University of California, Berkeley
Physics 110B, Fall 2008 (*Strovink*)
PROBLEM SET 3

7.
Griffiths Problem 12.55. Don't get fooled by the typo – he means "$\partial^\mu \equiv \partial/\partial x_\mu$".

8.
An object a^μ is a (contravariant) four-vector if it transforms (between frames as defined in Short Course in Special Relativity (SCSR) Fig. 2) according to

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu \ ,$$

where Λ is the (symmetric) 4×4 Lorentz transformation matrix. (Conventionally, the first (superscript) index labels the row and the second (subscript) index labels the column, but this makes no difference for a symmetric matrix.) Covariant four-vectors instead transform according to

$$a'_\mu = a_\nu (\Lambda^{-1})^\nu{}_\mu$$

(otherwise the scalar product $a_\mu a^\mu = a'_\mu a'^\mu$ would not remain invariant for different Lorentz frames). Consider now an (arbitrary) four-tensor $H^{\mu\nu}$. In frame $\mathcal{S}$, $H^{\mu\nu}$ contracts with covariant four-vector a_ν to yield contravariant four-vector b^μ, according to

$$b^\mu = H^{\mu\nu} a_\nu \ .$$

In the frame $\mathcal{S}'$, requiring $H^{\mu\nu}$ to satisfy the transformation properties of a four-tensor, we define $H'^{\mu\nu}$ so that

$$b'^\mu = H'^{\mu\nu} a'_\nu \ .$$

Prove that

$$H'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma H^{\rho\sigma} \ .$$

This defines the Lorentz transformation property of a four-tensor.

9. Consider the antisymmetric *electromagnetic field strength tensor*

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ ,$$

where both ∂^μ and A^μ are (contravariant) four-vectors. Prove that $F^{\mu\nu}$ is a four-tensor, *i.e.* it transforms according to the result of Problem 8.

10. (Light cone)
(a.)
Event A occurs at spacetime point $(ct, x, y, z) = (0, 1, 1, 1)$; event B occurs at $(1, 0, 0, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events A and B occur at the same spatial coordinates? If so, find $c|t'_A - t'_B|$, c times the magnitude of the time interval in $\mathcal{S}'$ between the two events.
(b.)
Is there an inertial system $\mathcal{S}''$ in which events A and B occur simultaneously? If so, find $|\vec{r}''_A - \vec{r}''_B|$, the distance in $\mathcal{S}''$ between the two events.
(c.)
Can event A be the cause of event B, or vice versa? Explain.
(d.)
Event D occurs at spacetime point $(ct, x, y, z) = (-1, 0, 0, 0)$; event E occurs at $(2, 1, 1, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events D and E occur simultaneously? If so, find $|\vec{r}'_E - \vec{r}'_D|$, the magnitude of the distance in $\mathcal{S}'$ between the two events.
(e.)
Is there an inertial system $\mathcal{S}''$ in which events D and E occur at the same spatial coordinates? If so, find $c|t''_E - t''_D|$, c times the magnitude of the time interval in $\mathcal{S}''$ between the two events.

11.
Using *e.g.* the method of Short Course in Special Relativity [SCSR] §7, obtain the 4×4 Lorentz transformation matrix for the case in which frame $\mathcal{S}'$ moves with respect to frame $\mathcal{S}$ with speed $\beta_0 c$ in an *arbitrary* direction $(n_1, n_2, 0)$ in the x-y plane, where $\vec{n}$ is a unit vector.

12.
(a.)
In SCSR §8, clock time intervals measured in a

frame in which the clock is not at rest are shown to be *dilated* by the factor γ_0. This analysis used the *inverse* Lorentz transformation. Reanalyze the same problem using the *direct* Lorentz transformation. Is the answer the same?
(b.)
In SCSR §9, the length of a rod measured in a frame in which the rod is not at rest is shown to be *contracted* by the factor $1/\gamma_0$. This analysis used the *direct* Lorentz transformation. Reanalyze the same problem using the *inverse* Lorentz transformation. Is the answer the same?

13. (Addition of velocities)
In texts that do not emphasize the rapidity or boost parameter η, the Einstein law for the addition of velocities is derived less elegantly as follows (see SCSR Fig. 7). Denote by x^1 (x'^1) the x coordinate of the origin of $\mathcal{S}''$ as observed in the lab frame $\mathcal{S}$ (moving frame $\mathcal{S}'$). Write a standard inverse Lorentz transformation

$$\begin{aligned} x^0 &= \gamma x'^0 + \gamma\beta x'^1 \\ x^1 &= \gamma x'^1 + \gamma\beta x'^0 \ . \end{aligned}$$

Then take the differential of it: $dx^0 = \ldots$; $dx^1 = \ldots$. Divide the bottom by the top equation and identify

$$\begin{aligned} \frac{dx^1}{dx^0} &= \beta'' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S} \\ \frac{dx'^1}{dx'^0} &= \beta' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S}' \ . \end{aligned}$$

Obtain the Einstein law for the addition of velocities (SCSR Eq. (24)):

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'} \ .$$

14.
Consider the standard case in which two Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$ coincide at $t = t' = 0$, with frame $\mathcal{S}'$ moving at velocity $\beta c\,\hat{x}$ with respect to frame $\mathcal{S}$. As seen in a third frame $\mathcal{S}''$, also moving along $\hat{x}$ with respect to $\mathcal{S}$, two clocks fixed to the origins of frames $\mathcal{S}$ and $\mathcal{S}'$, respectively, appear to agree. With respect to frame $\mathcal{S}$, considering that *rapidity ("boost") is the additive parameter of the Lorentz transformation*, show that the speed $\beta'' c$ of frame $\mathcal{S}''$ is given by

$$\beta'' = \tanh\left(\tfrac{1}{2}\tanh^{-1}\beta\right) \ .$$