University of California, Berkeley
Physics 110B, Fall 2008 (*Strovink*)
PROBLEM SET 10

60. Interference of two beams following different paths.
Consider two beams A and B. At (early) plane P, the relative properties of the two beams are well understood; for example, a single laser beam may be split into two. Between plane P and (late) plane Q, the beams follow different paths A and B through a nondispersive medium ($v_{\text{group}} = v_{\text{phase}}$); by the time they reach plane Q they have recombined. (For example, a Michelson interferometer may be interposed between the two planes.) At P and Q define

$$\text{physical } \vec{E}_{A,B}(P,Q) \equiv \text{Re}\big(\vec{E}^{A,B}_{P,Q}\, e^{-i\omega t}\big)$$

(this is four equations). On the left-hand side are physical fields that vary rapidly ($\approx$ sinusoidally) with time t; on the right-hand side are complex fields $\vec{E}^{A,B}_{P,Q}$ having magnitudes that are fixed, but phases that vary more slowly, over many sinusoidal periods. This slow variation may occur separately for the x and y components of a beam's electric field – in which case the beam is completely unpolarized – or it may occur in lockstep for the x and y components together, in which case the beam remains fully polarized.

The optical phase shifts for paths A and B are equal to

$$\omega\tau_{A,B} \equiv \int_P^Q \vec{k}_{A,B} \cdot d\vec{r}_{A,B}$$
$$\tau \equiv \tau_B - \tau_A \ ,$$

where $\vec{k}_{A,B}(\vec{r})$ is the wave vector for beam A or B, respectively, and $d\vec{r}_{A,B}$ lies along the path for beam A or B.

(a.)
The (undispersed) physical waves remain functions of $(\vec{k}_{A,B} \cdot \vec{r}_{A,B} - \omega t)$, even as these slow phase variations occur. Use this fact to show that

$$\text{physical } \vec{E}_{A,B}(Q)(t+\tau_{A,B}) = \text{physical } \vec{E}_{A,B}(P)(t) \ .$$

(b.)
Using the result of part (a.), show that

$$\vec{E}^{A,B}_Q(t+\tau_{A,B}) = \vec{E}^{A,B}_P(t)\exp\left(i\omega\tau_{A,B}\right) .$$

(c.)
At any other time t', the result of (b.) also holds. Choose $t' = t - \tau$. Show that

$$\vec{E}^B_Q(t+\tau_A) = \vec{E}^B_P(t-\tau)\exp\left(i\omega\tau_B\right) .$$

(d.)
The irradiance

$$I = \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\,|\vec{E}^A + \vec{E}^B|^2$$

for the superposition of the two beams satisfies

$$2\sqrt{\tfrac{\mu}{\epsilon}}I_{P,Q} = |\vec{E}^A_{P,Q}|^2 + |\vec{E}^B_{P,Q}|^2 + 2\text{Re}\big(\vec{E}^{A*}_{P,Q}\cdot\vec{E}^B_{P,Q}\big)$$

Using the results of (b.) and (c.), show that

$$I_Q(t+\tau_A) = I^A + I^B + \\ + \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\,2\,\text{Re}\big(\vec{E}^{A*}_P(t)\cdot\vec{E}^B_P(t-\tau)\exp\left(i\omega\tau\right)\big) \ ,$$

where $I^{A,B}$ are the (time-independent and space-independent) single-beam irradiances.

(e.)
Taking a long-time average (long compared to the characteristic time over which the complex electric field phases vary), obtain as a final step the *master equation for two-beam interference:*

$$\langle I^{A+B}_Q\rangle(\tau) = I^A + I^B + \\ + \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\,\langle 2\,\text{Re}\big(\vec{E}^{A*}_P(t)\cdot\vec{E}^B_P(t-\tau)\exp\left(i\omega\tau\right)\big)\rangle \ ,$$

where $\langle\rangle$ denotes a long-time average, and I^{A+B}_Q is the combined irradiance at plane Q.

61.
Please refer to the notation and results of the previous problem. Define the *correlation* $\Gamma^{AB}(\tau)$ as

$$\Gamma^{AB}(\tau) \equiv \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,\langle \vec{E}_P^{A*}(t)\cdot \vec{E}_P^{B}(t-\tau)\exp\left(i\omega\tau\right)\rangle\ ,$$

and define the *degree of partial coherence* $\gamma^{AB}(\tau)$ as

$$\gamma^{AB}(\tau) \equiv \frac{\Gamma^{AB}(\tau)}{\sqrt{I^A I^B}}\ .$$

(a.)
Show that the result of the last part of the previous problem can be written

$$\langle I_Q^{A+B}\rangle(\tau) = I^A + I^B + 2\sqrt{I^A I^B}\,\mathrm{Re}\,\gamma^{AB}(\tau)\ .$$

(b.)
If the screen Q in a two-beam interference setup deviates slightly from perfect perpendicularity to the beams, deviations of order 10-100 occur in $\omega\tau$ across the screen. For most sources these deviations do not cause a significant change in $\vec{E}_P^B(t-\tau)$, but they do cause the phase of $\exp\left(i\omega\tau\right)$ to change dramatically. Correspondingly there appear on the screen many light and dark bands ("fringes"), at the center of which the respective irradiances are $I_{\max}$ and $I_{\min}$. Define the *fringe visibility* $\mathcal{V}$ as

$$\mathcal{V} \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}\ .$$

If $I^A = I^B$, show that

$$\mathcal{V} = |\gamma^{AB}(\tau)|\ .$$

(c.)
As an experimentalist, suppose that you are required to analyze the extent to which a mystery beam is polarized.

A standard approach would be to measure the elements of its Stokes vector (by observing the reduction in irradiance caused by four different optical devices – see Problem 59); knowing the Stokes vector, you could calculate the degree of polarization V (Problem 59(b.)).

Instead you decide to send the beam into a Michelson interferometer with two exactly equal-length paths A and B. Observing the resulting fringe pattern on screen Q, you measure the fringe visibility $\mathcal{V}$ (as defined in part (b.) of this problem).

Do you obtain any useful information about V by measuring $\mathcal{V}$? If so, what is the relationship between the two?

62.
We wish to use the light of Betelgeuse (angular diameter 0.047 arc second), passed through a 600 nm filter, as the source for a double-thin-slit Young's interference experiment.
(a.)
Assuming an adequately narrow filter bandpass, roughly estimate the maximum slit separation (in m) that would yield an interference pattern which isn't too badly washed out, *i.e.* with a fringe visibility $\mathcal{V}$ of order $\frac{1}{2}$.
(b.)
Assuming an adequately small slit separation, roughly estimate the maximum filter bandpass (in nm) that would allow us to observe at least 20 fringes. With this choice of bandpass, what is the coherence length of the transmitted light?

63.
A monochromatic beam traveling in medium "0" is normally incident upon a substrate *"T"*. A single film "1" is interposed between the two media. The refractive indices are, respectively, n_0, n_1, and n_T. You may assume that all materials have the same magnetic permeability.
(a.)
Show that a film of thickness $\lambda_1/4$ (where λ_1 is the wavelength of light in the material i of which the film is made) will reduce the reflectance of the substrate to zero, provided that $n_1 = \sqrt{n_0 n_T}$.
(b.)
Prove that interposing a single film of thickness $\lambda_1/4$ will *always* reduce the reflectance of the substrate, provided that $n_0 < n_1 < n_T$.

64.
Referring to the conditions of the previous problem, consider next the case of three films ("1", "2", and "3") interposed between the two media, such that film 1 adjoins medium 0 and film 3 adjoins medium T. Again, assume that all materials have the same magnetic permeability.
(a.)
Suppose that each film has thickness $\lambda_i/4$ (where λ_i is the wavelength of the beam in the particular material of which that film is made). Show that the reflectance of the substrate is reduced to zero when

$$\frac{n_1 n_3}{n_2} = \sqrt{n_0 n_T} \ .$$

(b.)
An advantage of using three films instead of one (as in the previous problem) is that the band of wavelengths over which the reflectance is heavily suppressed can be made much broader. (Your expensive eyeglasses are coated with at least two films.) According to {Pedrotti×2 ed2, ed3} Fig. {19-7, 22-7}, this benefit may be enhanced further if the middle film (2) is doubled in thickness from $\lambda_2/4$ to $\lambda_2/2$. In this case, what condition on n_0, n_1, n_2, n_3, and n_T reduces the reflectance to zero?

65.
Consider a high-reflectance stack of the type depicted in {Pedrotti×2 ed2, ed3} Fig. {19-8, 22-8}. For specificity, assume that the stack consists of six double layers of MGF_2 ($n = 1.38$) and ZnS ($n = 2.35$). For simplicity, assume that the medium from which the light enters the stack (medium 0) and the medium into which the light exits the stack (medium T) are vacuum. Again, assume that all materials have the same magnetic permeability.
(a.)
Numerically, what fraction T of the incident irradiance is transmitted by the stack?
(b.)
The stack is now modified as follows: the upstreammost three double layers are flipped around so that the stack indices are L(ow) H(igh) L H L H H L H L H L. This is a *Fabry-Perot interference filter*. It has a transmission *maximum* at the wavelength for which it was designed, as opposed to the transmission *minimum* achieved by the configuration of part (a.). Calculate the fraction T of the incident irradiance that is transmitted by the modified stack.