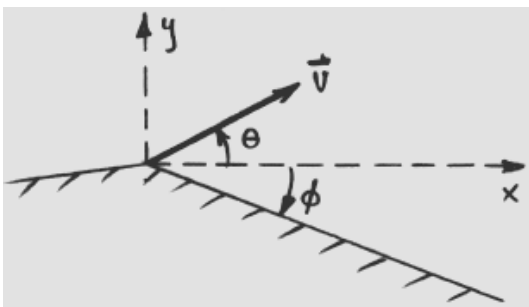
$$\frac{dt}{dt_0} = \frac{1}{2} - \frac{l_0}{gt_0^2} = 0 \;\Rightarrow\; t_0 = \sqrt{\frac{2l_0}{g}}$$

Notice that this is just the time that it takes the first object to fall a distance l_0. The endpoints here are $t_0 = 0$, where the separation remains at $l = 0$ forever and $t_0 = \infty$, where the time to reach a separation of l_0 is also infinity. Thus this best time to drop it is at $t_0 = \sqrt{2l_0/g}$. This means that the best thing to do is to drop the

second object when the first object has already fallen a distance l_0.

5. K&K problem 1.21

This is another maximization problem. We want to know the optimal angle to throw a ball down a hill with slope angle ϕ. Splitting this into the x and y directions is the easiest way to do the problem. First put the origin at the top of the hill. If the ball is thrown up at an angle θ with speed v, the initial velocities are $v_x = v\cos\theta$ and $v_y = v\sin\theta$.



Taking into account the acceleration of gravity, the positions are given by

$$x = vt\cos\theta$$
$$y = vt\sin\theta - \frac{1}{2}gt^2$$

We need to know where the ground is in these coordinates. At a position x on the ground, the y coordinate is given by $y_{\text{ground}} = -x\tan\phi$. We can now find the time at which the ball hits the ground. Plugging into the equation for distance traveled in y, we get

$$y_{\text{ground}} = -x\tan\phi = vt\sin\theta - \frac{1}{2}gt^2$$
$$\Rightarrow\ v\sin\theta - \frac{1}{2}gt = -v\cos\theta\tan\phi$$

This gives the time at which the ball hits the ground:

$$t = \frac{2v}{g}\left(\sin\theta + \cos\theta\tan\phi\right)$$

We now plug this time into the equation for distance traveled in the x direction, giving the distance that the ball traveled:

$$x = \frac{2v^2}{g}(\cos\theta\sin\theta + \cos^2\theta\tan\phi)$$
$$= \frac{2v^2}{g}(\frac{1}{2}\sin 2\theta + \cos^2\theta\tan\phi)$$

Treating this as a function of θ, we can maximize the range by differentiating with respect to θ. Note that the endpoints in this problem are not interesting. Throwing the ball straight up ($\theta = 90°$) and throwing it at an angle $-\phi$ both result in the ball traveling no distance in the x direction.

$$\frac{dx}{d\theta} = \frac{2v^2}{g}(\cos(2\theta) - 2\cos\theta\sin\theta\tan\phi)$$
$$= \frac{2v^2}{g}(\cos(2\theta) - \sin(2\theta)\tan\phi) = 0$$

Solving for θ, we get

$$\cos(2\theta) = \sin(2\theta)\tan\phi\ \Rightarrow\ \cot(2\theta) = \tan\phi$$

Remembering that $\cot\alpha = \tan(\pi/2 - \alpha)$, we see the final result:

$$\tan\left(\frac{\pi}{2} - 2\theta\right) = \tan\phi\ \Rightarrow\ \theta = \frac{\pi}{4} - \frac{\phi}{2}$$

Note that on a level surface, when $\phi = 0$, the optimal angle is 45°, as you might already know.

6. K&K problem 2.1

This is the first problem where you are asked to consider the forces causing acceleration. The force on a 5 kg mass is given by $\mathbf{F} = 4t^2\hat{\mathbf{x}} - 3t\hat{\mathbf{y}}$ Newtons. Apply Newton's second law of motion, namely $\mathbf{F} = m\mathbf{a}$, to get the acceleration, $\mathbf{a} = (4t^2/5)\hat{\mathbf{x}} - (3t/5)\hat{\mathbf{y}}$m/sec.

(a.) We can get velocity from acceleration by integrating

$$\mathbf{v}(t) - \mathbf{v}(t_0) = \int_{t_0}^{t}\mathbf{a}(t')dt'$$

Plugging the acceleration we just determined into this integral, and knowing that the velocity at $t = 0$ is zero, we get the velocity as a function of time:

$$\mathbf{v}(t) = \frac{4}{15}t^3\hat{\mathbf{x}} - \frac{3}{10}t^2\hat{\mathbf{y}}\ \text{m/sec}$$

(b.) We get the position by integrating again:

$$\mathbf{r}(t) - \mathbf{r}(t_0) = \int_{t_0}^{t} \mathbf{v}(t')dt'$$

Applying this to the result of part (a.), and remembering that at $t = 0$ the mass is at the origin so $\mathbf{r}(0) = 0$, we get the position as a function of time:

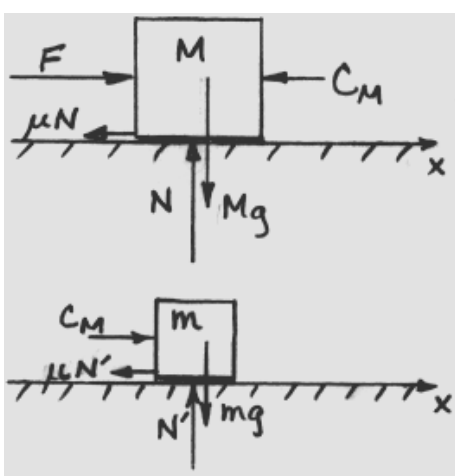
$$\mathbf{r}(t) = \frac{1}{15}t^4\hat{\mathbf{x}} - \frac{1}{10}t^3\hat{\mathbf{y}} \ \text{ m}$$

(c.) Now all that is left is to take the cross product of the position with the velocity. We find that

$$\mathbf{r} \times \mathbf{v} = \left(-\frac{3}{150}t^6 + \frac{4}{150}t^6\right)\hat{\mathbf{z}} = \frac{t^6}{150}\hat{\mathbf{z}}$$

7. This problem asks you to consider two blocks sliding on a table together. The larger block, with mass M, has five forces acting on it. They are $F\hat{\mathbf{x}}$, the applied force, a contact force that I will call $-C_M\hat{\mathbf{x}}$, the force of gravity $-g\hat{\mathbf{y}}$, the normal force $N\hat{\mathbf{y}}$, and the force of friction. Because there is no acceleration in the $\hat{\mathbf{y}}$ direction, we can easily find that $N = Mg$, so that there is no net force in the $\hat{\mathbf{y}}$ direction. From the normal force we can determine the force of friction. An object that is sliding with friction along a surface is acted upon by a force opposing the direction of motion with magnitude $\mu N (= \mu Mg)$, where μ is the coefficient of sliding friction. We now can write an expression for the acceleration of the large block in the $\hat{\mathbf{x}}$ direction

$$a_x = \frac{F_x}{M} = \frac{F}{M} - \frac{C_M}{M} - \mu g$$



There are two unknowns here, a_x and C_M. We need another equation. Luckily, there is another block that we can write equations about. The small block, having mass m, is affected by four forces. They are the contact force, the force of gravity, the normal force, and the force of friction. The contact force is exactly opposite to the contact force on the first block. This is due to Newton's third law of motion, which states that every force has an equal and opposite force. Thus $C_m = C_M$ and the contact force is $C_M\hat{\mathbf{x}}$. The force of gravity is just $-mg\hat{\mathbf{y}}$, the normal force exactly opposes gravity as before, $+mg\hat{\mathbf{y}}$, and the force of friction is again $-\mu mg$. We get the equation of motion for the second block, noticing that acceleration of the second block is the same as for the first block because they are moving together:

$$a_x = \frac{C_M}{m} - \mu g$$

We are only concerned with C_M here, so we can simplify the solution of these equations. Subtract the second equation from the first to get

$$\frac{F}{M} - C_M\left(\frac{1}{M} + \frac{1}{m}\right) = 0$$

We can now solve for C_M in terms of F:

$$C_M = F\left(\frac{m}{M+m}\right)$$

If we follow the same procedure when the force is acting on the second block, we get a very similar answer, but the mass in the numerator is the larger mass

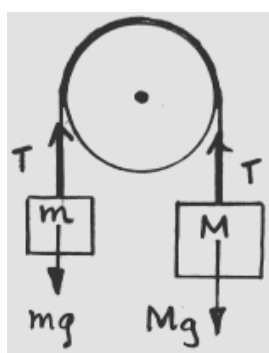
$$C_M = F\left(\frac{M}{M+m}\right)$$

This is a factor of M/m larger, which is what we expect. The only force pushing on the second block in each case is the contact force, and the acceleration doesn't depend on which side we push the combined system from. The

force of friction acts in proportion to the mass in this case, so it does not affect this argument. It can be thought of as a force that acts on the combined system, not on the individual blocks, because it is proportional to the mass. Notice that neither of these expressions depend on the value of μ, which would indicate that the friction was affecting the contact force.

8. K&K problem 2.5

This is the first pulley problem, and it won't be the last. The pulley is massless and frictionless, and supports two masses M and m by a massless rope connecting them. The first thing to notice about this problem is that the tension in the rope must be the same on both sides of the pulley. If the different sides had different tensions, there would be a tendency to cause an angular acceleration in the pulley. Since it is massless, this acceleration would be infinite, which is unphysical, so the tensions must be equal. You will see this in more detail when you do rigid body motion later this term.



The second thing to notice is that the more massive block will fall and the less massive block will rise, and their accelerations will be the same, but in opposite directions. This just means that the rope isn't stretching. For this problem I will set "down" to be the positive direction. The equation for the larger mass is

$$Ma = Mg - T \;\Rightarrow\; T = M(g - a)$$

Using the expression for tension derived in the above equation, the equation for the smaller mass is the following:

$$ma = -mg + T \;\Rightarrow\; m(a + g) = M(g - a)$$

Solving for the acceleration,

$$a = g\left(\frac{M - m}{M + m}\right)$$

The larger mass accelerates with magnitude a downward. The tension is found by plugging the acceleration into either of the starting equations

$$T = Mg - Mg\left(\frac{M - m}{M + m}\right) = g\left(\frac{2Mm}{M + m}\right)$$

9. Nope, this isn't the last pulley problem either. Again, the pulley and cords are massless and the pulley is frictionless. A force F is applied upward, and various things will happen depending on what F is. The first thing to notice is that the pulley is massless. This means that the tensions on the two ropes must be equal, otherwise a finite angular force would be applied to a massless object, which again is unphysical. The second thing to notice is that the upward force must exactly balance the sum of the tensions. If this weren't the case, there would be a net force applied to a massless object. This can't happen, so to balance the forces we just need $F = 2T$. With these two points in mind, we can do the rest of the problem.

(a.) The boundary between regimes (i) and (ii) is where the lighter block lifts off the ground. Consider the forces on this block. They are gravity $-mg$, the tension T, and the normal force N. The equation of motion is $ma = T + N - mg$. At the boundary between regimes (i) and (ii) when the block just barely can be lifted upward, the normal force N is zero, but so is the acceleration. This gives us $T = mg$. We know also that $T = F/2$, so the minimum force to lift the lighter block is $F = 2mg$. This is just twice the weight of the lighter block, which we expect because the force applied gets divided in half by the pulley. The boundary between regimes (ii) and (iii) is found in a similar way. The equation of motion is $T + N - Mg = Ma$. Again both a and N are zero at the transition point, so $T = Mg$, which gives the force $F = 2Mg$. The final results are

$$\begin{aligned} \text{regime(i)} &\quad \{F < 2mg\} \\ \text{regime(ii)} &\quad \{2mg < F < 2Mg\} \\ \text{regime(iii)} &\quad \{2Mg < F\} \end{aligned}$$

(b.) Now we want to find the accelerations for regimes (ii) and (iii). This is easy because we have already determined the equations of motion. For regime (ii), only the lighter block accelerates. The equation of motion is $F/2 - mg = ma$. This gives the result

$$\text{regime (ii)} \quad a_M = 0$$
$$a_m = \frac{F}{2m} - g$$

In regime (iii), the equation of motion of the first block is the same, so we get the same result for the acceleration. The equation of motion of the larger block is $F/2 - Mg = Ma$. These give the results

$$\text{regime (iii)} \quad a_M = \frac{F}{2M} - g$$
$$a_m = \frac{F}{2m} - g$$

10. K&K problem 2.6

The cement mixer's drum has a radius R. We want to know how fast it can rotate so that the material will not stick to the walls all of the time. We just need to figure out at what speed the drum can oppose gravity all of the time. For a glob of material of mass m, the worst case is at the top. To remain in contact with the drum, at that point the glob must feel a downward force from the drum that is positive. In addition it feels the downward force mg due to gravity. So the total downward force on it must be at least mg. Now, what acceleration accompanies this force? We don't want the glob to leave the drum, so its radial velocity must remain equal to zero. The only remaining possible downward acceleration is the centripetal acceleration due to the circular motion. This is $\omega^2 R$. Equating the mass m times this acceleration to the total downward force, we conclude that $m\omega^2 R \geq mg$, or $\omega \geq \sqrt{g/R}$. For the material to not always stick, we need the final result

$$\omega < \sqrt{\frac{g}{R}}$$