University of California, Berkeley
Physics 110B, Fall 2008 (*Strovink*)
PROBLEM SET 8

43.
This problem is omitted intentionally.

44.
Griffiths Problem 11.15.

45.
Start from the expression derived in class for the energy radiated by an accelerating point charge per steradian per unit of *retarded* time t':

$$\frac{dW}{d\Omega\, dt'} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{\epsilon_0}{c} \frac{\left|\hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \dot{\vec{\beta}}\right]\right|^2}{(1 - \hat{R} \cdot \vec{\beta})^5} \ .$$

Consider *synchrotron radiation* by a particle of charge q moving in a circular orbit of radius b in a coordinate system where

$$\hat{\beta} = \hat{z}$$
$$\dot{\hat{\beta}} = \hat{x} \ ,$$

i.e. $\hat{x}$ points toward the center of the circle and $\hat{z}$ points along its circumference in the particle's direction of motion. Define

$$\hat{R} \equiv (n_x, n_y, n_z) \ ,$$

where $\hat{n}$ is a unit vector extending from the particle in an arbitrary direction towards which an element of radiation is emitted.
(**a.**)
Show that

$$\hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \dot{\hat{\beta}}\right] = \hat{n} n_x - \hat{x} - \beta \hat{n} \times \hat{y} \ .$$

(**b.**)
Using this result, show that

$$\left|\hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \dot{\hat{\beta}}\right]\right|^2 = 1 - 2\beta n_z + \beta^2 n_z^2 - (1 - \beta^2) n_x^2$$

(**c.**)
Consider a set of spherical polar coordinates centered at the particle (*not* at the center of the beam circle). Taking θ to be the polar angle of $\hat{n}$ relative to $\hat{z}$, and ϕ to be its azimuth about $\hat{z}$, express n_x and n_z in terms of θ and ϕ.
(**d.**)
Using the results of (**b.**) and (**c.**), show that

$$\frac{dW}{d\Omega\, dt'} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{\epsilon_0}{c} \times$$
$$\times \frac{\dot{\beta}^2}{(1 - \beta\cos\theta)^3} \left(1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2 (1 - \beta\cos\theta)^2}\right) \ .$$

46.
Consider the final result of the previous problem.
(**a.**)
Perform the integration over $d\Omega = d(\cos\theta)\, d\phi$ to show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2}{3c^3} (q\dot{\beta}c)^2 \gamma^4 \ .$$

[Note that $(q\dot{\beta}c)^2$ is equivalent to $\ddot{p}^2$, where p is the electric dipole moment of the point charge relative to the origin. Therefore this result is the same as the (nonrelativistic) Larmor formula, except for the additional factor γ^4.]
(**b.**)
In terms of the |momentum| P of the point charge and its rest mass m, show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2q^2}{3c^3} \frac{P^4}{m^4 b^2} \ ,$$

and thus that the power lost to synchrotron radiation depends on the fourth power of P, the inverse fourth power of m (making it usually negligible for all but electrons), and the inverse square of b.
(**c.**)
Suppose that you use an electron synchrotron that taxpayers can afford. It circulates highly relativistic electrons with $\beta \approx 1$. You want to build a new synchrotron with the same beam current, the same power lost to synchrotron radiation, but twice the beam momentum. Show that the radius b of the new synchrotron must increase by a factor of 16.

47.
A free-electron laser consists of a beam of electrons (with constant velocity βc) passing through a structure known as an *undulator*. (Stronger versions of these structures called *wigglers* are used also in sections of a circular electron synchrotron such as the ALS.) Take the beam direction to be $\hat{z}$. Consider an alternating set of magnets (for compactness, these are often permanent magnets, made of samarium cobalt as developed at LBL by the late Klaus Halbach). With a full period Δz, they produce a strong magnetic field that points alternately in the $+\hat{x}$ and $-\hat{x}$ directions. For this undulator we (unrealistically) assume that this magnetic field is weak enough so that, as seen in a frame moving with velocity $\beta c\hat{z}$, the electrons' velocity *relative* to that frame is $\ll c$.
(**a.**)
In the rest frame $\mathcal{S}'$ of the electron, with what fundamental angular frequency ω' does the magnetic field from the undulator appear to oscillate?
(**b.**)
In $\mathcal{S}'$, the oscillating electron produces electromagnetic radiation with angular frequency ω'. Applying the relativistic Doppler shift to ("forward") radiation emitted along the beam direction, what angular frequency ω does that radiation have in the laboratory frame?
(**c.**)
Express λ, the wavelength of the forward radiation, as a multiple of Δz.
(**d.**)
At LBL's ALS, using an alternating set of magnets with $\Delta z = 10$ cm, an experimenter wishes to study the effect upon condensed-matter samples of a soft X-ray beam of wavelength 5 nm. Use this information to estimate the ALS beam energy (in GeV).

48.
This problem is omitted intentionally.

49.
(**a.**)
This part is omitted intentionally.
(**b.**)
This part is omitted intentionally.
(**c.**)
In a more microscopic and detailed treatment of plane EM waves propagating along $\hat{z}$ in materials that potentially are conducting, one assumes that N valence electrons per m^3 having charge $-e$ and mass m move in a potential well with effective spring constant $m\omega_0^2$ and damping coefficient Γm. One defines the *complex dielectric constant* $\tilde{\epsilon}$ via

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 \equiv \frac{\tilde{P}}{\epsilon_0 \tilde{E}} ,$$

where $\tilde{P}$ is the complex polarization, defined by

$$\vec{P}(z,t) = \mathrm{Re}\Big(\vec{\tilde{P}} \exp\big(i(\tilde{k}z - \omega t)\big)\Big) ,$$

in analogy to complex $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$. For not-too-dense media in which the electric field felt by the electron is approximately the same as the average field, it is straightforward to solve the force equation for these oscillating electrons and determine the complex polarization $\tilde{P}$ they create. One obtains

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\Gamma\omega} ,$$

where the *plasma frequency*[2] is

$$\omega_p^2 \equiv \frac{Ne^2}{m\epsilon_0} .$$

The complex dielectric constant $\tilde{\epsilon}$ includes the effects of all electrons (free and bound). It is related to the ordinary dielectric constant ϵ (which includes the effects only of bound electrons) by

$$\tilde{\epsilon} = \epsilon(1 + i\gamma) = \frac{\tilde{k}^2}{\mu\omega^2} ,$$

where as usual $\gamma = \frac{\sigma}{\epsilon\omega}$. Represent a *good conductor* by $\omega_0 = 0$ (unbound) and $\Gamma \gg \omega$ (overdamped). Using these results, show that the conductivity σ is approximately

$$\sigma \approx \frac{\epsilon_0 \omega_p^2}{\Gamma} ,$$

i.e. measuring the low-frequency conductivity is a simple way to determine the damping coefficient.

(**d.**)

Represent the *ionosphere* by $\omega_0 = 0$ (unbound), and $\Gamma \ll \omega$ (underdamped). Specialize to AM radio waves, for which $\omega < \omega_p$. Show that the amplitude reflection coefficient $|\tilde{R}| \approx 1$, *i.e.* that AM radio waves are nearly fully reflected by the ionosphere. (At dusk, the ionosphere drops to sufficiently low altitude that reflection off it enables AM stations hundreds of miles away to be received.)

50.

This problem is omitted intentionally.

51. Jones vectors.

For a plane transverse wave propagating in the $\hat{z}$ direction through a (not necessarily insulating) material with constant ϵ and μ, a (co)sinusoidal solution is represented by

$$\vec{E}(\vec{r},t) = \mathrm{Re}\Big(\vec{E}_0(x,y)\, e^{i(\tilde{k}z-\omega t)}\Big)$$
$$\vec{H}(\vec{r},t) = \mathrm{Re}\Big(\vec{H}_0(x,y)\, e^{i(\tilde{k}z-\omega t)}\Big) \ ,$$

where $\tilde{k}$ is the (not necessarily real) "wave vector" – here a scalar because we know it is directed along $\hat{z}$. Faraday's law causes $\vec{H}_0$ to be completely determined by $\vec{E}_0$:

$$\begin{aligned}\vec{H}_0 &\equiv \tilde{Z}^{-1}\hat{z}\times\vec{E}_0 \\ &= \frac{\tilde{k}}{\mu\omega}\hat{z}\times\vec{E}_0 \ ,\end{aligned}$$

so we focus on $\vec{E}_0$ as the sole independent variable. For a transverse wave $\vec{E}_0$ has no z component. Here we assume that the phase relationship between E_{0x} and E_{0y} is *fixed* – the wave is *fully polarized*. Then $\vec{E}_0$ is a complex transverse vector, completely specified by four components. In the Jones convention, all information carried by $\vec{E}_0$ except for its magnitude is written as a 2×1 column vector with the x component on top:

$$\begin{aligned}\vec{E}_0 &= \begin{pmatrix} E_{0x} \\ E_{0y}\end{pmatrix} \\ &\equiv \frac{1}{\sqrt{|\alpha^2|+|\beta^2|}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}|\vec{E}_0| \\ &\equiv \vec{J}\,|\vec{E}_0| \ ,\end{aligned}$$

where $\vec{J}$ is the *Jones vector*. Jones vectors are defined only within an overall phase (because the absolute phase of an optical-frequency EM wave can't conveniently be measured); therefore one has the freedom to set α equal to unity (unless it vanishes, in which case β is set to unity). The above form involving the complex constants α and β is a general Jones vector, corresponding to elliptical polarization. More common Jones vectors are

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \ ,$$

corresponding, respectively, to linear x, linear y, RH circular, and LH circular polarization.

(**a.**)

At $z = 0$, show (counterintuitively!) that the electric field vector for RH polarized light precesses *clockwise* around $\hat{z}$, *i.e.* it precesses according to the LH rule.

(**b.**)

Suppose that a particular state of elliptical polarization has nonvanishing x and y electric field components. Then, within an arbitrary overall phase, it may be represented by the Jones vector

$$\vec{J}_1 = \frac{1}{\sqrt{1+|\beta|^2}}\begin{pmatrix}1\\ \beta\end{pmatrix} \ ,$$

where β is a complex constant. You wish to characterize this state of polarization as "RH elliptical" or "LH elliptical", depending on whether (at $z = 0$) the electric field vector precesses clockwise or counterclockwise around $\hat{z}$. What property of β would you use to decide whether this state is RH or LH elliptical?

(**c.**)

For the conditions of part (**b.**), decompose $\vec{J}_1$ into a linear sum (with real coefficients) of a

wave with linear polarization plus a wave with RH circular polarization. Perform this same task with "RH" replaced by "LH". If you are successful in both tasks, you might wonder whether there really exists a unique association of RH or LH behavior with $\vec{J}_1$. Would this concern invalidate your answer to (**b.**)?