University of California, Berkeley
Physics 110B, Fall 2008 (*Strovink*)
PROBLEM SET 7

35.
Liénard's equation for the Poynting vector

$$\vec{S}_{\rm a} = \frac{1}{\mu_0}\vec{E}_{\rm a}\times\vec{B}_{\rm a}$$

arising from acceleration of a point particle of charge q is

$$\vec{S}_{\rm a} = (\frac{q}{4\pi\epsilon_0})^2\frac{\epsilon_0}{c}\Big\{\frac{\hat{\mathcal{R}}}{\mathcal{R}^2}\big[\frac{\hat{\mathcal{R}}\times[(\hat{\mathcal{R}}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\big]^2\Big\}_{\rm ret}.$$

(**a.**)
Show that Liénard's equation follows directly from the electric and magnetic fields arising from acceleration of a point particle, using the *acceleration fields*

$$\vec{E}_{\rm a} = \frac{q}{4\pi\epsilon_0}\frac{1}{c}\Big\{\frac{1}{\mathcal{R}}\frac{\hat{\mathcal{R}}\times[(\hat{\mathcal{R}}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\Big\}_{\rm ret}$$
$$c\vec{B}_{\rm a} = \big\{\hat{\mathcal{R}}\times\vec{E}_{\rm a}\big\}_{\rm ret}\ .$$

(**b.**)
Suppose that the particle is in uniform motion around a circle of radius b in the plane $z=0$ centered at the origin. The motion is ultrarelativistic, *i.e.* $(1-\beta^2)^{-1/2}\gg 1$. To lowest order, calculate the radiated power per unit area observed at $(0,0,z)$, where $z\gg b$.
(**c.**)
Is $\hat{z}$ a direction in which the power radiated per unit solid angle is near the maximum for this motion? Explain.

36.
As an intermediate step in the derivation of the velocity and acceleration fields $\vec{E}_{\rm v}$ and $\vec{E}_{\rm a}$, in class we derived the expression

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\Big\{\frac{1}{1-\hat{\mathcal{R}}\cdot\vec{\beta}}\big[\frac{\hat{\mathcal{R}}}{\mathcal{R}^2}+\frac{d}{c\,dt}\frac{\hat{\mathcal{R}}-\vec{\beta}}{\mathcal{R}(1-\hat{\mathcal{R}}\cdot\vec{\beta})}\big]\Big\}_{\rm ret}$$

where the subscript "ret" means that the differentiation should be done first, and afterward all time-dependent quantities should be evaluated at time $t_{\rm ret}=t-\mathcal{R}/c$.

Define $\dot{\vec{\beta}}\equiv d\vec{\beta}/dt$. Use two relations worked out in class:

$$\frac{d\mathcal{R}}{c\,dt} = -\hat{\mathcal{R}}\cdot\vec{\beta}$$
$$\frac{d\hat{\mathcal{R}}}{c\,dt} = \frac{\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})}{\mathcal{R}}\ .$$

With these tools, finish the derivation to obtain $\vec{E}_{\rm v}$ (as given in an earlier problem) and $\vec{E}_{\rm a}$ (as given in the previous problem).

37.
Griffiths Problem 3.40.

38.
The electrostatic potential created by a static point charge can take a nontrivial form when the coordinate system is chosen to have an origin which, for some other reason, must be centered at point that does not coincide with the charge's position.

This problem concerns the potential $V(\vec{r})$ created by a localized charge distribution $\rho(\vec{r}\,')$. With the observation point located outside the charge distribution ($r>r'_{\rm max}$), use the standard expansion in spherical harmonics

$$\epsilon_0 V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{Y_{lm}(\theta,\phi)}{(2l+1)r^{l+1}}\,q_{lm}\ ,$$

where the multipole moments q_{lm} are defined in the hint for the previous problem. In spherical polar coordinates, consider a point charge e located at (r',θ',ϕ') with respect to a certain origin. Determine the electrostatic potential that it creates at an observation point (r,θ,ϕ), with $r>r'_{\rm max}$.
(**a.**)
Write down the exact value of $\mathrm{V}(r,\theta,\phi,r',\theta',\phi')$ as an infinite sum over l and m.
(**b.**)
Explicitly evaluating the spherical harmonics as

functions of θ and ϕ (or θ' and ϕ'), write down all the monopole, dipole, and quadrupole terms ($l = 0$, 1, and 2).

39.
Arrange five finite point charges at five different positions so that all $l \leq 4$ moments of the charge distribution vanish, except for the $m = 0$ hexadecapole moment

$$q_{40} \equiv \int d\tau' \rho(\vec{r}\,') r'^4 Y_{40}^*(\theta', \phi') \ .$$

40.
Consider the dimensionless operator

$$\vec{L} \equiv \tfrac{1}{i} \vec{r} \times \nabla$$

(apart from a missing factor of $\hbar$, this is the same as the angular momentum operator used in quantum mechanics).
(**a.**)
In spherical polar coordinates, show that

$$i\vec{L} = \hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin\theta} \frac{\partial}{\partial \phi} \ .$$

(**b.**)
Express $\hat{\theta}$ and $\hat{\phi}$ in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$, θ, and ϕ.
(**c.**)
Show that

$$iL_z = \frac{\partial}{\partial \phi}$$
$$L_\pm \equiv L_x \pm iL_y = e^{\pm i\phi} \Big(\pm \frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \Big) \ .$$

[$L_\pm$ are *raising* and *lowering* operators, which, within a factor, change Y_{lm} into $Y_{l,m\pm1}$.]
(**d.**)
Show that

$$L^2 = L_z^2 + \tfrac{1}{2}\{L_+, L_-\} \ ,$$

where $\{a, b\}$ is the anticommutator $ab + ba$.
(**e.**)
Finally, show that

$$-L^2 = r^2 \nabla_{\text{ang}}^2 \ ,$$

where ∇_{ang}^2 is the part of ∇^2 which involves derivatives in θ and ϕ.

41.
This problem is omitted intentionally.

42.
Working in the far zone $r' \ll \lambda \ll r$, consider azimuthally symmetric ($m = 0$) electric quadrupole ($\text{E}2_0$) radiation. At a particular angular frequency ω, work with the complex fields $\vec{\tilde{B}}(\vec{r})$ and $\vec{\tilde{E}}(\vec{r})$ defined by

$$\vec{B}(\vec{r}, t) \equiv \text{Re}\left(\vec{\tilde{B}}(\vec{r}) e^{-i\omega t}\right)$$
$$\vec{E}(\vec{r}, t) \equiv \text{Re}\left(\vec{\tilde{E}}(\vec{r}) e^{-i\omega t}\right) \ .$$

For E-type radiation, the magnetic field $\vec{\tilde{B}}$ ($\perp \hat{r}$) is proportional to the vector spherical harmonic $\vec{X}$:

$$\vec{\tilde{B}} \propto \vec{X}_{20}(\theta, \phi) \equiv \vec{L}\, Y_{20}(\theta, \phi) \ ,$$

with $i\vec{L} \equiv \vec{r} \times \nabla$. Use the fact that

$$\vec{\tilde{E}} \approx c\vec{\tilde{B}} \times \hat{r}$$

in the far zone. Obtain a function $f(\theta, \phi)$ such that the radiated power P in the far zone is proportional to it:

$$\frac{dP}{d\Omega} \propto f(\theta, \phi) \ .$$