University of California, Berkeley
Physics 110B, Fall 2008 (*Strovink*)
PROBLEM SET 4

15. (Taylor & Wheeler problem 51)
The clock paradox, version 3.
Can one go to a point 7000 light years away – *and return* – without aging more than 40 years? "Yes" is the conclusion reached by an engineer on the staff of a large aviation firm in a recent report. In his analysis the traveler experiences a constant "1-g" acceleration (or deceleration, depending on the stage reached in her journey). Assuming this limitation, is the engineer right in his conclusion? (For simplicity, limit attention to the first phase of the motion, during which the astronaut accelerates for 10 years – then double the distance covered in that time to find how far it is to the most remote point reached in the course of the journey.)
(a.)
The acceleration is *not* $g = 9.8$ meters per second per second relative to the laboratory frame. If it were, how many times faster than light would the spaceship be moving at the end of ten years (1 year $= 31.6 \times 10^6$ seconds)? *If the acceleration is not specified with respect to the laboratory, then with respect to what is it specified?* Discussion: Look at the bathroom scales on which one is standing! The rocket jet is always turned up to the point where these scales read one's *correct* weight. Under these conditions one is being accelerated at 9.8 meters per second per second with respect to a spaceship that (1) instantaneously happens to be riding alongside with identical velocity, but (2) is *not* being accelerated, and, therefore (3) *provides the* (momentary) *inertial frame of reference relative to which the acceleration is* g.
(b.)
How much velocity does the spaceship have after a given time? This is the moment to object to the question and to rephrase it. *Velocity* βc is not the simple quantity to analyze. The simple quantity is the *boost parameter* η. This parameter is simple because it is *additive* in this sense: Let the boost parameter of the spaceship with respect to the imaginary instantaneously comoving inertial frame change from 0 to $d\eta$ in an astronaut time $d\tau$. Then the boost parameter of the spaceship with respect to the *laboratory* frame changes in the same astronaut time from its initial value η to the subsequent value $\eta + d\eta$. Now relate $d\eta$ to the acceleration g in the instantaneously comoving inertial frame. In this frame $g\,d\tau = c\,d\beta = c\,d(\tanh\eta) = c\,\tanh(d\eta) \approx c\,d\eta$ so that

$$c\,d\eta = g\,d\tau$$

Each lapse of time $d\tau$ on the astronaut's watch is accompanied by an additional increase $d\eta = \frac{g}{c}d\tau$ in the boost parameter of the spaceship. In the laboratory frame the total boost parameter of the spaceship is simply the sum of these additional increases in the boost parameter. Assume that the spaceship starts from rest. Then its boost parameter will increase linearly with *astronaut* time according to the equation

$$c\eta = g\tau$$

This expression gives the boost parameter η of the spaceship in the *laboratory* frame at any time τ in the *astronaut's* frame.
(c.)
What laboratory distance x does the spaceship cover in a given astronaut time τ? At any instant the velocity of the spaceship in the laboratory frame is related to its boost parameter by the equation $dx/dt = c\,\tanh\eta$ so that the distance dx covered in *laboratory* time dt is

$$dx = c\,\tanh\eta\,dt$$

Remember that the time between ticks of the astronaut's watch $d\tau$ appear to have the larger value dt in the laboratory frame (time dilation) given by the expression

$$dt = \cosh\eta\,d\tau$$

Hence the laboratory distance dx covered in *astronaut* time $d\tau$ is

$$dx = c\,\tanh\eta\,\cosh\eta\,d\tau = c\,\sinh\eta\,d\tau$$

Use the expression $c\eta = g\tau$ from part (b.) to obtain

$$dx = c \sinh\left(\frac{g\tau}{c}\right) d\tau$$

Sum (integrate) all these small displacements dx from zero astronaut time to a final astronaut time to find

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$

This expression gives the laboratory *distance* x covered by the spaceship at any time τ in the astronaut's frame.

(d.)

Plugging in the appropriate numerical values, determine whether the engineer is correct in his conclusion reported at the beginning of this exercise.

16. (Surface muons)

"Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays to an (anti)muon (μ^+, a heavy electron-like particle) and a muon neutrino (ν_μ) via

$$\pi^+ \to \mu^+ + \nu_\mu \ .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed (and, as it turns out, 100% polarization as well). For the purposes of this problem, consider a muon to have $\frac{3}{4}$ of the rest mass of a pion; neglect the neutrino mass.

(a.)

Show that the surface muons travel at a speed which is a fraction $\beta_0 = \frac{7}{25}$ of the speed of light.

(b.)

If a muon's mean proper lifetime is τ, what fraction of the muons will decay during a flight path of length L in the laboratory? Express your answer in terms of β_0.

17.

In the lab frame $\mathcal{S}$, a particle with velocity βc $= \frac{4}{5}c$ decays into two massless particles with the same energy each.

(a.)

If the parent particle has mean (proper) lifetime τ in its *own* rest frame $\mathcal{S}'$, calculate its mean flight path L in the lab frame $\mathcal{S}$.

(b.)

In the lab frame $\mathcal{S}$, calculate the opening angle $\psi = \cos^{-1} \hat{p}_1 \cdot \hat{p}_2$ between the two daughter particles.

18.

Here's an adult version of Griffiths' Problem 12.35. In a pair annihilation experiment, a positron (mass m) with total energy $E = \gamma mc^2$ hits an electron (same mass, but opposite charge) at rest. (Griffiths has it the other way around, but that's unrealistic – it's easy to make a positron beam, but hard to make a positron target.) The two particles annihilate, producing two photons. (If only one photon were produced, energy-momentum conservation would force it to be a massive particle traveling at a velocity less than c.) If one of the photons emerges at angle θ relative to the incident positron direction, show that its energy ϵ is given by

$$\frac{mc^2}{\epsilon} = 1 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} \cos\theta \ .$$

(In particular, if the photon emerges perpendicular to the beam, its energy is equal to mc^2, independent of the beam energy. Similar results have been used to design clever experiments.)

19.

If you have studied Rutherford scattering (elastic scattering of a nonrelativistic He nucleus from an Au nucleus), you have seen the *differential cross section* for this process written in the form

$$\frac{d\sigma}{d\Omega} \propto \frac{(zZe^2)^2}{\sin^4 \frac{\Theta}{2}} \ ,$$

where ze (Ze) is the electric charge of the He (Au) nucleus; Θ is the angle by which the He nucleus is elastically scattered, measured in the

CM frame; and $d\Omega$ is an element of solid angle within which the He nucleus emerges in that frame. [A century ago, Rutherford-scattering data collected by graduate students who were used as particle detectors demonstrated that atoms contain charged point-like constituents (nuclei).] Here we revisit Rutherford scattering for *relativistic* particles.

Consider the 2+2 relativistic scattering process

$$p + a \to q + b \ ,$$

where p, a, q, and b denote both the particles and their 4-momenta. The *Mandelstam variables*, first written down by Berkeley emeritus professor Stanley Mandelstam, are

$$s \equiv (p+a)\cdot(p+a) \equiv \text{CM energy}^2$$
$$t \equiv (q-p)\cdot(q-p) \equiv 4 \text{ momentum transfer}^2$$
$$u \equiv (b-p)\cdot(b-p) \equiv \text{cross channel transfer}^2$$

In this problem we are concerned with the Mandelstam variable t.

(a.)
Further assuming that the masses of particles p and q are negligible, show that

$$-t = 4\tfrac{E}{c}\tfrac{E'}{c}\sin^2\tfrac{\Theta}{2} \ ,$$

where E (E') is the energy of particle p (q), and Θ is the angle between $\vec{p}$ and $\vec{q}$.
(b.)
Why is $d(-t)$ a Lorentz invariant? $d\sigma$ is an area transverse to the beam direction. Why is $d\sigma$ invariant to Lorentz transformations along that direction? In a system of units where $\hbar = c = 1$, all quantities have dimensions that can be expressed in units of Joules. In those units, what are the dimensions of $d\sigma$?
(c.)
In part (b.) you showed that $d\sigma/d(-t)$ is a Lorentz invariant. If particle p (which becomes particle q) and particle a (which becomes particle b) both are structureless, and if the scattering is elastic (particles a and b both have the same mass), the only relevant Lorentz-invariant variable that is available to us is $-t$. On purely dimensional grounds, show that

$$\frac{d\sigma}{d(-t)} \propto \frac{1}{t^2} \ .$$

[If p (a) has electric charge ze (Ze), and they interact electromagnetically, the constant of proportionality is $4\pi z^2 Z^2 \alpha^2$, where the fine structure constant α is given as usual by $4\pi\epsilon_0\alpha = e^2/\hbar c$. This formula is correct to the extent that Z or $z \times (\alpha \approx 1/137)$ can be neglected relative to unity.]
(d.)
Under all of these conditions, using the results of (a.) and (c.) and working in the center of mass, show that the nonrelativistic elastic scattering result

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\frac{\Theta}{2}}$$

does remain valid even when relativistic effects are taken into account.

20. (Relations used in particle physics)
(a.) Lorentz-invariant phase space [LIPS]
By transforming dp_x and E while keeping p_y and p_z fixed, show that

$$\frac{c\,dp_x}{E}dp_y dp_z$$

is invariant to a boost along x. (Since one can always define x to be the boost direction, and $d^3p \equiv dp_x dp_y dp_z$, a LIPS element

$$\frac{c\,d^3p}{E}$$

is invariant to a boost in *any* direction and therefore is Lorentz invariant.)
(b.)
Suppose a particle has momentum $\vec{p}$ and energy E. Define the particle's *longitudinal rapidity* y to be the boost along x that would be needed to make $p'_x = 0$ in the new frame $\mathcal{S}'$. Show that

$$y = \tanh^{-1}\frac{cp_x}{E} \ .$$

If your calculator doesn't have an arc hyperbolic tangent button, use the equivalent definitions

$$y = \tfrac{1}{2}\ln\frac{E/c + p_x}{E/c - p_x}$$
$$y = \ln\frac{E/c + p_x}{\sqrt{p_y^2 + p_z^2 + m^2c^2}} \ .$$

(c.)
Using the fact that the rapidity (boost) is the additive parameter for the Lorentz transformation, and that y is defined to be a boost along $\hat{x}$, argue that an increment dy in longitudinal rapidity must be the same in two Lorentz frames that differ only by a relative boost along $\hat{x}$. Use this argument to conclude that

$$dy\, dp_y dp_z$$

is invariant to boosts along $\hat{x}$, as was the LIPS element

$$\frac{c\, d^3p}{E}$$

in part (a.). (In fact, these two expressions are equal.) Invariance of the longitudinal rapidity interval dy is a godsend for proton collider users. Since the proton's interacting constituents (quarks or gluons) carry only a variable fraction of the proton momentum, the center of mass (CM) of the colliding constituents is boosted along the beam direction by a variable amount (typically of order unity). However, the *difference* in longitudinal rapidity between any pair of emitted particles is unaffected by this unwelcome CM boost.)
(d.)
Define the *pseudorapidity* y_{pseudo} as the longitudinal rapidity that a particle would have if it were ultrarelativistic. Show that

$$y_{\text{pseudo}} = \tanh^{-1}(\cos\theta)\ ,$$

where θ is the angle between the particle's direction and the x axis. (This is another godsend: if a particle is known to be ultrarelativistic, its longitudinal rapidity can be approximated by its pseudorapidity, which can be measured by knowing only the particle's direction.) An equivalent definition of pseudorapidity is

$$y_{\text{pseudo}} = -\ln\left(\tan\tfrac{\theta}{2}\right)\ .$$