University of California, Berkeley
Physics 110B, Fall 2008 (*Strovink*)
HINTS FOR ASSIGNED EXERCISES

-6.
(**a.**)
Taking $\hat{z}$ to be its direction of propagation, a plane monochromatic electromagnetic wave with angular frequency ω is described by the electric and magnetic fields

$$\vec{E} = \mathrm{Re}\Big(\tilde{\mathbf{E}} \exp\big(i(\tilde{k}z - \omega t)\big)\Big)$$
$$\vec{H} = \mathrm{Re}\Big(\tilde{\mathbf{H}} \exp\big(i(\tilde{k}z - \omega t)\big)\Big) \ ,$$

where $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are complex constant vectors, and $\tilde{k}$ is a constant scalar, complex if the medium is conducting.

In an *insulating* medium with fixed uniform dielectric constant ϵ and magnetic permeability μ, $\tilde{k}$ is real. Show that Maxwell's equations reduce to

$$\tilde{E}_z = 0$$
$$\tilde{H}_z = 0$$
$$k\hat{z} \times \tilde{\mathbf{E}} = \mu\omega\tilde{\mathbf{H}}$$
$$k\hat{z} \times \tilde{\mathbf{H}} = -\epsilon\omega\tilde{\mathbf{E}} \ .$$

Hint: *Generically, to get one of these resulting equations, plug the above expressions for $\vec{E}, \vec{H}$ into one of the six Maxwell equations. Choose either a source-free Maxwell equation, such as Faraday's law, or a Maxwell equation for which the source term is a* **free** *charge or current, which vanishes in an insulator. Use the fact that $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$ with ϵ and μ strictly constant. Note that Maxwell's equations, like any equations involving only physical quantities like EM fields, must be real; as usual, choose to set equal the imaginary parts as well as the real parts of the complex equations of which Maxwell's equations are the real parts. Use the fact that* **differentiating** *commutes with* **taking the real part of**.

-5.
(**a.**)
Under the conditions of Problem **-6**, how many different real constants must be supplied in order to fully specify *e.g.* $\tilde{\mathbf{E}}$? Once $\tilde{\mathbf{E}}$ is specified, how many more additional real constants are needed to specify $\tilde{\mathbf{H}}$?
Hint: *How many constants are required to specify a complex vector in two dimensions?*

(**b.**)
Under these same conditions, show that the time-averaged energy flux density of the electromagnetic wave is

$$\langle \vec{S} \rangle = \tfrac{1}{2}\mathrm{Re}(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*) \ .$$

Hint: *Define $\tilde{\mathbf{E}} \equiv \hat{x}E_x e^{i\theta_x} + \hat{y}E_y e^{i\theta_y}$ where E_x, E_y are real (similarly for $\tilde{\mathbf{H}}$). By brute force, compute the time average $\langle \vec{S} \rangle \equiv \langle \vec{E} \times \vec{H} \rangle$ over one period at a fixed position, for example $z = 0$. By similar brute force, compute $\frac{1}{2}\mathrm{Re}(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*)$ and compare the result.*

Would this be true in a *conducting* medium?
Hint: *By definition, $\tilde{\mathbf{E}}, \tilde{\mathbf{H}}$ are independent of z. Is $\langle \vec{S} \rangle$?*

-4.
Consider a plane EM wave in medium 1 that is normally incident on a plane interface with medium 2. In lecture we derived the amplitude reflection coefficient

$$\tilde{R} = \frac{\tilde{Z}_1^{-1} - \tilde{Z}_2^{-1}}{\tilde{Z}_1^{-1} + \tilde{Z}_2^{-1}} \ ,$$

where, using the notation of Problem **-6**, the *inverse characteristic impedance* of the medium is

$$\tilde{Z}^{-1} \equiv \frac{\tilde{H}}{\tilde{E}} \ ,$$

and $\vec{E}$ is assumed to lie along $\hat{x}$ and $\vec{H}$ along $\hat{y}$.
(**a.**)
In a *conducting* medium with fixed uniform magnetic permeability μ, show that Faraday's law still requires, as in Problem -6,

$$\tilde{k}\tilde{E} = \mu\omega\tilde{H}$$
$$\Rightarrow \tilde{Z}^{-1} = \frac{\tilde{k}}{\mu\omega} \ .$$

Hint: *Repeat your derivation of the 3rd resulting equation of Problem* **-6**, *this time allowing* $\tilde{k}$ *to become complex.*

(**b.**)
Using the full Maxwell equations and allowing for nonzero constant conductivity σ (but neglecting any volume free charge density), we showed in lecture that

$$\frac{\tilde{k}}{\omega/v} \equiv r + is \ , \text{ where}$$
$$r = \sqrt{\frac{\sqrt{1+\gamma^2}+1}{2}}$$
$$s = \sqrt{\frac{\sqrt{1+\gamma^2}-1}{2}}$$
$$\gamma \equiv \frac{\sigma}{\epsilon\omega}$$
$$v \equiv \sqrt{\frac{1}{\epsilon\mu}} \ .$$

Considering the special case in which medium 1 is an insulator and medium 2 is a conductor, solve for the reflectivity $\tilde{R}$ and show that your result is consistent with Griffiths Eq. 9.147(a).
Hint: *In an insulator,* $\tilde{k}/(\mu\omega) = k/(\mu\omega) = 1/(\mu v)$, *where* v *is the phase velocity.*

-3.
(**a.**)
Using the results of Problem **-4**, and considering the limiting case of a poor conductor ($\gamma \ll 1$), show that both $\vec{E}$ and $\vec{H}$ are attenuated by a factor e for every distance

$$\Delta z = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$$

that is traversed by the plane wave.
Hint: *Retain the leading term in a Taylor series expansion of* $\sqrt{1+\gamma^2} - 1$.

(**b.**)
Consider again the problem of reflection at normal incidence. Suppose that both media have the same ϵ/μ, but medium 1 is an insulator while medium 2 has a small but nonzero conductivity. In the poor-conductivity limit show that

$$\tilde{R} = -\frac{i\gamma}{4} \ .$$

Hint: *Retain the leading term in a Taylor series expansion of* $\sqrt{1+\gamma^2} - 1$. *Use the fact that* $\sqrt{\epsilon/\mu} = 1/(\mu v)$ *in an insulator.*

-2.
Seawater has a resistivity of ≈ 0.2 ohm-meters, and the dielectric constant of seawater is $\epsilon/\epsilon_0 \approx 80$.
(**a.**)
Using reasonable approximations, estimate the angular frequency ω at which the amplitude of electromagnetic waves in seawater will diminish by a factor e for every 10 meters traversed.
Hint: *To reach a skin depth as large as 10 m, should we be in the "good conductor" limit* $\gamma \gg 1$, *i.e.* $\omega \ll \sigma/\epsilon$, *or the "poor conductor" limit* $\gamma \ll 1$? *It's easiest initially to consider the second option, given that the result of Problem* **-3**(**a**) *is frequency-independent. Note that* $\sqrt{\mu_0/\epsilon_0}$, *the* **characteristic impedance of free space**, *is 377 ohms.*

(**b.**)
A dial-up modem allows communication at a maximum of 56 kilobits per second over telephone lines working at audio frequencies ($\omega \leq\approx 2\pi \times 10^4$ Hz). If electromagnetic waves exchanged between satellites and submarines must penetrate ≈ 30 m of seawater without being attenuated by more than a factor ≈ 400 in power, estimate the maximum number of bits per second that can be exchanged.
Hint: *How should the maximum bit rate depend on the working frequency?*

-1.
You shine a 1W flashlight on a crow, who absorbs its beam. (Model the mechanical and thermal properties of a crow as those of 100 g of water.)
(**a.**)
How long must the flashlight shine to heat the crow by 1 degree Celsius?
Hint: *The specific heat of water is 1 calorie per gram per* $^\circ$*K, and 1 calorie = 4.186 joules.*

(**b.**)
How long must the flashlight shine to accelerate

the crow from rest to a velocity of 1 mm/sec (assuming no other forces are acting)?
Hint: *The flashlight beam is composed of massless photons. The standard equation of relativistic kinematics is $E^2/c^2 = p^2 + m^2c^2$, where E is the energy and p is the momentum.*

0.
Griffiths Problem 9.18.
(a) **Hint:** *Use Eq. (9.120) of Griffiths. Characteristically, the refractive index of glass is* $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} \approx 1.5$.

(b) **Hint:** *In the good-conductor limit $\gamma \gg 1$, the thickness of the conductive coating should be several times its skin depth $\Delta z \equiv 1/\mathrm{Im}(\tilde{k})$.*

(c) **Hint:** *Use the formulæ of Problem* **-4(b)**. *As always, the wavelength is defined as $2\pi/\mathrm{Re}(\tilde{k})$ even when significant attenuation occurs within one wavelength.*

1.
Consider a uniform static magnetic field

$$\vec{B} = \hat{z}\, B_0 \ ,$$

where B_0 is a constant.
(**a.**)
Show that $\vec{B}$ can arise from the vector potential

$$\vec{A}_a = -B_0 y\, \hat{x} \ .$$

Hint:
In cartesian coordinates, take the curl of $\vec{A}_a$.
(**b.**)
Show that $\vec{B}$ can arise from the vector potential

$$\vec{A}_b = \tfrac{1}{2} B_0 s\, \hat{\phi}$$

(s and ϕ are cylindrical coordinates).
Hint:
In cylindrical coordinates, take the curl of $\vec{A}_b$. Note that s is the perpendicular distance from the point of interest to the z axis.
(**c.**)
By coordinate-system-independent vector analysis, show that $\vec{B}$ can arise from the vector potential

$$\vec{A} = \tfrac{1}{2}\vec{B} \times \vec{r}$$

(remember that $\vec{B}$ is constant).
Hint:
Use Identity #8 on Inside Cover (IC) #2 of Griffiths (G). Does $\vec{B}$ have any nonzero derivatives? It's easy to evaluate $\nabla \cdot \vec{r}$ and $(\vec{B} \cdot \nabla)\vec{r}$ in cartesian coordinates. Do these (trivial) results depend on the coordinate system chosen?
(**d.**)
Referring to G Eq. (10.7), find the gauge function λ that accomplishes the gauge transformation from $\vec{A}_a$ to $\vec{A}_b$.
Hint:
If you set $\vec{A}_a$ equal to G's $\vec{A}$, and $\vec{A}_b$ equal to his $\vec{A}'$, G Eq. (10.7) becomes

$$\vec{A}_b - \vec{A}_a = \nabla\lambda \ .$$

You'll find it convenient to express both $\vec{A}_a$ and $\vec{A}_b$ in the same coordinate system – probably cartesian is easier. (Remember that $\hat{\phi}$ is not a constant – it depends on where you are. See GIC #4.) Integrate

$$\int_0^{\vec{r}} (\nabla\lambda) \cdot d\vec{\ell}$$

and use the Gradient Theorem (GIC #2) to isolate $\lambda(\vec{r}) - \lambda(0)$; to evaluate this difference, similarly integrate the left-hand side (LHS) over any path you like.

2.
Griffiths Problem 10.3.
Hint:
The fields come from G Eq. (10.2) and (10.3) (spherical polar coordinates are convenient). As usual, get the charge and current distributions from the sourceful Maxwell equations (G Eq. (2.16) and (7.36)).

3.
Griffiths Problem 10.5.
Hint:
Use G *Eq. (10.7) to do the transformation. The result should be very familiar!*

4.
In free space with $\rho = 0$ and $\vec{J} = 0$, show that

all four Maxwell equations can be obtained correctly if the scalar potential V is assumed to vanish, while the vector potential $\vec{A}$ satisfies

$$0 = \nabla \cdot \vec{A}$$
$$0 = \big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\big)\vec{A} \,.$$

Hint:
What do G *Eqs. (10.2) and (10.3) tell you about the sourceless Maxwell equations for any $\vec{A}$ and V? So let's consider the sourceful Maxwell equations. To show that $\nabla \cdot \vec{E} = 0$ as required by Gauss's law, use Eq. (10.2) to express $\vec{E}$ in terms of $\vec{A}$; then use the first of the above equations. To show that $c^2 \nabla \times \vec{B} = \partial \vec{E}/\partial t$ as required by Maxwell's version of Ampère's law, use Eq. (10.3) to express $\vec{B}$ in terms of $\vec{A}$, then apply Identity #11 (*GIC *#2). Use the first of the above equations to eliminate one term and the second plus Eq. (10.3) to reexpress what remains in terms of $\vec{E}$.*

5.
Griffiths Problem 10.7.
Hint:
We worked this problem in class – please consult your notes. We assumed that

$$\nabla \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial V}{\partial t} = s(\vec{r}, t) \,,$$

where s is any nonzero time-dependent scalar field. What we wanted was

$$\nabla \cdot \vec{A}' + \epsilon_0 \mu_0 \frac{\partial V'}{\partial t} = 0$$

for a different (primed) set of potentials related to the first set by G *Eq. (10.7). Using Eq. (10.7) to substitute primed potentials for the unprimed potentials in the first equation above, and taking advantage of a cancellation from the second equation, we were left with a result of the form* G *(10.14) with λ replacing V and s replacing ρ/ϵ_0. If we know how to solve (10.14) for V, do we know how to solve the equation you got for λ?*

6.
Consider the Levi-Civita density $\epsilon_{ijk} \equiv 1$ ($ijk =$ even permutation of 123); $\equiv -1$ (odd permutation of 123); $\equiv 0$ (otherwise). It is found, for example, in the cross product

$$(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k \,.$$

Note that summation over the repeated indices j and k is implied; their domain is $1 \le j, k \le 3$.
(a.)
Show that

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \,,$$

where δ is the Kronecker delta function (whose elements are those of the unit matrix).
Hint:
ϵ_{ijk} and ϵ_{klm} vanish unless $i \neq j \neq k$ and $k \neq l \neq m$. Therefore only one value of k yields a nonzero term, which occurs either if $i = l$ and $j = m$ or if $i = m$ and $j = l$. Suppose that k takes on its one useful value, that $i = l$ and $j = m$, and that ijk is an even permutation (even number of adjacent swaps) of 123. What value does $\epsilon_{ijk}\epsilon_{klm}$ take under those circumstances? Work out the other three possibilities.
(b.)
The determinant of a 3×3 matrix is given by

$$\det A \propto \epsilon_{ijk} A_{il} A_{jm} A_{kn} \epsilon_{lmn} \,.$$

By considering the number of nonzero terms on the RHS, and comparing it to the number of terms you would have expected for a 3×3 determinant, deduce the constant of proportionality. Express it in terms of a factorial.
Hint:
Consider the special case $A_{ij} = \delta_{ij}$, i.e. A is the unit matrix. Then $\det A = \epsilon_{ijk}\epsilon_{ijk}$. An individual term in this sum is equal to 1 if $i \neq j \neq k$ and 0 otherwise. How many different combinations of i, j, and k are there in which i, j, and k are all different?
(c.)
Guessing the explicit constant of proportionality, write a similar equation for the determinant of a 4×4 matrix. How should ϵ_{ijkl} be defined?
Hint:
How many different combinations of i, j, k, and l are there in which i, j, k, and l are all different? In defining ϵ_{ijkl}, consider the fact that swapping

two adjacent indices of ϵ_{ijkl} is like swapping two adjacent rows of the matrix whose determinant is calculated; swapping two adjacent indices of ϵ_{mnop} is like swapping two adjacent columns of that matrix.

7.
Griffiths Problem 12.55. Don't get fooled by the typo – he means "$\partial^\mu \equiv \partial/\partial x_\mu$".
Hint:
Use the chain rule to express $\partial/\partial x'_\mu$ in terms of $\partial x_\nu/\partial x'_\mu$ and $\partial/\partial x_\nu$. The problem then becomes one of expressing $\partial x_\nu/\partial x'_\mu$ in terms of $\Lambda^\mu{}_\nu$.

Consider the **direct** *Lorentz transformation of the* **covariant** *spacetime four-vector x:*

$$x'_\nu = x_\mu \left(\Lambda^{-1}\right)^\mu{}_\nu \ .$$

Now, for any linear transformation

$$x'_\nu = \frac{\partial x'_\nu}{\partial x_\mu} x_\mu \ .$$

Therefore

$$\frac{\partial x'_\nu}{\partial x_\mu} = \left(\Lambda^{-1}\right)^\mu{}_\nu \ .$$

Likewise, considering the **inverse** *transformation of the* **covariant** *four-vector x, show that*

$$\frac{\partial x_\nu}{\partial x'_\mu} = \Lambda^\mu{}_\nu \ .$$

8.
An object a^μ is a (contravariant) four-vector if it transforms (between frames as defined in Short Course in Special Relativity (SCSR) Fig. 2) according to

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu \ ,$$

where Λ is the (symmetric) 4×4 Lorentz transformation matrix. (Conventionally, the first (superscript) index labels the row and the second (subscript) index labels the column, but this makes no difference for a symmetric matrix.) Covariant four-vectors instead transform according to

$$a'_\mu = a_\nu (\Lambda^{-1})^\nu{}_\mu$$

(otherwise the scalar product $a_\mu a^\mu = a'_\mu a'^\mu$ would not remain invariant for different Lorentz frames). Consider now an (arbitrary) four-tensor $H^{\mu\nu}$. In frame $\mathcal{S}$, $H^{\mu\nu}$ contracts with covariant four-vector a_ν to yield contravariant four-vector b^μ, according to

$$b^\mu = H^{\mu\nu} a_\nu \ .$$

In the frame $\mathcal{S}'$, requiring $H^{\mu\nu}$ to satisfy the transformation properties of a four-tensor, we define $H'^{\mu\nu}$ so that

$$b'^\mu = H'^{\mu\nu} a'_\nu \ .$$

Prove that

$$H'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma H^{\rho\sigma} \ .$$

This defines the Lorentz transformation property of a four-tensor.
Hint:
Start from

$$b^\rho = H^{\rho\sigma} a_\sigma \ .$$

Substitute

$$b^\rho = (\Lambda^{-1})^\rho{}_\alpha b'^\alpha$$

and

$$a_\sigma = a'_\nu \Lambda^\nu{}_\sigma \ .$$

Then multiply both sides of the resulting equation by $\Lambda^\mu{}_\rho$ and use the fact that

$$\Lambda^\mu{}_\rho (\Lambda^{-1})^\rho{}_\alpha = \delta^\mu{}_\alpha \ ,$$

where $\delta^\mu{}_\alpha$ is an element of the unit matrix. Use this fact to simplify the left-hand side. Then compare your equation to

$$b'^\mu = H'^{\mu\nu} a'_\nu$$

and draw the desired conclusion.

9.
Consider the antisymmetric *electromagnetic field strength tensor*

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ ,$$

where both ∂^μ and A^μ are (contravariant) four-vectors. Prove that $F^{\mu\nu}$ is a four-tensor, *i.e.* it

transforms according to the result of Problem 8.
Hint:
Start from

$$F'^{\mu\nu} = \partial'^{\mu} A'^{\nu} - \partial'^{\nu} A'^{\mu} \ .$$

Write

$$\partial'^{\mu} = \Lambda^{\mu}{}_{\rho} \partial^{\rho}$$
$$A'^{\nu} = \Lambda^{\nu}{}_{\sigma} A^{\sigma}$$

(similarly for ∂'^{ν} and A'^{μ}), plug in, and collect terms.

10. (Light cone)
(a.)
Event A occurs at spacetime point $(ct, x, y, z) = (0, 1, 1, 1)$; event B occurs at $(1, 0, 0, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events A and B occur at the same spatial coordinates? If so, find $c|t'_A - t'_B|$, c times the magnitude of the time interval in $\mathcal{S}'$ between the two events.
Hint:
Denote by x_A (x_B) the spacetime coordinate of event A (B); $\Delta x \equiv x_B - x_A$. $\Delta x \cdot \Delta x$ is the same in any Lorentz frame. Is Δx timelike? If so, how is $\Delta x' \cdot \Delta x'$ related to $c|t'_A - t'_B|$?
(b.)
Is there an inertial system $\mathcal{S}''$ in which events A and B occur simultaneously? If so, find $|\vec{r}''_A - \vec{r}''_B|$, the distance in $\mathcal{S}''$ between the two events.
Hint:
Is Δx spacelike? If so, how is $\Delta x'' \cdot \Delta x''$ related to $|\vec{r}''_A - \vec{r}''_B|$?
(c.)
Can event A be the cause of event B, or vice versa? Explain.
Hint:
Is Δx timelike?
(d.)
Event D occurs at spacetime point $(ct, x, y, z) = (-1, 0, 0, 0)$; event E occurs at $(2, 1, 1, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events D and E occur simultaneously? If so, find $|\vec{r}'_E - \vec{r}'_D|$, the magnitude of the distance in $\mathcal{S}'$ between the two events.
Hint:
Is Δx spacelike? If so, how is $\Delta x' \cdot \Delta x'$ related to $|\vec{r}'_E - \vec{r}'_D|$?
(e.)
Is there an inertial system $\mathcal{S}''$ in which events D and E occur at the same spatial coordinates? If so, find $c|t''_E - t''_D|$, c times the magnitude of the time interval in $\mathcal{S}''$ between the two events.
Hint:
Is Δx timelike? If so, how is $\Delta x'' \cdot \Delta x''$ related to $c|t''_E - t''_D|$?

11.
Using *e.g.* the method of Short Course in Special Relativity [SCSR] §7, obtain the 4×4 Lorentz transformation matrix for the case in which frame $\mathcal{S}'$ moves with respect to frame $\mathcal{S}$ with speed $\beta_0 c$ in an *arbitrary* direction $(n_1, n_2, 0)$ in the x-y plane, where $\vec{n}$ is a unit vector.
Hint:
Consult SCSR *§7.*

12.
(a.)
In SCSR §8, clock time intervals measured in a frame in which the clock is not at rest are shown to be *dilated* by the factor γ_0. This analysis used the *inverse* Lorentz transformation. Reanalyze the same problem using the *direct* Lorentz transformation. Is the answer the same?
Hint:
The temporal direct Lorentz transformation is

$$ct'_2 = \gamma_0 ct_2 - \gamma_0 \beta_0 x_2$$
$$ct'_1 = \gamma_0 ct_1 - \gamma_0 \beta_0 x_1 \ .$$

Subtracting,

$$\gamma_0 c\Delta t = c\Delta t' + \gamma_0 \beta_0 \Delta x \ .$$

In order to eliminate Δx, apply the spatial direct Lorentz transformation and use the fact that $\Delta x' = 0$.
(b.)
In SCSR §9, the length of a rod measured in a frame in which the rod is not at rest is shown to be *contracted* by the factor $1/\gamma_0$. This analysis used the *direct* Lorentz transformation. Reanalyze the same problem using the *inverse* Lorentz transformation. Is the answer the same?
Hint:
The spatial inverse Lorentz transformation is

$$x_2 = \gamma_0 x'_2 + \gamma_0 \beta_0 ct'_2$$
$$x_1 = \gamma_0 x'_1 + \gamma_0 \beta_0 ct'_1 \ .$$

Subtracting,

$$\Delta x = \gamma_0 \Delta x' + \gamma_0 \beta_0 c \Delta t' \ .$$

In order to eliminate $\Delta t'$, apply the temporal inverse Lorentz transformation and use the fact that $\Delta t = 0$.

13. (Addition of velocities)
In texts that do not emphasize the rapidity or boost parameter η, the Einstein law for the addition of velocities is derived less elegantly as follows (see SCSR Fig. 7). Denote by x^1 (x'^1) the x coordinate of the origin of $\mathcal{S}''$ as observed in the lab frame $\mathcal{S}$ (moving frame $\mathcal{S}'$). Write a standard inverse Lorentz transformation

$$x^0 = \gamma x'^0 + \gamma\beta x'^1$$
$$x^1 = \gamma x'^1 + \gamma\beta x'^0 \ .$$

Then take the differential of it: $dx^0 = \ldots$; $dx^1 = \ldots$. Divide the bottom by the top equation and identify

$$\frac{dx^1}{dx^0} = \beta'' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S}$$
$$\frac{dx'^1}{dx'^0} = \beta' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S}' \ .$$

Obtain the Einstein law for the addition of velocities (SCSR Eq. (24)):

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'} \ .$$

Hints *are already embedded in the statement of this problem.*

14.
Consider the standard case in which two Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$ coincide at $t = t' = 0$, with frame $\mathcal{S}'$ moving at velocity $\beta c\,\hat{x}$ with respect to frame $\mathcal{S}$. As seen in a third frame $\mathcal{S}''$, also moving along $\hat{x}$ with respect to $\mathcal{S}$, two clocks fixed to the origins of frames $\mathcal{S}$ and $\mathcal{S}'$, respectively, appear to agree. With respect to frame $\mathcal{S}$, considering that *rapidity ("boost") is the additive parameter of the Lorentz transformation*, show that the speed $\beta'' c$ of frame $\mathcal{S}''$ is given by

$$\beta'' = \tanh\left(\tfrac{1}{2}\tanh^{-1}\beta\right) \ .$$

Hint:
If the clocks in $\mathcal{S}$ and $\mathcal{S}'$ agree, what does that say about the $|\text{boost}|$ of each frame relative to frame $\mathcal{S}''$? If frames $\mathcal{S}$ and $\mathcal{S}'$ are different, can the signs of their boosts relative to frame $\mathcal{S}''$ be the same? Keeping in mind that rapidity (boost) is the additive parameter of the Lorentz transformation, write a simple equation describing how the boosts η'' and η are related. Then convert that equation to an equation in β'' and β.

15. (Taylor & Wheeler problem 51)
The clock paradox, version 3.
Hints *are already embedded in the statement of this problem, which is lengthy and not repeated here.*

16. (Surface muons)
"Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays to an (anti)muon (μ^+, a heavy electron-like particle) and a muon neutrino (ν_μ) via

$$\pi^+ \to \mu^+ + \nu_\mu \ .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed (and, as it turns out, 100% polarization as well). For the purposes of this problem, consider a muon to have $\frac{3}{4}$ of the rest mass of a pion; neglect the neutrino mass.
(a.)
Show that the surface muons travel at a speed which is a fraction $\beta_0 = \frac{7}{25}$ of the speed of light.
Hint:
Take π, μ, and ν to be the four-momenta of the pion, muon, and neutrino, respectively. Express energy-momentum conservation two ways:

$$\pi - \mu = \nu$$
$$\pi - \nu = \mu \ .$$

For each equation take the dot product of the LHS *and* RHS *with itself. Solve the first equation for E_μ and the second for E_ν. Given that the neutrino is massless, what can you say about the relationship between E_ν and $c|\vec{p}_\nu|$? Given that momentum is conserved, what can you say about the relationship between $\vec{p}_\nu$ and $\vec{p}_\mu$? How is β_μ related to E_μ and $c|\vec{p}_\mu|$?*

(**b.**)

If a muon's mean proper lifetime is τ, what fraction of the muons will decay during a flight path of length L in the laboratory? Express your answer in terms of β_0.

Hint:

Apply time dilation to find the muon's mean lab lifetime, and use the muon's lab velocity to convert its mean lab lifetime to a mean lab path length. For spontaneous decay, remember that the survival probability depends exponentially on the appropriate parameter (time or distance).

17.

In the lab frame $\mathcal{S}$, a particle with velocity $\beta c = \frac{4}{5}c$ decays into two massless particles with the same energy each.

(**a.**)

If the parent particle has mean (proper) lifetime τ in its *own* rest frame $\mathcal{S}'$, calculate its mean flight path L in the lab frame $\mathcal{S}$.

Hint:

Apply time dilation to find the parent particle's mean lab lifetime, and use the particle's lab velocity to convert its mean lab lifetime to a mean lab flight path.

(**b.**)

In the lab frame $\mathcal{S}$, calculate the opening angle $\psi = \cos^{-1} \hat{p}_1 \cdot \hat{p}_2$ between the two daughter particles.

Hint:

Take P, p_1, and p_2 to be the four-momenta of the parent and two daughters. Expressing energy-momentum conservation as $P = p_1 + p_2$, take the dot product of the LHS *and* RHS *with itself. Given that the daughter particles are massless, how are $c|\vec{p}_{1,2}|$ related to $E_{1,2}$? Given that energy is conserved, how are $E_{1,2}$ related to the parent energy E? Given the parent's speed βc, how is $c|\vec{P}|$ related to E? Solve for $\cos^{-1} \hat{p}_1 \cdot \hat{p}_2$.*

18.

Here's an adult version of Griffiths' Problem 12.35. In a pair annihilation experiment, a positron (mass m) with total energy $E = \gamma mc^2$ hits an electron (same mass, but opposite charge) at rest. (Griffiths has it the other way around, but that's unrealistic – it's easy to make a positron beam, but hard to make a positron target.) The two particles annihilate, producing two photons. (If only one photon were produced, energy-momentum conservation would force it to be a massive particle traveling at a velocity less than c.) If one of the photons emerges at angle θ relative to the incident positron direction, show that its energy ϵ is given by

$$\frac{mc^2}{\epsilon} = 1 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} \cos\theta \ .$$

(In particular, if the photon emerges perpendicular to the beam, its energy is equal to mc^2, independent of the beam energy. Similar results have been used to design clever experiments.)

Hint:

Take a, b, d, and e to be the four-momenta in this 2+2 reaction $a + b \to d + e$; note that particles a and b have mass m, that particle b is at rest, and that particle d emerges at angle θ with respect to the direction of a. Because the least is known about particle e, express energy-momentum conservation as

$$a + b - d = e$$

and take the dot product of the LHS *and* RHS *with itself. Your expression should involve E_a, $\vec{p}_a$, and $\vec{p}_d$ as well as m and ϵ (the energy of d). Use the fact that particle d is massless to express $c|\vec{p}_d|$ in terms of ϵ. You are given the Lorentz factor γ of particle a. Eliminate E_a by expressing it in terms of m and γ, and eliminate $|\vec{p}_a|$ by expressing it in terms of m, γ, and $\beta(\gamma)$.*

19.

If you have studied Rutherford scattering (elastic scattering of a nonrelativistic He nucleus from an Au nucleus), you have seen the *differential cross section* for this process written in the form

$$\frac{d\sigma}{d\Omega} \propto \frac{(zZe^2)^2}{\sin^4 \frac{\Theta}{2}} \ ,$$

where ze (Ze) is the electric charge of the He (Au) nucleus; Θ is the angle by which the He nucleus is elastically scattered, measured in the CM frame; and $d\Omega$ is an element of solid angle within which the He nucleus emerges in that frame. [A century ago, Rutherford-scattering data collected by graduate students who were used as particle detectors demonstrated that atoms contain charged point-like constituents (nuclei).] Here we revisit Rutherford scattering for *relativistic* particles.

Consider the 2+2 relativistic scattering process

$$p + a \to q + b \ ,$$

where p, a, q, and b denote both the particles and their 4-momenta. The *Mandelstam variables*, first written down by Berkeley emeritus professor Stanley Mandelstam, are

$$\begin{aligned} s &\equiv (p+a)\cdot(p+a) \equiv \text{CM energy}^2 \\ t &\equiv (q-p)\cdot(q-p) \equiv 4 \text{ momentum transfer}^2 \\ u &\equiv (b-p)\cdot(b-p) \equiv \text{cross channel transfer}^2 \end{aligned}$$

In this problem we are concerned with the Mandelstam variable t.

(a.)
Further assuming that the masses of particles p and q are negligible, show that

$$-t = 4\tfrac{E}{c}\tfrac{E'}{c}\sin^2\tfrac{\Theta}{2} \ ,$$

where E (E') is the energy of particle p (q), and Θ is the angle between $\vec{p}$ and $\vec{q}$.
Hint:
Take the dot product of $(q-p)$ with itself; use the fact that particles q and p are massless to express $c|\vec{p}|$ and $c|\vec{q}|$ in terms of E and E'.
(b.)
Why is $d(-t)$ a Lorentz invariant? $d\sigma$ is an area transverse to the beam direction. Why is $d\sigma$ invariant to Lorentz transformations along that direction? In a system of units where $\hbar = c = 1$, all quantities have dimensions that can be expressed in units of Joules. In those units, what are the dimensions of $d\sigma$?
Hint:
The constant $\hbar c$ has dimensions of (energy) $\times$ (distance) [$\hbar c \approx 197 \times 10^{-9}$ eV m in SI]. However, in this problem's system of units $\hbar c$ is equal to unity, allowing you to multiply or divide by it at will.
(c.)
In part (**b.**) you showed that $d\sigma/d(-t)$ is a Lorentz invariant. If particle p (which becomes particle q) and particle a (which becomes particle b) both are structureless, and if the scattering is elastic (particles a and b both have the same mass), the only relevant Lorentz-invariant variable that is available to us is $-t$. On purely dimensional grounds, show that

$$\frac{d\sigma}{d(-t)} \propto \frac{1}{t^2} \ .$$

[If p (a) has electric charge ze (Ze), and they interact electromagnetically, the constant of proportionality is $4\pi z^2 Z^2 \alpha^2$, where the fine structure constant α is given as usual by $4\pi\epsilon_0\alpha = e^2/\hbar c$. This formula is correct to the extent that Z or $z \times (\alpha \approx 1/137)$ can be neglected relative to unity.]
(d.)
Under all of these conditions, using the results of (**a.**) and (**c.**) and working in the center of mass, show that the nonrelativistic elastic scattering result

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\frac{\Theta}{2}}$$

does remain valid even when relativistic effects are taken into account.
Hint:
In the CM, *the elastic scattering process is specified completely by the single variable Θ. Express $-t$ and $d(-t)$ in terms of Θ and $d\Theta$. Then, integrating over azimuth, express $d\Theta$ in terms of $d\Omega$.*

20. (Relations used in particle physics)
(a.) Lorentz-invariant phase space [LIPS]
By transforming dp_x and E while keeping p_y and p_z fixed, show that

$$\frac{c\,dp_x}{E}dp_y dp_z$$

is invariant to a boost along x. (Since one can always define x to be the boost direction, and $d^3p \equiv dp_x dp_y dp_z$, a LIPS element

$$\frac{c\,d^3p}{E}$$

is invariant to a boost in *any* direction and therefore is Lorentz invariant.)

Hint:

You are trying to show

$$\frac{c\,dp'_x dp'_y dp'_z}{E'} = \frac{c\,dp_x dp_y dp_z}{E} \ .$$

Use a direct Lorentz transformation along x to express dp'_x in terms of dp_x and dE, and to express E' in terms of E and p_x. Consider the fact that the four-momentum^2 is equal to m^2c^2:

$$\tfrac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = m^2c^2 \ .$$

Holding p_y and p_z constant, take the differential of each side to obtain a relation between dE and dp_x. Use this relation to eliminate dE from your expression for dp'_x; a welcome cancellation should result.

(**b.**)

Suppose a particle has momentum $\vec{p}$ and energy E. Define the particle's *longitudinal rapidity* y to be the boost along x that would be needed to make $p'_x = 0$ in the new frame $\mathcal{S}'$. Show that

$$y = \tanh^{-1}\frac{cp_x}{E} \ .$$

If your calculator doesn't have an arc hyperbolic tangent button, use the equivalent definitions

$$y = \tfrac{1}{2}\ln\frac{E/c + p_x}{E/c - p_x}$$
$$y = \ln\frac{E/c + p_x}{\sqrt{p_y^2 + p_z^2 + m^2c^2}} \ .$$

Hint:

How is the particle's |velocity| $|\vec{\beta}|c$ related to $|\vec{p}|$ and E? How is the x component $\beta_x c$ of that velocity related to p_x and E? What relative velocity β_0 of $\mathcal{S}'$ along the x direction would cause β'_x and therefore p'_x to vanish? How is β_0 related to the required boost?

(**c.**)

Using the fact that the rapidity (boost) is the additive parameter for the Lorentz transformation, and that y is defined to be a boost along $\hat{x}$, argue that an increment dy in longitudinal rapidity must be the same in two Lorentz frames that differ only by a relative boost along $\hat{x}$. Use this argument to conclude that

$$dy\,dp_y dp_z$$

is invariant to boosts along $\hat{x}$, as was the LIPS element

$$\frac{c\,d^3p}{E}$$

in part (**a.**). (In fact, these two expressions are equal.) Invariance of the longitudinal rapidity interval dy is a godsend for proton collider users. Since the proton's interacting constituents (quarks or gluons) carry only a variable fraction of the proton momentum, the center of mass (CM) of the colliding constituents is boosted along the beam direction by a variable amount (typically of order unity). However, the *difference* in longitudinal rapidity between any pair of emitted particles is unaffected by this unwelcome CM boost.)

(**d.**)

Define the *pseudorapidity* y_{pseudo} as the longitudinal rapidity that a particle would have if it were ultrarelativistic. Show that

$$y_{\text{pseudo}} = \tanh^{-1}(\cos\theta) \ ,$$

where θ is the angle between the particle's direction and the x axis. (This is another godsend: if a particle is known to be ultrarelativistic, its longitudinal rapidity can be approximated by its pseudorapidity, which can be measured by knowing only the particle's direction.) An equivalent definition of pseudorapidity is

$$y_{\text{pseudo}} = -\ln\left(\tan\tfrac{\theta}{2}\right) \ .$$

Hint:

If the particle is ultrarelativistic, what is the relation between $c|\vec{p}|$ and E?

21. (Effect of inefficient rocket engine)

All of the energy put out by the rocket engine depicted in SCSR Fig. 9 consists of particles emitted straight out the back. Consider the more realistic case in which only a fraction ϵ of the energy output consists of such particles; as seen in a Lorentz frame comoving with the rocket, the balance of

the energy is emitted isotropically, for example as thermal photons. Therefore ϵ is the engine's *efficiency*. Otherwise adopt the conditions of SCSR Fig. 9 and carry out a derivation analogous to that found in SCSR §14. Assuming that it starts from rest, show that the final boost of the rocket is reduced directly by this efficiency factor:

$$\eta_{\text{final}} = \epsilon\,|\vec{\beta}_1|\ln\frac{m_0}{m_{\text{final}}}\ .$$

Hint:
For a perfectly efficient engine, the first set of equations in SCSR *§14 is*

$$\begin{aligned} P_0 &= (mc, \vec{0}) \\ P' &\approx \big((m-dm)c + \tfrac{1}{2}mc|d\vec{\beta}|^2, mc\,d\vec{\beta}\big) \\ p_1 &= \big(\frac{dE_1}{c}, \vec{\beta}_1\frac{dE_1}{c}\big)\ . \end{aligned}$$

When the engine becomes inefficient, a third class of final-state elements is created: photons that carry off energy but are emitted isotropically. Define another four-momentum

$$q_1 = \big(\frac{dF_1}{c}, \vec{q}_1\big)$$

to represent the sum of these photons. What is the value of their net momentum $\vec{q}_1$? How is their total energy dF_1 related to dE_1?

22. (Wave aberration)
Please refer to SCSR Fig. 10. Consider Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with spatial origins coincident at $t = t' = 0$. As usual, frame $\mathcal{S}'$ moves in the $\hat{x} = \hat{x}'$ direction with velocity $\beta_0 c$ relative to frame $\mathcal{S}$. A wave is emitted by a source that is at rest with respect to $\mathcal{S}'$. As seen by an observer in the lab frame $\mathcal{S}$, the wave travels with phase velocity $\beta_{\text{ph}}c$ at an angle θ with respect to the $\hat{x}$ direction ($\theta = 0$ if directly approaching, $\theta = \pi$ if directly receding). However, as seen by an observer who is at rest with respect to the frame $\mathcal{S}'$, show that the wave makes a different angle θ' with respect to the $\hat{x}'$ direction, where

$$\tan\theta' = \frac{\sin\theta}{\gamma_0(\cos\theta - \beta_0\beta_{\text{ph}})}\ .$$

Hint:
Defining the y axis so that the wave travels in the xy plane, write direct Lorentz transformations for k'_y and k'_x.

23. (Lorentz transformation of EM fields)
Consider Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with frame $\mathcal{S}'$ moving in the $\hat{x} = \hat{x}'$ direction with velocity $\beta_0 c$ relative to frame $\mathcal{S}$. Using the Lorentz transformation for the field strength tensor,

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma}\ ,$$

and considering explicitly the values of the elements of $F^{\mu\nu}$, as given by SCSR Eq. (62), show that

$$E'_y = \gamma_0(E_y - \beta_0 c B_z)\ ,$$

as claimed by Griffiths' Eq. (12.102).
Hint: *Evaluate F'^{02} using the elements of Λ and F.*

24. (Relativistic electron-positron beams)
In a straight channel oriented along the $\hat{z}$ axis there are two opposing beams:

- a beam of positrons (charge $+e$) with velocity $+\hat{z}\beta_0 c$.
- a beam of electrons (charge $-e$) with velocity $-\hat{z}\beta_0 c$.

Each beam is confined to a small cylindrical volume of cross sectional area A centered on the $\hat{z}$ axis. Within that volume, there is a uniform number density $= n$ positrons/m^3 and n electrons/m^3.
(a.)
In terms of n, A, e, and β_0, calculate the total current I in the channel due to the sum of both beams (note $I \neq 0$).
Hint:
For a positron plasma of uniform velocity $\vec{v}$ and number density n, the current density $\vec{J} = +en\vec{v}$, and the current $I = \iint \vec{J}\cdot d\vec{a}$.
(b.)
Use Ampère's Law to calculate the (azimuthal) magnetic field $\vec{B}$ outside the channel a distance s from the $\hat{z}$ axis.

Consider now a Lorentz frame $\mathcal{S}'$ traveling in the $\hat{z}$ direction with velocity $\beta_0 c$ relative to the lab

frame described above. (This β_0 is the same β_0 as above.)

(**c.**)

As seen in $\mathcal{S}'$, calculate the number density n'_+ of *positrons* within the cylindrical volume. (You may use elementary arguments involving space contraction, or you may use the fact that $(c\rho, \vec{J})$ is a 4-vector, where ρ is the charge density (coul/m^3) and $\vec{J}$ is the current density (amps/m^2).)

Hint:

Considering only the positrons, write an inverse Lorentz transformation for $c\rho_+$. For this particular relative velocity between $\mathcal{S}'$ and $\mathcal{S}$, what is $\vec{J}'_+$? How is $\rho_+^{(\prime)}$ related to $n_+^{(\prime)}$?

(**d.**)

As seen in $\mathcal{S}'$, calculate the number density n'_- of *electrons* within the cylindrical volume.

Hint:

Considering only the electrons, write a direct Lorentz transformation for $c\rho'_-$. Because $\mathcal{S}'$ is not at rest with respect to the electrons, your result will be slightly less elementary than your result for n'_+.

(**e.**)

Calculate the (cylindrically radial) electric field $\vec{E}'$ seen outside the channel in $\mathcal{S}'$. Do this both

- by using the results of (**c.**) and (**d.**) plus Gauss's law, and
- by using the results of (**b.**) plus the rules for relativistic $\vec{E}$ and $\vec{B}$ field transformations.

Hint:

Taking advantage of the trivial value of $\vec{E}$ in $\mathcal{S}$, write a direct Lorentz transformation for $\vec{E}'_\perp$ (cf. SCSR *Eq. (42)). Use $\epsilon_0\mu_0 = 1/c^2$ to check that both answers agree.*

25.

(**a.**)

Express $\mu_0 J^\nu$ as the four-divergence of the field strength tensor $F^{\mu\nu}$. Exploiting the antisymmetry of $F^{\mu\nu}$ under interchange of its indices, prove without reference to the specific values of the elements of F that

$$\partial_\mu J^\mu = 0$$

and thus that electric charge must be conserved. (The basic structure of Maxwell's equations would have to be completely reformulated if even the tiniest violation of electric charge conservation were to be observed anywhere in the universe.)

Hint:

Consider the sum $A_{\mu\nu}B^{\mu\nu}$, where A and B are any two Lorentz four-tensors. Suppose that A is even under the interchange of μ and ν, while B is odd. Consider a particular set of values of μ and ν, for example 0 and 2, yielding a term $A_{02}B^{02}$ in the sum. However, since B but not A is odd under interchange of indices, this term will be cancelled by another term $A_{20}B^{20}$. In this argument, if B is well-behaved under differentiation, can $\partial_\mu\partial_\nu$ play the same role as $A_{\mu\nu}$?

(**b.**)

Define

$$\epsilon_{\mu\nu\rho\sigma} \equiv g_{\mu\alpha}g_{\nu\beta}g_{\rho\kappa}g_{\sigma\lambda}\epsilon^{\alpha\beta\kappa\lambda} ,$$

where ϵ is as defined after SCSR Eq. (64). Prove that

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma} .$$

Hint:

When an index μ is changed from a superscript (contravariant) to a subscript (covariant), or vice versa, a sign change is required if the index is spacelike ($1 \le \mu \le 3$) but not if it is timelike ($\mu = 0$). If $\epsilon^{\mu\nu\rho\sigma}$ is nonzero, how many of its indices are spacelike?

(**c.**)

Without making reference to the specific values of the dual field strength tensor

$$G^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} ,$$

using the antisymmetry of ϵ, prove that

$$\partial_\mu G^{\mu\nu} = 0 .$$

(This is equivalent to the sourceless Maxwell equations.)

Hint:

*Substitute for $G^{\mu\nu}$ its definition in terms of $F_{\rho\sigma}$, and substitute for $F_{\rho\sigma}$ its definition in terms of $\partial_{\rho,\sigma}$ and $A_{\rho,\sigma}$. Recalling the argument made in part (**a.**), what is the behavior of ϵ when μ and ρ are interchanged?*

26.
The field strength and dual field strength tensors are

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$G^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ ,$$

where $\epsilon^{\mu\nu\rho\sigma} = 1$ (-1) when $\mu\nu\rho\sigma$ are even (odd) permutations of 0123, and 0 otherwise. You may use their explicit elements as given in SCSR Eqs. (62) and (65).
(**a.**)
By explicit evaluation, show that $F^{\mu\nu}F_{\mu\nu}$ is proportional to $E^2 - c^2B^2$, and find the constant of proportionality. (Because $F^{\mu\nu}F_{\mu\nu}$ is obviously a Lorentz scalar, the Lorentz invariance of $E^2 - c^2B^2$ is therefore said to be *manifest*.)
Hint:
Start with SCSR *Eq. (62) for the* $F^{\mu\nu}$. *Use* SCSR *Eqs. (58) (or the information in the top right quarter of* SCSR *page 17) to get the* $F_{\mu\nu}$. *Now,*

$$F^{\mu\nu}F_{\mu\nu} = F^{00}F_{00} + F^{01}F_{01} + \ldots + F^{33}F_{33}$$

(16 terms). Add up the terms.
(**b.**)
By explicit evaluation, show that $F^{\mu\nu}G_{\mu\nu}$ is proportional to $\vec{E}\cdot\vec{B}$, and find the constant of proportionality. (Likewise the Lorentz invariance of $\vec{E}\cdot\vec{B}$ is manifest.)
Hint:
Start with SCSR *Eq. (65) for the* $G^{\mu\nu}$.
(**c.**)
What two criteria must (uniform nonzero) $\vec{E}$ and $\vec{B}$ satisfy in the lab frame so that, in a different inertial frame, $\vec{B}$ is allowed to vanish?
Hint:
If $\vec{B}$ *is to vanish in one Lorentz frame, can* $\vec{E}\cdot\vec{B}$ *(a Lorentz invariant) be nonzero in any Lorentz frame? If* $|\vec{E}|$ *is to exceed* $c|\vec{B}|$ *in one Lorentz frame, can* $E^2 - c^2B^2$ *(a Lorentz invariant) be negative or zero in any Lorentz frame?*

27.
Griffiths Problem 12.36.
Hint:
Using the notation employed in class, and dividing through by mc, Griffiths asks you to prove

$$\frac{d}{dt}\gamma\vec{\beta} = \gamma\big(\dot{\vec{\beta}} + \gamma^2\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\big) \ .$$

Writing γ *in terms of* β *and routinely differentiating the* LHS, *cancelling the term* $\gamma\dot{\vec{\beta}}$ *on both sides, and ignoring the common factor* $\gamma^3\vec{\beta}$, *this reduces to proving*

$$\beta\frac{d\beta}{dt} = \vec{\beta}\cdot\frac{d\vec{\beta}}{dt} \ .$$

To do this, consider $\frac{d}{dt}(\vec{\beta}\cdot\vec{\beta})$.

28.
You are an indefagitable runner of rest mass m, whose feet generate a constant force F_0 in the x direction (as observed in the *laboratory*). This force causes your (relativistic) momentum to increase linearly with laboratory time t.
(**a.**)
At $t = 0$, when you are at rest at the origin, your feet begin to exert this force. Thereafter, show that $\sinh\eta$, where η is your rapidity, is proportional to t, and find the constant of proportionality.
Hint:
Along $\hat{x}$, *the time derivative of*

$$p = \gamma\beta mc = \cosh\eta\tanh\eta\, mc$$

is equal to F_0.
(**b.**)
At $t = t_1$, a laser pulse is shot from the origin in the x direction. How much of a head start t_1 do you require in order for the laser pulse never to catch up with you?
Hint:
From part (**a.**) *you know that* $\sinh\eta = \omega t$, *where* ω *is the constant of proportionality that you obtained there. Using* $\cosh^2\eta - \sinh^2\eta = 1$, *you also know the time dependence of* $\cosh\eta$. *Taking the ratio, you know the time dependence of* $\beta = \tanh\eta$. *Integrate* $c\beta(t)$ *(a perfect differential) to obtain the runner's equation of motion*

$$\tfrac{\omega}{c}x(t) = \sqrt{1+\omega^2t^2} - 1 \ .$$

Compare this to the laser pulse's equation of motion

$$\tfrac{\omega}{c}x_p(t) = \omega(t - t_1) \ .$$

For what value of t_1 will x stay larger than x_p (infinitesimally so when ωt is $\gg 1$ but finite)?

29.
At $t = 0$, a particle of rest mass m and charge e is at rest at the origin. It accelerates under the influence of a uniform static electric field $\vec{E} = \hat{z}E_0$.
(**a.**)
For $t > 0$, show that the relativistic solution for $z(t)$ is given by

$$z(t) = \int_0^t c\beta_z(t)\,dt$$
$$\beta_z(t) = \tanh\eta(t)$$
$$\eta(t) = \sinh^{-1}\frac{eE_0}{mc}t\ .$$

Hint:
*Use the hint for part (**a.**) of the previous problem.*
(**b.**)
Suppose instead that, at $t = 0$, the particle has an initial momentum $\vec{p}(0) = \hat{x}p_\perp$. Show that the solution for $z(t)$ is the same as in (**a.**), except that m is replaced by m_{eff}, where

$$m_{\text{eff}} = \sqrt{m^2 + p_\perp^2/c^2}\ .$$

This says that, under the influence of a uniform electrostatic field, the longitudinal motion of a particle with nonzero transverse momentum is the same as that of a heavier particle with zero transverse momentum.
Hint:
Why is $p_\perp$ constant? When $p_\perp = 0$, the equation of motion along $\hat{z}$ is determined by

$$\frac{d}{dt}\gamma\beta mc = eE_0\ ,$$

where

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{mc^2}$$

and E is the particle's total energy. When $\vec{p}_\perp \neq 0$, if you can show that

$$\frac{d}{dt}\gamma_{\text{eff}}\beta_z m_{\text{eff}}c = eE_0\ , \qquad \text{(A)}$$

where

$$\gamma_{\text{eff}} \equiv \frac{E}{m_{\text{eff}}c^2} = \frac{1}{\sqrt{1-\beta_z^2}}\ , \qquad \text{(B)}$$

the assertion to be proved will be justified. To prove Eq. (A), consider the force equation $dp_z/dt = eE_0$ and use the fact that $\gamma m = \gamma_{\text{eff}}m_{\text{eff}}$. To prove Eq. (B), write $\beta_z = cp_z/E$.
(**c.**)
Under the conditions of part (**b.**), for $t > 0$ does $x(t)$ increase linearly with t? Explain.
Hint:
You have argued that $p_\perp = \gamma\beta_x mc$ is constant. So β_x is constant if γ is constant.

30.
At $t = 0$, a particle of rest mass m and charge e is at the origin, with initial momentum $\vec{p}(0) = \hat{x}p_\perp$. It accelerates under the influence of a uniform static magnetic field $\vec{B} = \hat{z}B_0$.
(**a.**)
For $t > 0$, show that the relativistic solution for $x(t)$ and $y(t)$ is given by

$$x(t) = \int_0^t c\beta_x(t)\,dt$$
$$y(t) = \int_0^t c\beta_y(t)\,dt$$
$$\gamma_{\text{a}}mc\beta_x(t) = p_\perp\cos\omega_c t$$
$$\gamma_{\text{a}}mc\beta_y(t) = p_\perp\sin\omega_c t$$
$$\omega_c = -\frac{eB_0}{\gamma_{\text{a}}m}$$
$$\gamma_{\text{a}}m = \sqrt{m^2 + p_\perp^2/c^2}\ .$$

Hint:
First, by considering the power $\vec{F}\cdot c\vec{\beta}$ and the nature of the Lorentz force in a pure magnetic field, show that $|\vec{p}_\perp|$ and therefore γ_{a} is constant. Show that the Lorentz force equation can be written

$$\frac{d\vec{\beta}}{dt} = (\hat{z}\omega_c)\times\vec{\beta}\ .$$

This is an equation for (CCW) precession of $\vec{\beta}$ about $\hat{z}$ with angular frequency ω_c. (Considering the negative sign of ω_c, the precession is actually CW).
(**b.**)

Suppose instead that, at $t = 0$, the particle has an initial momentum $\vec{p}(0) = \hat{x}p_\perp + \hat{z}p_\parallel$. Show that the solution for $x(t)$ and $y(t)$ is the same as in (**a.**), except that $\gamma_{\rm a}$ is replaced by γ, where

$$\gamma m = \sqrt{m^2 + p_\perp^2/c^2 + p_\parallel^2/c^2}\ .$$

Hint:
Show that the solution is the same as in part (a), except that

$$\omega_c = -\frac{eB_0}{\gamma m}\ .$$

(**c.**)
Show that the result of (**b.**) alternatively can be expressed as the result of (**a.**) with m replaced by

$$m_{\rm eff} = \sqrt{m^2 + p_\parallel^2/c^2}\ .$$

This says that, under the influence of a uniform magnetostatic field, the transverse motion of a particle with nonzero longitudinal momentum is the same as that of a heavier particle with zero longitudinal momentum.
Hint:
Show that

$$\gamma m = \sqrt{m_{\rm eff}^2 + p_\perp^2/c^2}\ .$$

(**d.**)
Under the conditions of part (**b.**), for $t > 0$ does $z(t)$ increase linearly with t? Explain.
Hint:
Use the fact that $p_\parallel = \gamma\beta_z mc$. Are $p_\parallel$ and γ both constant?

31.
At $t = 0$ at the origin of a spherical polar coordinate system in the lab, a point particle of charge q has velocity βc directed along the $\hat{z}$ (north polar) axis. It has been moving with that constant velocity for a long time.
(**a.**)
Starting with the Coulomb field in the particle's rest frame, and using the rules for relativistic transformation of EM fields, show that the electric field observed in the lab at $t = 0$ and $\vec{r} = (r, \theta, \phi)$ is

$$4\pi\epsilon_0\vec{E} = \hat{r}\,\frac{q}{r^2}\,\frac{\gamma}{(\gamma^2\cos^2\theta + \sin^2\theta)^{3/2}}\ ,$$

where as usual $\gamma = 1/\sqrt{1-\beta^2}$.
Hint:
Work in Cartesian coordinates. Put the observer in frame $\mathcal{S}$ at $(x, 0, z)$, with $\hat{z}$ the direction of the charge's motion. Put the charge at the origin $(x', y', z') = (0, 0, 0)$ of frame $\mathcal{S}'$. Write E'_x and E'_z in terms of x', y', and z'. Then transform E'_x, E'_z, and z' to frame $\mathcal{S}$. Finally, convert to spherical polar coordinates in the lab frame.
(**b.**)
Show that this result is equivalent to Griffiths Eq. (10.68).
Hint:
*In Griffiths' notation, $\mathbf{R}$ is the same as our $\vec{r}$. Remove a factor γ^3 from the denominator of the answer to part (**a.**), and substitute $\cos^2\theta = 1 - \sin^2\theta$.*

32.
Griffiths Problem 10.9(b).
Hint:
Substitute $I(t_{\rm ret}) = q_0\delta(t_{\rm ret}) = q_0\delta(t - \mathcal{R}/c)$ in Griffiths' first equation of his solution to Example 10.2. Since the integrand is even, integrate from 0 to ∞ rather than from $-\infty$ to ∞. Using $z = \sqrt{\mathcal{R}^2 - s^2}$, where s is the cylindrical radial coordinate, convert the integral over dz to an integral over $d\mathcal{R}$. Convert the delta function $\delta(t - \mathcal{R}/c)$ to $\delta(\mathcal{R} - ct)$ using the rule $\delta(f(x)) = \delta(x)/|df/dx|$ (true if $f = 0$ when $x = 0$). The delta function trivializes the integration. Differentiate your answer for $\vec{A}$ to get $\vec{E}$ and $\vec{B}$.

33.
The general expression for the electromagnetic fields arising from a point particle of charge q moving with velocity $\vec{\beta}c$ and acceleration $\dot{\vec{\beta}}c$ is

$$\begin{aligned}
\vec{E} &= \vec{E}_{\rm v} + \vec{E}_{\rm a} \ \text{ with} \\
\vec{E}_{\rm v} &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\mathcal{R}^2}\,\frac{(\hat{\mathcal{R}} - \vec{\beta})(1-\beta^2)}{(1 - \hat{\mathcal{R}}\cdot\vec{\beta})^3}\right\}_{\rm ret} \\
c\vec{B} &= \left\{\hat{\mathcal{R}} \times \vec{E}\right\}_{\rm ret}\ ,
\end{aligned}$$

where $\vec{E}_{\rm v}$ is the *velocity field*, and the *acceleration field* $\vec{E}_{\rm a}$ is given in a later problem. Here $\vec{r}$ is

a vector from the origin to the observer, $\vec{w}(t)$ is a vector from the origin to the particle, $\vec{\mathcal{R}} \equiv \vec{r} - \vec{w}$, and the subscript "ret" means that quantities are to be evaluated at time $t_{\rm ret} = t - \mathcal{R}/c$.

Assume that $\vec{\beta}$ lies in the z direction and is a constant, so that the acceleration field vanishes. As usual $\theta = \cos^{-1} \hat{z} \cdot \hat{r}$. Choose the origin of coordinates to be the position of the particle at $t = 0$. At that time, show that...

(a.)

$$-ct_{\rm ret} = \gamma\big(\gamma\beta z + \sqrt{(\gamma\beta z)^2 + r^2}\big) \ ;$$

Hint:
At $t = 0$,

$$ct_{\rm ret} = -\mathcal{R} = |\vec{r} - \hat{z}\beta ct_{\rm ret}| \ .$$

Solve this quadratic equation for $ct_{\rm ret}$ and choose the solution that yields $-ct_{\rm ret} = r$ when $\beta = 0$.

(b.)

$$\mathcal{R}(1 - \hat{\mathcal{R}} \cdot \vec{\beta}) = r\sqrt{1 - \beta^2 \sin^2\theta} \ ;$$

Hint:
Write the LHS *as $\mathcal{R} - \vec{\mathcal{R}} \cdot \vec{\beta}$. At $t = 0$, $\mathcal{R} = -ct_{\rm ret}$ (which is positive). Combining terms, you should obtain*

$$\mathcal{R}(1 - \hat{\mathcal{R}} \cdot \vec{\beta}) = -ct_{\rm ret}(1 - \beta^2) - \beta z \ .$$

Substituting the result of part (**a.**) *for $ct_{\rm ret}$ results in a lovely cancellation.*

(c.)

$$\vec{r} = \mathcal{R}(\hat{\mathcal{R}} - \vec{\beta}) \ .$$

Hint:
Write the RHS *as $\vec{\mathcal{R}} - \mathcal{R}\vec{\beta}$. You have already encountered all the ingredients you need: at $t = 0$, $\vec{\mathcal{R}} = \vec{r} - \vec{\beta}ct_{\rm ret}$ and $ct_{\rm ret} = -\mathcal{R}$.*

34.
Under the conditions of the previous problem, and using the tools developed there, show that $\vec{E}_{\rm v}$ is equivalent to Griffiths Eq. (10.68).

Hint:
Tools (**b.**) *and* (**c.**) *from the previous problem are sufficient.*

35.
Liénard's equation for the Poynting vector

$$\vec{S}_{\rm a} = \frac{1}{\mu_0}\vec{E}_{\rm a} \times \vec{B}_{\rm a}$$

arising from acceleration of a point particle of charge q is

$$\vec{S}_{\rm a} = \Big(\frac{q}{4\pi\epsilon_0}\Big)^2 \frac{\epsilon_0}{c}\Big\{\frac{\hat{\mathcal{R}}}{\mathcal{R}^2}\Big[\frac{\hat{\mathcal{R}} \times [(\hat{\mathcal{R}} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \hat{\mathcal{R}} \cdot \vec{\beta})^3}\Big]^2\Big\}_{\rm ret} .$$

(a.)
Show that Liénard's equation follows directly from the electric and magnetic fields arising from acceleration of a point particle, using the *acceleration fields*

$$\vec{E}_{\rm a} = \frac{q}{4\pi\epsilon_0}\frac{1}{c}\Big\{\frac{1}{\mathcal{R}}\frac{\hat{\mathcal{R}} \times [(\hat{\mathcal{R}} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \hat{\mathcal{R}} \cdot \vec{\beta})^3}\Big\}_{\rm ret}$$
$$c\vec{B}_{\rm a} = \big\{\hat{\mathcal{R}} \times \vec{E}_{\rm a}\big\}_{\rm ret} \ .$$

Hint:
After applying the bac − cab rule, only one term survives.

(b.)
Suppose that the particle is in uniform motion around a circle of radius b in the plane $z = 0$ centered at the origin. The motion is ultrarelativistic, *i.e.* $(1 - \beta^2)^{-1/2} \gg 1$. To lowest order, calculate the radiated power per unit area observed at $(0, 0, z)$, where $z \gg b$.

Hint:
To an excellent approximation, $\hat{\mathcal{R}}$ is perpendicular to both $\vec{\beta}$ and $\dot{\vec{\beta}}$. This makes $\vec{S}_a$ trivial to evaluate.

(c.)
Is $\hat{z}$ a direction in which the power radiated per unit solid angle is near the maximum for this motion? Explain.

Hint:
Does the "train factor" $(1 - \hat{\mathcal{R}} \cdot \vec{\beta})^{-6}$ provide any enhancement at this observation point?

36.
As an intermediate step in the derivation of the

velocity and acceleration fields $\vec{E}_{\rm v}$ and $\vec{E}_{\rm a}$, in class we derived the expression

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\Big\{\frac{1}{1-\hat{\mathcal{R}}\cdot\vec{\beta}}\big[\frac{\hat{\mathcal{R}}}{\mathcal{R}^2} + \frac{d}{c\,dt}\frac{\hat{\mathcal{R}}-\vec{\beta}}{\mathcal{R}(1-\hat{\mathcal{R}}\cdot\vec{\beta})}\big]\Big\}_{\rm ret}$$

where the subscript "ret" means that the differentiation should be done first, and afterward all time-dependent quantities should be evaluated at time $t_{\rm ret} = t - \mathcal{R}/c$.

Define $\dot{\vec{\beta}} \equiv d\vec{\beta}/dt$. Use two relations worked out in class:

$$\frac{d\mathcal{R}}{c\,dt} = -\hat{\mathcal{R}}\cdot\vec{\beta}$$
$$\frac{d\hat{\mathcal{R}}}{c\,dt} = \frac{\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})}{\mathcal{R}}\ .$$

With these tools, finish the derivation to obtain $\vec{E}_{\rm v}$ (as given in an earlier problem) and $\vec{E}_{\rm a}$ (as given in the previous problem).

Hint:

For the terms involving $\dot{\vec{\beta}}$, apply the bac − cab rule in reverse. For the terms not involving $\dot{\vec{\beta}}$, apply the bac − cab rule directly to $\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})$.

37.

Griffiths Problem 3.40.

Hint:

Start with the definition

$$q_{lm} \equiv \int d\tau'\rho(\vec{r}\,')r'^l\,Y^*_{lm}(\theta',\phi')$$

of the multipole moments. A line charge is azimuthally symmetric, causing the $m \neq 0$ moments to vanish since

$$\int d\phi\, e^{im\phi} = 0\ .$$

For a line charge along the z axis, $\int\rho\,d\tau'$ reduces to $\int\lambda\,dz$.

38.

The electrostatic potential created by a static point charge can take a nontrivial form when the coordinate system is chosen to have an origin which, for some other reason, must be centered at point that does not coincide with the charge's position.

This problem concerns the potential $V(\vec{r})$ created by a localized charge distribution $\rho(\vec{r}\,')$. With the observation point located outside the charge distribution ($r > r'_{\rm max}$), use the standard expansion in spherical harmonics

$$\epsilon_0 V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{Y_{lm}(\theta,\phi)}{(2l+1)r^{l+1}}\,q_{lm}\ ,$$

where the multipole moments q_{lm} are defined in the hint for the previous problem. In spherical polar coordinates, consider a point charge e located at (r',θ',ϕ') with respect to a certain origin. Determine the electrostatic potential that it creates at an observation point (r,θ,ϕ), with $r > r'_{\rm max}$.

(a.)

Write down the exact value of $\mathrm{V}(r,\theta,\phi,r',\theta',\phi')$ as an infinite sum over l and m.

Hint:

The charge distribution for a point charge is a delta function that trivializes the q_{lm} integral. Compare your result to the expression in your lecture notes for the expansion in spherical harmonics of $1/|\vec{r}-\vec{r}\,'|$.

(b.)

Explicitly evaluating the spherical harmonics as functions of θ and ϕ (or θ' and ϕ'), write down all the monopole, dipole, and quadrupole terms ($l = 0$, 1, and 2).

Hint:

Explicit spherical harmonics are found in Jackson (p. 109) or in the Particle Physics Booklet (p. 299 in the 2006 edition). Note that

$$Y_{l\ -m} = (-1)^m Y^*_{lm}\ .$$

39.

Arrange five finite point charges at five different positions so that all $l \leq 4$ moments of the charge distribution vanish, except for the $m = 0$ hexadecapole moment

$$q_{40} \equiv \int d\tau'\rho(\vec{r}\,')r'^4 Y^*_{40}(\theta',\phi')\ .$$

Hint:

As noted in an earlier hint, the $m \neq 0$ moments

vanish if the charge distribution is cylindrically symmetric about the z axis; for point charges this can be accomplished only by placing the charges on the axis. The $m = 0$, $l =$ odd moments vanish if the charge distribution is even in z. Place four charges away from the origin to cancel the $m = 0$ quadrupole ($l = 2$) moment; place a fifth charge at the origin to cancel the monopole ($l = 0$) moment.

40.
Consider the dimensionless operator

$$\vec{L} \equiv \tfrac{1}{i}\vec{r} \times \nabla$$

(apart from a missing factor of $\hbar$, this is the same as the angular momentum operator used in quantum mechanics).
(**a.**)
In spherical polar coordinates, show that

$$i\vec{L} = \hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi} \ .$$

Hint:
Consult Griffiths' inside cover (GIC) *#1.*
(**b.**)
Express $\hat{\theta}$ and $\hat{\phi}$ in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$, θ, and ϕ.
Hint:
Consult GIC *#4.*
(**c.**)
Show that

$$iL_z = \frac{\partial}{\partial\phi}$$

$$L_\pm \equiv L_x \pm iL_y = e^{\pm i\phi}\Big(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Big) \ .$$

[$L_\pm$ are *raising* and *lowering* operators, which, within a factor, change Y_{lm} into $Y_{l,m\pm1}$.]
Hint:
Plug the result of (**b.**) *into the result of* (**a.**)*. To identify L_x, L_y, and L_z, collect the terms multiplying $\hat{x}$, $\hat{y}$, and $\hat{z}$.*
(**d.**)
Show that

$$L^2 = L_z^2 + \tfrac{1}{2}\{L_+, L_-\} \ ,$$

where $\{a, b\}$ is the anticommutator $ab + ba$.
Hint:
Write down L_+L_- and L_-L_+ in terms of L_x and L_y (remember to preserve the order of the operators).
(**e.**)
Finally, show that

$$-L^2 = r^2\nabla^2_{\text{ang}} \ ,$$

where ∇^2_{ang} is the part of ∇^2 which involves derivatives in θ and ϕ.
Hint:
Plug the results of (**c.**) *into the result of* (**d.**)*. Consult* GIC *#1.*

41.
This problem is omitted intentionally.

42.
Working in the far zone $r' \ll \lambda \ll r$, consider azimuthally symmetric ($m = 0$) electric quadrupole ($\mathrm{E2}_0$) radiation. At a particular angular frequency ω, work with the complex fields $\vec{\tilde{B}}(\vec{r})$ and $\vec{\tilde{E}}(\vec{r})$ defined by

$$\vec{B}(\vec{r}, t) \equiv \mathrm{Re}\left(\vec{\tilde{B}}(\vec{r})e^{-i\omega t}\right)$$
$$\vec{E}(\vec{r}, t) \equiv \mathrm{Re}\left(\vec{\tilde{E}}(\vec{r})e^{-i\omega t}\right) \ .$$

For E-type radiation, the magnetic field $\vec{\tilde{B}}$ ($\perp \hat{r}$) is proportional to the vector spherical harmonic $\vec{X}$:

$$\vec{\tilde{B}} \propto \vec{X}_{20}(\theta, \phi) \equiv \vec{L}\, Y_{20}(\theta, \phi) \ ,$$

with $i\vec{L} \equiv \vec{r} \times \nabla$. Use the fact that

$$\vec{\tilde{E}} \approx c\vec{\tilde{B}} \times \hat{r}$$

in the far zone. Obtain a function $f(\theta, \phi)$ such that the radiated power P in the far zone is proportional to it:

$$\frac{dP}{d\Omega} \propto f(\theta, \phi) \ .$$

Hint:
Since Y_{l0} is independent of ϕ, the only part of $\vec{r} \times \nabla$ that is relevant to this problem is $\hat{\phi}\frac{\partial}{\partial\theta}$ (see problem 40). This makes it easy to compute the

angular dependence of $\vec{\tilde{B}}$. *The time-averaged Poynting vector is*

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}(\vec{\tilde{E}} \times \vec{\tilde{B}}^{\,*})$$

(see Griffiths Problem 11.15). Combine this with the equation

$$\vec{\tilde{E}} \approx c\vec{\tilde{B}} \times \hat{r}$$

and apply the $bac - cab$ *rule. This should convince you that* $P \propto |\vec{\tilde{B}}|^2$.

43.
This problem is omitted intentionally.

44.
Griffiths Problem 11.15.
Hint:
The algebra for this problem can be simplified somewhat by defining $u \equiv \cos\theta$ *and maximizing Griffiths' Eq. (11.74) with respect to* u. *Solve the resulting quadratic equation for* u. *In the ultrarelativistic limit it will also be convenient to define* $\epsilon \equiv 1 - \beta$. *In that limit* $\epsilon \ll 1$ *and, near where the maximum radiation is emitted,* $\theta \ll 1$. *To obtain Griffiths' approximate result you will need the Taylor expansions*

$$\cos\theta \approx 1 - \tfrac{\theta^2}{2}$$
$$(1+\epsilon)^n \approx 1 + n\epsilon \ .$$

45.
Start from the expression derived in class for the energy radiated by an accelerating point charge per steradian per unit of *retarded* time t':

$$\frac{dW}{d\Omega\, dt'} = \Big(\frac{q}{4\pi\epsilon_0}\Big)^2 \frac{\epsilon_0}{c} \frac{\big|\hat{R} \times \big[(\hat{R} - \vec{\beta}) \times \dot{\vec{\beta}}\big]\big|^2}{(1 - \hat{R}\cdot\vec{\beta})^5} \ .$$

Consider *synchrotron radiation* by a particle of charge q moving in a circular orbit of radius b in a coordinate system where

$$\hat{\beta} = \hat{z}$$
$$\dot{\hat{\beta}} = \hat{x} \ ,$$

i.e. $\hat{x}$ points toward the center of the circle and $\hat{z}$ points along its circumference in the particle's direction of motion. Define

$$\hat{R} \equiv (n_x, n_y, n_z) \ ,$$

where $\hat{n}$ is a unit vector extending from the particle in an arbitrary direction towards which an element of radiation is emitted.
(a.)
Show that

$$\hat{R} \times \big[(\hat{R} - \vec{\beta}) \times \dot{\hat{\beta}}\big] = \hat{n} n_x - \hat{x} - \beta \hat{n} \times \hat{y} \ .$$

Hint:
Apply the $bac - cab$ *rule.*
(b.)
Using this result, show that

$$\big|\hat{R} \times \big[(\hat{R} - \vec{\beta}) \times \dot{\hat{\beta}}\big]\big|^2 = 1 - 2\beta n_z + \beta^2 n_z^2 - (1-\beta^2) n_x^2$$

(c.)
Consider a set of spherical polar coordinates centered at the particle (*not* at the center of the beam circle). Taking θ to be the polar angle of $\hat{n}$ relative to $\hat{z}$, and ϕ to be its azimuth about $\hat{z}$, express n_x and n_z in terms of θ and ϕ.
Hint:
Consult GIC *#4.*
(d.)
Using the results of (**b.**) and (**c.**), show that

$$\frac{dW}{d\Omega\, dt'} = \Big(\frac{q}{4\pi\epsilon_0}\Big)^2 \frac{\epsilon_0}{c} \times$$
$$\times \frac{\dot{\beta}^2}{(1 - \beta\cos\theta)^3} \Big(1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2 (1 - \beta\cos\theta)^2}\Big) \ .$$

46.
Consider the final result of the previous problem.
(a.)
Perform the integration over $d\Omega = d(\cos\theta)\, d\phi$ to show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2}{3c^3} (q\dot{\beta}c)^2 \gamma^4 \ .$$

[Note that $(q\dot{\beta}c)^2$ is equivalent to $\ddot{p}^2$, where p is the electric dipole moment of the point charge

relative to the origin. Therefore this result is the same as the (nonrelativistic) Larmor formula, except for the additional factor γ^4.]

Hint:

Define $u \equiv 1 - \beta\cos\theta$ and express the integrand as a function of u rather than θ. It becomes a polynomial in u that may trivially be integrated.

(**b.**)

In terms of the |momentum| P of the point charge and its rest mass m, show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2q^2}{3c^3}\,\frac{P^4}{m^4 b^2}\ ,$$

and thus that the power lost to synchrotron radiation depends on the fourth power of P, the inverse fourth power of m (making it usually negligible for all but electrons), and the inverse square of b.

Hint:

For motion around a circle of radius b, the centripetal acceleration $\dot{\beta}c$ is equal to $(\beta c)^2/b$. Use $P = \gamma\beta mc$.

(**c.**)

Suppose that you use an electron synchrotron that taxpayers can afford. It circulates highly relativistic electrons with $\beta \approx 1$. You want to build a new synchrotron with the same beam current, the same power lost to synchrotron radiation, but twice the beam momentum. Show that the radius b of the new synchrotron must increase by a factor of 16.

Hint:

Keep in mind that if the beam current stays the same while the synchrotron radius increases by a factor λ, the total number of radiating electrons increases by λ as well.

47.

A free-electron laser consists of a beam of electrons (with constant velocity βc) passing through a structure known as an *undulator*. (Stronger versions of these structures called *wigglers* are used also in sections of a circular electron synchrotron such as the ALS.) Take the beam direction to be $\hat{z}$. Consider an alternating set of magnets (for compactness, these are often permanent magnets, made of samarium cobalt as developed at LBL by the late Klaus Halbach). With a full period Δz, they produce a strong magnetic field that points alternately in the $+\hat{x}$ and $-\hat{x}$ directions. For this undulator we (unrealistically) assume that this magnetic field is weak enough so that, as seen in a frame moving with velocity $\beta c\hat{z}$, the electrons' velocity *relative* to that frame is $\ll c$.

(**a.**)

In the rest frame $\mathcal{S}'$ of the electron, with what fundamental angular frequency ω' does the magnetic field from the undulator appear to oscillate?

Hint:

Apply a Lorentz contraction to get the apparent spacing of the magnets as seen by the electron. Divide it by the relative velocity (between the electron and the magnets) to obtain the time period of the magnetic field as seen by the electron.

(**b.**)

In $\mathcal{S}'$, the oscillating electron produces electromagnetic radiation with angular frequency ω'. Applying the relativistic Doppler shift to ("forward") radiation emitted along the beam direction, what angular frequency ω does that radiation have in the laboratory frame?

Hint:

As an alternative to the relativistic Doppler formula, you may simply perform an inverse Lorentz transformation on the 0th component of the wave four-vector $(\omega/c, \vec{k})$. You are describing a massless photon, so $|\vec{k}| = \omega/c$ in either frame.

(**c.**)

Express λ, the wavelength of the forward radiation, as a multiple of Δz.

(**d.**)

At LBL's ALS, using an alternating set of magnets with $\Delta z = 10$ cm, an experimenter wishes to study the effect upon condensed-matter samples of a soft X-ray beam of wavelength 5 nm. Use this information to estimate the ALS beam energy (in GeV).

48.

This problem is omitted intentionally.

49.

(**a.**)

This part is omitted intentionally.
(**b.**)
This part is omitted intentionally.
(**c.**)
In a more microscopic and detailed treatment of plane EM waves propagating along $\hat{z}$ in materials that potentially are conducting, one assumes that N valence electrons per m^3 having charge $-e$ and mass m move in a potential well with effective spring constant $m\omega_0^2$ and damping coefficient Γm. One defines the *complex dielectric constant* $\tilde{\epsilon}$ via

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 \equiv \frac{\tilde{P}}{\epsilon_0 \tilde{E}} ,$$

where $\tilde{P}$ is the complex polarization, defined by

$$\vec{P}(z,t) = \mathrm{Re}\Big(\vec{\tilde{P}} \exp\big(i(\tilde{k}z - \omega t)\big)\Big) ,$$

in analogy to complex $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$. For not-too-dense media in which the electric field felt by the electron is approximately the same as the average field, it is straightforward to solve the force equation for these oscillating electrons and determine the complex polarization $\tilde{P}$ they create. One obtains

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\Gamma\omega} ,$$

where the *plasma frequency*[2] is

$$\omega_p^2 \equiv \frac{Ne^2}{m\epsilon_0} .$$

The complex dielectric constant $\tilde{\epsilon}$ includes the effects of all electrons (free and bound). It is related to the ordinary dielectric constant ϵ (which includes the effects only of bound electrons) by

$$\tilde{\epsilon} = \epsilon(1 + i\gamma) = \frac{\tilde{k}^2}{\mu\omega^2} ,$$

where as usual $\gamma = \frac{\sigma}{\epsilon\omega}$. Represent a *good conductor* by $\omega_0 = 0$ (unbound) and $\Gamma \gg \omega$ (overdamped). Using these results, show that the conductivity σ is approximately

$$\sigma \approx \frac{\epsilon_0 \omega_p^2}{\Gamma} ,$$

i.e. measuring the low-frequency conductivity is a simple way to determine the damping coefficient.
Hint:
Consider the imaginary part of $\tilde{\epsilon}/\epsilon_0$.
(**d.**)
Represent the *ionosphere* by $\omega_0 = 0$ (unbound), and $\Gamma \ll \omega$ (underdamped). Specialize to AM radio waves, for which $\omega < \omega_p$. Show that the amplitude reflection coefficient $|\tilde{R}| \approx 1$, *i.e.* that AM radio waves are nearly fully reflected by the ionosphere. (At dusk, the ionosphere drops to sufficiently low altitude that reflection off it enables AM stations hundreds of miles away to be received.)
Hint:
Use the amplitude reflection coefficient $\tilde{R}$ *from Problem -4. First show that*

$$\tilde{k}^2 = \mu\omega^2\epsilon_0\big(1 + \frac{\omega_w^2}{\omega_0^2 - \omega^2 - i\Gamma\omega}\big) .$$

Then, for the underdamped case and for $\omega < \omega_p$ *, argue that* $\tilde{k}$ *is almost pure imaginary, causing the numerator of* $\tilde{\mathcal{R}}$ *to be the complex conjugate of the denominator.*

50.
This problem is omitted intentionally.

51. Jones vectors.
For a plane transverse wave propagating in the $\hat{z}$ direction through a (not necessarily insulating) material with constant ϵ and μ, a (co)sinusoidal solution is represented by

$$\vec{E}(\vec{r},t) = \mathrm{Re}\Big(\vec{E}_0(x,y)\, e^{i(\tilde{k}z - \omega t)}\Big)$$
$$\vec{H}(\vec{r},t) = \mathrm{Re}\Big(\vec{H}_0(x,y)\, e^{i(\tilde{k}z - \omega t)}\Big) ,$$

where $\tilde{k}$ is the (not necessarily real) "wave vector" – here a scalar because we know it is directed along $\hat{z}$. Faraday's law causes $\vec{H}_0$ to be completely determined by $\vec{E}_0$:

$$\vec{H}_0 \equiv \tilde{Z}^{-1}\hat{z} \times \vec{E}_0$$
$$= \frac{\tilde{k}}{\mu\omega}\hat{z} \times \vec{E}_0 ,$$

so we focus on $\vec{E}_0$ as the sole independent variable. For a transverse wave $\vec{E}_0$ has no z component. Here we assume that the phase relationship between E_{0x} and E_{0y} is *fixed* – the wave is *fully polarized*. Then $\vec{E}_0$ is a complex transverse vector, completely specified by four components. In the Jones convention, all information carried by $\vec{E}_0$ except for its magnitude is written as a 2×1 column vector with the x component on top:

$$\begin{aligned}\vec{E}_0 &= \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} \\ &\equiv \frac{1}{\sqrt{|\alpha^2| + |\beta^2|}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |\vec{E}_0| \\ &\equiv \vec{J}\,|\vec{E}_0| \ ,\end{aligned}$$

where $\vec{J}$ is the *Jones vector*. Jones vectors are defined only within an overall phase (because the absolute phase of an optical-frequency EM wave can't conveniently be measured); therefore one has the freedom to set α equal to unity (unless it vanishes, in which case β is set to unity). The above form involving the complex constants α and β is a general Jones vector, corresponding to elliptical polarization. More common Jones vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \ ,$$

corresponding, respectively, to linear x, linear y, RH circular, and LH circular polarization.

(**a.**)

At $z = 0$, show (counterintuitively!) that the electric field vector for RH polarized light precesses *clockwise* around $\hat{z}$, *i.e.* it precesses according to the LH rule.

Hint:

Evaluate $\mathrm{Re}\big(\vec{J}e^{-i\omega t}\big)$ *and examine the time evolution of the resulting x and y components.*

(**b.**)

Suppose that a particular state of elliptical polarization has nonvanishing x and y electric field components. Then, within an arbitrary overall phase, it may be represented by the Jones vector

$$\vec{J}_1 = \frac{1}{\sqrt{1 + |\beta|^2}} \begin{pmatrix} 1 \\ \beta \end{pmatrix} \ ,$$

where β is a complex constant. You wish to characterize this state of polarization as "RH elliptical" or "LH elliptical", depending on whether (at $z = 0$) the electric field vector precesses clockwise or counterclockwise around $\hat{z}$. What property of β would you use to decide whether this state is RH or LH elliptical?

Hint:

Again evaluate $\mathrm{Re}\big(\vec{J}_1 e^{-i\omega t}\big)$ *and examine the time evolution of the resulting x and y components. If the y component becomes negative as t increases from 0, the elliptical polarization is right-handed; if it becomes positive, the polarization is left-handed. What property of β controls this behavior?*

(**c.**)

For the conditions of part (**b.**), decompose $\vec{J}_1$ into a linear sum (with real coefficients) of a wave with linear polarization plus a wave with RH circular polarization. Perform this same task with "RH" replaced by "LH". If you are successful in both tasks, you might wonder whether there really exists a unique association of RH or LH behavior with $\vec{J}_1$. Would this concern invalidate your answer to (**b.**)?

Hint:

For example, to decompose $\vec{J}_1$ into a linearly polarized wave plus a RHCP *wave, take the difference*

$$\begin{pmatrix} 1 \\ \beta \end{pmatrix} - C \begin{pmatrix} 1 \\ -i \end{pmatrix} \ .$$

Determine C such that this difference (the linearly polarized part) has elements that both have the same phase.

52. Irradiance and Jones vectors.

Consider two transverse plane waves A and B that move in vacuum and are combined together (*i.e.* by a Michelson interferometer). The beams have complex electric fields

$$\begin{aligned}\begin{pmatrix} E^A_{0x} \\ E^A_{0y} \end{pmatrix} &= \frac{|\vec{E}^A_0|}{\sqrt{|\alpha|^2 + |\beta|^2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{pmatrix} E^B_{0x} \\ E^B_{0y} \end{pmatrix} &= \frac{|\vec{E}^B_0|}{\sqrt{|\gamma|^2 + |\delta|^2}} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \ .\end{aligned}$$

Express the combined irradiance

$$I_{A+B} \equiv \langle \vec{S}_{A+B} \cdot \hat{z} \rangle \ ,$$

where $\vec{S}$ is the Poynting vector and $\langle\ \rangle$ is a time average, as a function of the complex constants α, β, γ, δ, and the uncombined irradiances I_A and I_B of the individual beams.

Hint:

Start from the result of Griffiths' Problem 9.11, modified so that it is true in material as well as vacuum:

$$\langle\vec{S}\rangle = \frac{1}{2\mu}\mathrm{Re}(\vec{\tilde{E}} \times \vec{\tilde{B}}^{\,*}) \ .$$

Express Faraday's law in terms of the complex fields so that you can write $\vec{\tilde{B}}^{\,*}$ in terms of $\vec{\tilde{E}}^{\,*}$. You should obtain

$$\vec{\tilde{B}}^{\,*} = \frac{\tilde{k}^*}{\omega}\hat{z} \times \vec{\tilde{E}}^{\,*} \ .$$

By definition, the irradiance $I \equiv \langle\vec{S}\cdot\hat{z}\rangle$. For the individual beams, you should obtain

$$\begin{aligned} I_{A,B} &= \frac{1}{2\mu\omega}\,\mathrm{Re}\big(\tilde{k}^*\,\vec{\tilde{E}}_{A,B}\cdot\vec{\tilde{E}}^{\,*}_{A,B}\big) \\ &= \frac{\mathrm{Re}\,\tilde{k}}{2\mu\omega}\,|\vec{\tilde{E}}_{A,B}|^2 \ . \end{aligned}$$

For the combined beam (C), write $\vec{\tilde{E}}_C = \vec{\tilde{E}}_A + \vec{\tilde{E}}_A$. You should obtain

$$\begin{aligned} I_C &= I_A + I_B + \frac{1}{2\mu\omega}\,\mathrm{Re}\big(\tilde{k}^*(\vec{\tilde{E}}_A\cdot\vec{\tilde{E}}^{\,*}_B + \vec{\tilde{E}}_B\cdot\vec{\tilde{E}}^{\,*}_A)\big) \\ &= I_A + I_B + \frac{\mathrm{Re}\,\tilde{k}}{\mu\omega}\,\mathrm{Re}\big(\vec{\tilde{E}}_A\cdot\vec{\tilde{E}}^{\,*}_B\big) \ . \end{aligned}$$

Finally, substitute the Jones vectors supplied for $\vec{\tilde{E}}_A$ and $\vec{\tilde{E}}_B$:

$$\vec{\tilde{E}}_{A,B} = |\vec{\tilde{E}}_{A,B}|\,\vec{J}_{A,B} \ .$$

Solve for I_C as a function of the variables required by the problem.

53.

(a.)

A set of N ideal linear polarizers $\mathrm{L}_1 \ldots \mathrm{L}_N$ is arranged so that $\hat{x}$ polarized light passes through them in ascending order. The transmission axis of polarizer n is oriented along $(\hat{x}\cos\phi_n + \hat{y}\sin\phi_n)$, where $\phi_n = \frac{\pi n}{2N}$. In the limit $N \to \infty$, deduce the Jones matrix for this set.

Hint:

Consider the effect on $\hat{x}$ polarized light of the first polarizer only. Use Pedrotti×2 (≡ P×2) Eq. (14-15) with $\theta = \pi/2N$. Taking $N \gg 1$, show that the fractional reduction in electric field amplitude is only of order N^{-2}. In the limit $N \to \infty$, argue that the total reduction in electric field amplitude after the infinite set of ideal polarizers is negligible. Considering the effect of this set of polarizers upon $\hat{x}$ polarized light, what is the left-hand column of the corresponding Jones matrix? What about the right-hand column?

(b.)

Consider a *twisted nematic cell*, as found in an LCD display. It functions as a *rotator* (P×2 Eq. (14-21)). Show that if the rotator parameter $\beta = \frac{\pi}{2}$, the twisted cell will have the same effect on $\hat{x}$ polarized light as does the set of polarizers described in (**a.**).

(c.)

Do the devices in (**a.**) and (**b.**) also have equivalent effect on $\hat{y}$ polarized light? Explain.

Hint:

Their effect is equivalent iff their Jones matrices are the same.

54.

Apart from an experimentally irrelevant overall phase, an ideal wave plate of thickness D with phase retardation difference

$$\delta \equiv (n_x - n_y)\frac{\omega D}{c} \ ,$$

having its slow axis along $\hat{x}$, is represented by the Jones matrix

$$M_{\mathrm{W}}(\phi = 0) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix} \ .$$

If instead the wave plate has its slow axis along $(\hat{x}\cos\phi + \hat{y}\sin\phi)$, show that it is represented by the general Jones matrix

$$M_{\mathrm{W}}(\phi) = \begin{pmatrix} \cos\frac{\delta}{2} + i\sin\frac{\delta}{2}\cos 2\phi & i\sin\frac{\delta}{2}\sin 2\phi \\ i\sin\frac{\delta}{2}\sin 2\phi & \cos\frac{\delta}{2} - i\sin\frac{\delta}{2}\cos 2\phi \end{pmatrix}$$

Note that $\delta = \frac{\pi}{2}$ for a quarter-wave plate (QWP) and $\delta = \pi$ for a half-wave plate (HWP), which is equivalent to two QWPs. Note also that, like the general Jones matrix $M_{\rm L}(\phi)$ for the ideal linear polarizer (P×2 Eq. (14-15)), $M_{\rm W}(\phi)$ is *symmetric* and invariant to the transformation $\phi \to \phi + \pi$. However, unlike $M_{\rm L}(\phi)$, $M_{\rm W}(\phi)$ is also *unitary* ($M^{-1} = M^\dagger$) with unit determinant, preserving the irradiance.

Hint:

First make a 2D coordinate (passive) rotation $(x, y) \to (x', y')$ so that the wave plate's slow axis is along $\hat{x}'$. Then apply the Jones matrix $M_{\rm W}(\phi = 0)$. Finally, rotate back to the (x, y) frame.

55.

Use the result of the previous problem to do Pedrotti×2 Problem 14-11. To get their result you must assume, as they do [Eqs. (14-17)-(14-20)], that the wave plate's slow axis lies along either the x or y axis.

Hint:

The initial state is described by P×2 Eq. (14-6). After the final state is put in the same form (with α replaced by α'), the final angle of inclination is simply α'. Alternatively, P×2 Eq. (14-10) supplies a general result for the final angle of inclination, useful even for elliptical polarization; this problem involves only the linearly polarized case for which $\epsilon = 0$ in that equation.

56.

(**a.**)

Do Pedrotti×2 Problem 14-17. Does their Jones matrix really convert *any* state of incident polarization to a *finite* irradiance of RH polarized light? Explain.

(**b.**)

Devise a combination of ideal wave plate(s) and polarizer(s) that, within a multiplicative constant, yields the Jones matrix of part (**a.**). Supply the absolute magnitude of this constant. Congratulations! You have invented an ideal *homogeneous right-hand circular polarizer.*

Hint:

*As an alternative to multiplying random Jones matrices until you find a combination that works, you could reason physically. You want a device that emits only RH circularly polarized (*RHCP*) light. Recall the "points-of-the-compass" diagram drawn in class: applying a* QWP *with slow axis at +45° produces the transitions $\hat{x}$* LP $\to$ LHCP $\to \hat{y}$ LP $\to$ RHCP $\to \hat{x}$ LP *(replace $\to$ by $\leftarrow$ for a* QWP *with slow axis at −45°). Therefore, for example, an $\hat{x}$ linear polarizer upstream of a* QWP *with slow axis at −45° emits only* RHCP *light. However, this isn't the complete answer, because you want a device that emits some* RHCP *light unless the incident beam is* LHCP *(see part (**a.**)). What can you put upstream of the $\hat{x}$ linear polarizer that emits some $\hat{x}$* LP *light unless the beam incident upon it is* LHCP*?*

(**c.**)

Show that the result of part (**b.**) functions also as a *right-hand circular analyzer*, *i.e.* it fully transmits RH circularly polarized light and fully absorbs LH circularly polarized light.

57. Stokes vectors #1.

Using the standard definition of the complex electric field $\vec{E}_0$,

$$\vec{E}(z,t) = \mathrm{Re}\big(\vec{E}_0 \exp\left(i(\tilde{k}z - \omega t)\right)\big) \ ,$$

consider the case in which the phase difference between its x and y components

$$\epsilon(t) = \arg E_{0x} - \arg E_{0y}$$

is not necessarily fixed, as would be the case for fully polarized light, but rather is allowed to vary with time – slowly with respect to ω^{-1}, but rapidly with respect to experimenters' ability to measure it. The *Stokes vector* $\mathcal{S}$ is defined by the real elements

$$\mathcal{S} \equiv \begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix} \equiv \frac{\mathrm{Re}\,\tilde{k}}{2\mu\omega} \begin{pmatrix} |E_{0x}|^2 + |E_{0y}|^2 \\ |E_{0x}|^2 - |E_{0y}|^2 \\ \langle 2\,\mathrm{Re}(E_{0x}E_{0y}^*)\rangle \\ \langle 2\,\mathrm{Im}(E_{0x}E_{0y}^*)\rangle \end{pmatrix} \ ,$$

where $\langle\rangle$ denotes the time average.

(**a.**)

Show that

$$\mathcal{S} = \frac{\mathrm{Re}\,\tilde{k}}{2\mu\omega} \begin{pmatrix} |E_{0x}|^2 + |E_{0y}|^2 \\ |E_{0x}|^2 - |E_{0y}|^2 \\ \langle 2|E_{0x}||E_{0y}|\cos\epsilon\rangle \\ \langle 2|E_{0x}||E_{0y}|\sin\epsilon\rangle \end{pmatrix} \ .$$

Hint:
Without loss of generality, substitute

$$E_{0x} = |E_{0x}|\, e^{i(\eta(t)+\epsilon(t)/2)}$$
$$E_{0y} = |E_{0y}|\, e^{i(\eta(t)-\epsilon(t)/2)}$$

in the definition of $\mathcal{S}$.
(**b.**)
The *normalized Stokes vector* $\bar{\mathcal{S}}$ is defined to be the usual Stokes vector divided by $\mathcal{S}_0$, so that its topmost element is unity. Consider a fully polarized beam in an arbitrary state of polarization described by the general Jones vector

$$J = \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} .$$

Show that the normalized Stokes vector for this beam is

$$\bar{\mathcal{S}} = \frac{1}{|\alpha|^2 + |\beta|^2} \begin{pmatrix} |\alpha|^2 + |\beta|^2 \\ |\alpha|^2 - |\beta|^2 \\ 2\,\mathrm{Re}(\alpha\beta^*) \\ 2\,\mathrm{Im}(\alpha\beta^*) \end{pmatrix} .$$

Hint:
For a fully polarized beam, $\epsilon(t)$ *in the result of part (***a.***) becomes a constant equal to* $\arg(\alpha) - \arg(\beta)$.
(**c.**)
Using the result of (**b.**) and your knowledge of Jones vectors, show that fully linearly polarized beams in the $\hat{x}$, $\hat{y}$, $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$, and $\frac{1}{\sqrt{2}}(\hat{x}-\hat{y})$ directions are described, respectively, by the normalized Stokes vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} ,$$

and that fully circularly RH and LH polarized beams are described, respectively, by the normalized Stokes vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} .$$

58. Stokes vectors #2.
Please refer to the notation and results of the previous problem.
(**a.**)
For fully polarized ("p") light (ϵ fixed), show that

$$\mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2 = \mathcal{S}_0^2 .$$

Hint:
*Use the result of part (***a.***) of the previous problem; with* ϵ *fixed, it is unnecessary to average over time.*
(**b.**)
Natural ("n") light is completely unpolarized. It has $|E_{0x}| = |E_{0y}|$, but the phases of both E_{0x} and E_{0y} vary randomly with time so that $\langle\cos\epsilon\rangle = \langle\sin\epsilon\rangle = 0$. For natural light, show (conversely to (**a.**)) that

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = 0 .$$

59. Stokes vectors #3.
Please refer to the notation and results of the previous two problems. Consider four devices: (A) a grey filter passing half the incident irradiance; (B) an $\hat{x}$ polarizer; (C) an $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$ polarizer; (D) a RH circular analyzer. After passing through (only) device X, the beam has irradiance I_{X}. It can be shown that

$$\mathcal{S} = 2 \begin{pmatrix} I_{\mathrm{A}} \\ I_{\mathrm{B}} - I_{\mathrm{A}} \\ I_{\mathrm{C}} - I_{\mathrm{A}} \\ I_{\mathrm{D}} - I_{\mathrm{A}} \end{pmatrix} .$$

Therefore, a Stokes vector can be completely determined by *measuring only irradiances*. This reveals one extent to which Stokes "vectors" satisfy vector properties. The additive property normally associated with a vector, $\mathcal{S}_{\mathrm{tot}} = \mathcal{S}_A + \mathcal{S}_B$ for two beams A and B, holds only if their *irradiances* rather than their amplitudes add, *i.e.* only if the two beams are *completely mutually incoherent*. This is a total contrast to Jones vectors, which can be defined only for fully polarized beams and can be added only if the two beams are completely mutually coherent.
(**a.**)

Using the additive property for Stokes vectors in mutually incoherent beams, show that an arbitrary beam

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix}$$

is the (necessarily incoherent) superposition of a fully polarized beam p and a natural-light beam n. Show this by specifying the elements of the constituent Stokes vectors $\mathcal{S}_p$ and $\mathcal{S}_n$ in terms of the elements of the overall Stokes vector $\mathcal{S}$.

Hint:

*The bottom three elements of $\mathcal{S}_n$ vanish, so the bottom three elements of $\mathcal{S}_p$ are the same as those of $\mathcal{S}$. Using the result of part (**a.**) of the previous problem, deduce the top element of $\mathcal{S}_p$.*

(**b.**)

Define the *degree of polarization* V by

$$V \equiv \frac{I_p}{I_p + I_n} \ .$$

For the above arbitrary beam, show that

$$V = \frac{\sqrt{\mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2}}{\mathcal{S}_0} \ .$$

Hint:

From the original definition of $\mathcal{S}$, one sees that its top element is simply the irradiance of the beam.

Appendix: Mueller matrices

The *Mueller matrices* manipulate Stokes vectors in the same way that Jones matrices manipulate Jones vectors. For an $\hat{x}$ polarizer and for a $\frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ polarizer, the Mueller matrices are, respectively,

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

The Mueller matrices for a $\hat{y}$ polarizer and for a $\frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$ polarizer are, respectively,

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

For a QWP with slow axis along x and for a homogeneous right-hand circular polarizer, the Mueller matrices are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \ .$$

The Mueller matrices for a QWP with slow axis along y and for a homogeneous left-hand circular polarizer are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \ .$$

60. Interference of two beams following different paths.

Consider two beams A and B. At (early) plane P, the relative properties of the two beams are well understood; for example, a single laser beam may be split into two. Between plane P and (late) plane Q, the beams follow different paths A and B through a nondispersive medium ($v_{\text{group}} = v_{\text{phase}}$); by the time they reach plane Q they have recombined. (For example, a Michelson interferometer may be interposed between the two planes.) At P and Q define

$$\text{physical } \vec{E}_{A,B}(P,Q) \equiv \text{Re}\big(\vec{E}_{P,Q}^{A,B}\, e^{-i\omega t}\big)$$

(this is four equations). On the left-hand side are physical fields that vary rapidly ($\approx$ sinusoidally) with time t; on the right-hand side are complex fields $\vec{E}_{P,Q}^{A,B}$ having magnitudes that are fixed, but phases that vary more slowly, over many sinusoidal periods. This slow variation may occur separately for the x and y components of a beam's electric field – in which case the beam is completely unpolarized – or it may occur in lockstep for the x and y components together, in which case the beam remains fully polarized.

The optical phase shifts for paths A and B are equal to

$$\omega\tau_{A,B} \equiv \int_P^Q \vec{k}_{A,B} \cdot d\vec{r}_{A,B}$$
$$\tau \equiv \tau_B - \tau_A \ ,$$

where $\vec{k}_{A,B}(\vec{r})$ is the wave vector for beam A or B, respectively, and $d\vec{r}_{A,B}$ lies along the path for beam A or B.

(**a.**)

The (undispersed) physical waves remain functions of $(\vec{k}_{A,B} \cdot \vec{r}_{A,B} - \omega t)$, even as these slow phase variations occur. Use this fact to show that

$$\text{physical } \vec{E}_{A,B}(Q)(t + \tau_{A,B}) =$$
$$\text{physical } \vec{E}_{A,B}(P)(t) \ .$$

Hint:

Using the fact that the physical fields are functions only of the argument $(\vec{k}_{A,B} \cdot \vec{r}_{A,B} - \omega t)$ — actually $(\int \vec{k}_{A,B} \cdot d\vec{r}_{A,B} - \omega t)$ because the paths are not straight — find a time t' such that the physical field argument at point Q and time t' has the same value as the physical field argument at point P and time t.

(**b.**)

Using the result of part (**a.**), show that

$$\vec{E}_Q^{A,B}(t + \tau_{A,B}) = \vec{E}_P^{A,B}(t) \exp\left(i\omega\tau_{A,B}\right) \ .$$

Hint:

Express the physical fields in terms of the complex fields.

(**c.**)

At any other time t', the result of (**b.**) also holds. Choose $t' = t - \tau$. Show that

$$\vec{E}_Q^B(t + \tau_A) = \vec{E}_P^B(t - \tau) \exp\left(i\omega\tau_B\right) \ .$$

(**d.**)

The irradiance

$$I = \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\, |\vec{E}^A + \vec{E}^B|^2$$

for the superposition of the two beams satisfies

$$2\sqrt{\tfrac{\mu}{\epsilon}} I_{P,Q} = |\vec{E}_{P,Q}^A|^2 + |\vec{E}_{P,Q}^B|^2 + 2\mathrm{Re}\big(\vec{E}_{P,Q}^{A*} \cdot \vec{E}_{P,Q}^B\big)$$

Using the results of (**b.**) and (**c.**), show that

$$I_Q(t + \tau_A) = I^A + I^B +$$
$$+ \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\, 2\,\mathrm{Re}\big(\vec{E}_P^{A*}(t) \cdot \vec{E}_P^B(t - \tau) \exp\left(i\omega\tau\right)\big) \ ,$$

where $I^{A,B}$ are the (time-independent and space-independent) single-beam irradiances.

Hint:

In the supplied expression for $I_Q(t+\tau_A)$, express the $\vec{E}_Q^{A,B}$ in terms of the $\vec{E}_P^{A,B}$.

(**e.**)

Taking a long-time average (long compared to the characteristic time over which the complex electric field phases vary), obtain as a final step the *master equation for two-beam interference:*

$$\langle I_Q^{A+B}\rangle(\tau) = I^A + I^B +$$
$$+ \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\, \langle 2\,\mathrm{Re}\big(\vec{E}_P^{A*}(t) \cdot \vec{E}_P^B(t - \tau) \exp\left(i\omega\tau\right)\big)\rangle \ ,$$

where $\langle\rangle$ denotes a long-time average, and I_Q^{A+B} is the combined irradiance at plane Q.

Hint:

Note that $\langle I_Q^{A+B}\rangle(\tau)$ is not the combined irradiance at time τ. First, τ is not a time – it is the difference of the optical phase shifts for paths B and A. Second, $\langle I_Q^{A+B}\rangle$ can't be a function of time, because it is already a long-term time average. Rather, $\langle I_Q^{A+B}\rangle(\tau)$ is a time-independent irradiance that depends on the path difference that is parametrized by τ.

61.

Please refer to the notation and results of the previous problem. Define the *correlation* $\Gamma^{AB}(\tau)$ as

$$\Gamma^{AB}(\tau) \equiv \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\, \langle \vec{E}_P^{A*}(t) \cdot \vec{E}_P^B(t - \tau) \exp\left(i\omega\tau\right)\rangle \ ,$$

and define the *degree of partial coherence* $\gamma^{AB}(\tau)$ as

$$\gamma^{AB}(\tau) \equiv \frac{\Gamma^{AB}(\tau)}{\sqrt{I^A I^B}} \ .$$

(**a.**)

Show that the result of the last part of the previous problem can be written

$$\langle I_Q^{A+B}\rangle(\tau) = I^A + I^B + 2\sqrt{I^A I^B}\,\mathrm{Re}\,\gamma^{AB}(\tau) \ .$$

(**b.**)
If the screen Q in a two-beam interference setup deviates slightly from perfect perpendicularity to the beams, deviations of order 10-100 occur in $\omega\tau$ across the screen. For most sources these deviations do not cause a significant change in $\vec{E}_P^B(t-\tau)$, but they do cause the phase of $\exp(i\omega\tau)$ to change dramatically. Correspondingly there appear on the screen many light and dark bands ("fringes"), at the center of which the respective irradiances are $I_{\max}$ and $I_{\min}$. Define the *fringe visibility* $\mathcal{V}$ as

$$\mathcal{V} \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \ .$$

If $I^A = I^B$, show that

$$\mathcal{V} = |\gamma^{AB}(\tau)| \ .$$

Hint:
In the middle of a bright (dark) fringe, $\gamma^{AB} = +|\gamma^{AB}|$ $(\gamma^{AB} = -|\gamma^{AB}|)$.
(**c.**)
As an experimentalist, suppose that you are required to analyze the extent to which a mystery beam is polarized.

A standard approach would be to measure the elements of its Stokes vector (by observing the reduction in irradiance caused by four different optical devices – see Problem 59); knowing the Stokes vector, you could calculate the degree of polarization V (Problem 59(**b.**)).

Instead you decide to send the beam into a Michelson interferometer with two exactly equal-length paths A and B. Observing the resulting fringe pattern on screen Q, you measure the fringe visibility $\mathcal{V}$ (as defined in part (**b.**) of this problem).

Do you obtain any useful information about V by measuring $\mathcal{V}$? If so, what is the relationship between the two?
Hint:
If there are slow random phase variations of the complex field $\tilde{E}_y$ *with respect to* $\tilde{E}_x$*, a beam is completely unpolarized. If such phase variations were to begin, while the Michelson path lengths remained exactly equal, how would the observed fringe pattern change? That is, how would the interference of* $\tilde{E}_x^A$ *with* $\tilde{E}_x^B$ *, and* $\tilde{E}_y^A$ *with* $\tilde{E}_y^B$ *be affected?*

62.
We wish to use the light of Betelgeuse (angular diameter 0.047 arc second), passed through a 600 nm filter, as the source for a double-thin-slit Young's interference experiment.
(**a.**)
Assuming an adequately narrow filter bandpass, roughly estimate the maximum slit separation (in m) that would yield an interference pattern which isn't too badly washed out, *i.e.* with a fringe visibility $\mathcal{V}$ of order $\frac{1}{2}$.
Hint:
This part is a transverse coherence problem, involving a source rendered monochromatic by the narrow filter. In the paraxial approximation, first consider the classical two-thin-slit irradiance pattern $I(\psi) \propto \cos^2 \frac{kd}{2}\psi$ *that would occur if Betelgeuse were a point source. Now allow the two (point) halves of Betelgeuse to separate. This causes the two halves of the classical irradiance pattern likewise to move apart. As they do so, the primary irradiance maximum (at* $\psi = 0$*) and first minimum (at* $\psi = \pi/kd$*) remain at the same positions, but become less extreme. Parametrize this separation by a phase* $\pm\delta$ *that is added to the argument of* $\cos^2$*. Solve for* δ *such that* $\mathcal{V} = \frac{1}{2}$ *; then relate* 2δ *to the (supplied) angular separation of Betelgeuse's two halves. This fixes the slit separation* d*.*
(**b.**)
Assuming an adequately small slit separation, roughly estimate the maximum filter bandpass (in nm) that would allow us to observe at least 20 fringes. With this choice of bandpass, what is the coherence length of the transmitted light?
Hint:
This part is a temporal coherence problem, involving a slit separation so small that, with a monochromatic source, a very large number of fringes would be visible. Call the central (brightest) fringe the 0th *fringe maximum. About 10 fringes to either side, the pattern is mostly washed out because the source is polychromatic. This occurs when the 10*th *fringe maximum of light with wavelength* $\lambda_0 - \Delta\lambda/2$ *coincides with*

the 10th fringe minimum of light with wavelength $\lambda_0 + \Delta\lambda/2$. Use this fact to relate $\Delta\lambda$ to λ (see the related discussion in {P×2 ed2, ed3} section {11-2, 8-2}). The coherence length of the transmitted light follows from {P×2 ed2, ed3} Eq. {(12-18), (9-18)}.

63.
A monochromatic beam traveling in medium "0" is normally incident upon a substrate *"T"*. A single film "1" is interposed between the two media. The refractive indices are, respectively, n_0, n_1, and n_T. You may assume that all materials have the same magnetic permeability.
(**a.**)
Show that a film of thickness $\lambda_1/4$ (where λ_1 is the wavelength of light in the material i of which the film is made) will reduce the reflectance of the substrate to zero, provided that $n_1 = \sqrt{n_0 n_T}$.
Hint:
Remember that $\lambda_1 = \lambda_{\text{vacuum}}/n_1$. This allows the film's phase advance δ (e.g. in {P×2 ed2, ed3} Eq. {(19-24), (22-24)}) to take the simple value $\pi/2$. Substitute the resulting transfer matrix elements in the standard reflection formula, e.g. {P×2 ed2, ed3} Eq. {(19-36), (22-36)}.
(**b.**)
Prove that interposing a single film of thickness $\lambda_1/4$ will *always* reduce the reflectance of the substrate, provided that $n_0 < n_1 < n_T$.
Hint:
After obtaining the amplitude reflectance $r_{0,\text{b}}$ for the cases (no film) and (b), form the ratio r_0^2/r_b^2 and (tentatively) set it > 1. Applying brute force, multiply through by all denominators and make many cancellations. Distill the resulting inequality into one that is obviously true, given that $n_0 < n_1 < n_T$. To check your algebra, keep in mind that every inequality should become an equality if n_1 is set to n_0 or n_1 is set to n_T (as these cases correspond to no film at all).

64.
Referring to the conditions of the previous problem, consider next the case of three films ("1", "2", and "3") interposed between the two media, such that film 1 adjoins medium 0 and film 3 adjoins medium T. Again, assume that all materials have the same magnetic permeability.
(**a.**)
Suppose that each film has thickness $\lambda_i/4$ (where λ_i is the wavelength of the beam in the particular material of which that film is made). Show that the reflectance of the substrate is reduced to zero when

$$\frac{n_1 n_3}{n_2} = \sqrt{n_0 n_T}\ .$$

Hint:
After you take the product of the transfer matrices, the result should have diagonal elements that vanish.
(**b.**)
An advantage of using three films instead of one (as in the previous problem) is that the band of wavelengths over which the reflectance is heavily suppressed can be made much broader. (Your expensive eyeglasses are coated with at least two films.) According to {Pedrotti×2 ed2, ed3} Fig. {19-7, 22-7}, this benefit may be enhanced further if the middle film (2) is doubled in thickness from $\lambda_2/4$ to $\lambda_2/2$. In this case, what condition on n_0, n_1, n_2, n_3, and n_T reduces the reflectance to zero?
Hint:
Here the product of transfer matrices should be diagonal.

65.
Consider a high-reflectance stack of the type depicted in {Pedrotti×2 ed2, ed3} Fig. {19-8, 22-8}. For specificity, assume that the stack consists of six double layers of MGF_2 ($n = 1.38$) and ZnS ($n = 2.35$). For simplicity, assume that the medium from which the light enters the stack (medium 0) and the medium into which the light exits the stack (medium T) are vacuum. Again, assume that all materials have the same magnetic permeability.
(**a.**)
Numerically, what fraction T of the incident irradiance is transmitted by the stack?
Hint:
Consider one double layer. When both films in the double layer have thickness $\lambda_i/4$ (where λ_i is the wavelength in the medium of layer i), the

product of the transfer matrices for the two layers is diagonal. Therefore the transfer matrix for N double layers is simply the transfer matrix for one layer raised to the Nth power (see {P×2 ed2, ed3} Eq. {(19-50), (22-50)}).

(**b.**)

The stack is now modified as follows: the upstreammost three double layers are flipped around so that the stack indices are L(ow) H(igh) L H L H H L H L H L. This is a *Fabry-Perot interference filter*. It has a transmission *maximum* at the wavelength for which it was designed, as opposed to the transmission *minimum* achieved by the configuration of part (**a.**). Calculate the fraction T of the incident irradiance that is transmitted by the modified stack.

Hint:

When the order of the films is inverted, the diagonal elements of the transfer matrix for the double layer are interchanged. To what extremely simple form does the N-element transfer matrix reduce?

66. Green's theorem.

Denote by $\vec{G}$ a vector field, and start from the *divergence theorem*

$$\int \nabla \cdot \vec{G}\, d\tau = \oint \vec{G} \cdot \hat{n}\, da \ ,$$

where $\hat{n}$ is the (outward) direction of the surface area element $d\vec{a}$, and the left-hand integral extends over the volume enclosed by the right-hand surface.

(**a.**)

Substituting $\vec{G} = V\nabla U$, where V and U are scalar fields, show that

$$\int \big(\nabla V \cdot \nabla U + V\nabla^2 U\big)\, d\tau = \oint V \frac{\partial U}{\partial n}\, da \ .$$

Hint:

Use the fact that

$$(\nabla U) \cdot \hat{n} \equiv \frac{\partial U}{\partial n} \ .$$

(**b.**)

Show that

$$\int \big(V\nabla^2 U - U\nabla^2 V\big)\, d\tau = \oint \Big(V \frac{\partial U}{\partial n} - U \frac{\partial V}{\partial n}\Big)\, da \ .$$

Hint:

*Repeat part (**a.**) with $\vec{G} = U\nabla V$. Then subtract one of the two resulting equations from the other.*

(**c.**)

If V and U both satisfy the scalar Helmholtz equation,

$$\big(\nabla^2 + k^2\big)(U, V) = 0 \ ,$$

where k is a constant, show that

$$0 = \oint \Big(V \frac{\partial U}{\partial n} - U \frac{\partial V}{\partial n}\Big)\, da \ .$$

This is *Green's theorem for solutions to the scalar Helmholtz equation.*

Hint:

*Substitute $k^2\, U, V$ for $\nabla^2\, U, V$ in the result of part (**b.**).*

67. Fresnel-Kirchoff integral theorem.

Please use the notation and results of the previous problem.

(**a.**)

Consider a closed surface consisting of an inner sphere of radius R, centered at the origin, and an arbitrary closed outer surface $\mathcal{A}$. Apply the result of part (**c.**) to the combined surface. Take V to be an inward-propagating spherical wave

$$V = V_0 \frac{e^{i(kr+\omega t)}}{r} \ .$$

In the limit $R \to 0$, show that

$$U(0) = \tfrac{1}{4\pi} \oint \Big(\frac{e^{ikr}}{r} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n}\big(\frac{e^{ikr}}{r}\big)\Big)\, da \ ,$$

where the integral is taken only over $\mathcal{A}$. This is the *Kirchoff integral theorem.*

Hint:

Factoring out $V_0\, e^{i\omega t}$, this is equivalent to showing that, in the limit $R \to 0$, the integral over the inner spherical surface reduces to $-4\pi\, U(0)$. (Note that the normal to this surface is $-\hat{r}$.)

(**b.**)

Now punch a hole ("aperture") in $\mathcal{A}$. Place a

point source S outside $\mathcal{A}$; the origin (now called "observation point P") still lies inside $\mathcal{A}$. The source radiates an outward-propagating scalar spherical wave

$$U = U_0 \frac{e^{i(kr'-\omega t)}}{r'} \ ,$$

where $\vec{r}\,'$ is a vector from S to a point in space. Using the result of (**a.**), assume that the opacity of the remainder of $\mathcal{A}$ allows the integral to be carried out over only the aperture ("ap"). In the far zone limit $kr', kr \gg 1$, show that

$$U_P = \frac{-ikU_0 e^{-i\omega t}}{4\pi} \int_{\rm ap} \frac{e^{ik(r+r')}}{rr'} \big(\hat{r}\cdot\hat{n} - \hat{r}'\cdot\hat{n}\big)\, da \ ,$$

where $\vec{r}$ ($\vec{r}\,'$) is a vector from P (point S) to a point on the element of aperture da, and $\hat{n}$ is the (outward from P) normal to da. This is the *Fresnel-Kirchoff integral theorem*; it is the starting point for the study of diffraction in the *scalar field approximation.*

Hint:
Substitute

$$U = U_0 \frac{e^{i(kr'-\omega t)}}{r'}$$

*in the result of part (**a.**) and perform the indicated differentiation. In analogy to the hint to part (**a.**) of the previous problem, note that*

$$\frac{\partial}{\partial n}\frac{e^{ikr'}}{r'} = \big(\nabla' \frac{e^{ikr'}}{r'}\big)\cdot\hat{n} \ ,$$

where ∇' is the gradient with respect to the coordinate $\vec{r}\,'$.

68. Knife-edge diffraction.
A plane wave of initial irradiance I_0 propagating along $\hat{z}$ is incident upon a semi-infinite totally absorbing screen lying in the $z = 0$ plane. The screen extends from $-\infty < x < \infty$ and $-\infty < y < 0$. An observer stationed at $(0, 0, z)$, where $kz \gg 1$, detects an irradiance I'. What is I'/I_0, and why?

Hint:
*Using $z = 0$ as the aperture plane, equate the optical disturbance U_P to a Fresnel-Kirchoff integral over that plane, (**a.**) for the case in which there is no absorber, and (**b.**) for the knife-edge case in which the absorber covers $y < 0$. Considering that the source is a plane wave with no y dependence, and that the observer is stationed at the up-down symmetry point $y = 0$, how is integral (**b.**) related to integral (**a.**)? How is U related to the irradiance?*

69. Fourier diffraction.
The *convolution* of two functions $f(x)$ and $g(x)$, denoted by $(f \otimes g)(x)$, is defined by

$$f \otimes g \equiv \int_{-\infty}^{\infty} dx'\, f(x')\, g(x - x') \ .$$

Define the *Fourier transform* $\mathcal{F}_\mu\big(g(x)\big)$ by

$$\mathcal{F}_\mu\big(g(x)\big) \equiv \int_{-\infty}^{\infty} dx\, g(x)\, e^{-i\mu x} \ .$$

(**a.**)
As a warmup, prove that

$$f \otimes g = g \otimes f \ .$$

Hint:
Substitute $u' = x - x'$ in the convolution integral.

(**b.**)
For use in part (**d.**), prove that

$$\mathcal{F}_\mu\big(f(x) \otimes g(x)\big) = \mathcal{F}_\mu\big(f(x)\big)\, \mathcal{F}_\mu\big(g(x)\big) \ .$$

Hint:
Write $\mathcal{F}_\mu\big(f(x) \otimes g(x)\big)$ as a double integral $\int dx \int dx' \ldots$. Reverse the order of integration and substitute $u = x - x'$. Express the result as the product of an integral over u of purely u-dependent terms, $\times$ an integral over x' of purely x'-dependent terms.

(**c.**)
If $f(x)$ is the aperture function for a pair of thin slits separated by d,

$$f(x) \propto \delta(x - \tfrac{d}{2}) + \delta(x + \tfrac{d}{2}) \ ,$$

and if $g(x)$ is the aperture function of a single slit of thickness a,

$$g(x) \propto \theta(x + \tfrac{a}{2}) - \theta(x - \tfrac{a}{2}) \ ,$$

show that $f \otimes g$ is the aperture function corresponding to two slits of thickness a, separated (centerline-to-centerline) by d.

Hint:

As your intuition develops, this proposition will become obvious, but here you are asked to show it formally. Do so by carrying out the convolution integral, evaluating the integrand at the points where one of the δ-functions is nonzero. Note that the θ-function $\theta(u)$ steps from 0 at $u < 0$ to 1 at $u > 0$.

(**d.**)

In the Fraunhofer approximation, where $\vec{r}\,'$ and $\vec{r}$ (cf. Problem 67) are paraxial and the wavefront curvature across the aperture is negligible, the scalar "optical disturbance" amplitude is

$$U_P(\mu,\nu) \propto \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, g(x,y) e^{-i(\mu x + \nu y)} \ ,$$

where U_P is measured at the transform plane (X, Y), the aperture function g is measured at the aperture plane (x, y), μ and ν are defined by

$$\mu \equiv \frac{kX}{f} \quad \nu \equiv \frac{kY}{f} \ ,$$

and f is the focal length of the thin field lens located an equidistance f from the aperture and transform planes. Write down the diffraction pattern

$$\frac{I_N(\psi_x, \psi_y)}{I_1(0,0)}$$

for N slits of center-to-center separation $\Delta x = d$ and thicknesses $\delta x = a$ and $\delta y = b$, where

$$(\sin)\psi_x \equiv \frac{X}{f}$$
$$(\sin)\psi_y \equiv \frac{Y}{f} \ .$$

You may use the fact – directly obtainable by applying the Fourier transform – that

$$\frac{I_N(\psi_x)}{I_1(0)} = N^2 \frac{\sin^2\left(\frac{Nkd}{2}\sin\psi_x\right)}{\left(N \sin\left(\frac{kd}{2}\sin\psi_x\right)\right)^2}$$

for N thin slits of infinite length and separation d, and that

$$\frac{I(\psi_x)}{I(0)} = \mathrm{sinc}^2\left(\tfrac{ka}{2}\sin\psi_x\right)$$

for a single slit of infinite length and thickness a.

Hint:

*Building on the result of part (**c.**), use your intuition to express the aperture function for N thick slits as the convolution of the aperture function for N thin slits and the aperture function for one thick slit. Also use the fact that the Fourier transform of the product of two functions of different variables (x and y here) is the product of the Fourier transforms. Then exploit the result of part (**b.**).*

70. Quadruple slit.

Consider four equally spaced long ($\Delta y = \infty$) thin slits, located at $x = \pm\frac{d}{2}$ and $x = \pm\frac{3d}{2}$. As usual, $\tan\psi_x = \frac{dx}{dz}$ of the outgoing wavefront.

(**a.**)

Write down the standard result

$$\mathcal{R}(\psi_x) \equiv \frac{I(\psi_x)}{I(\psi_x = 0)}$$

for the Fraunhofer diffraction pattern from $N = 4$ equally spaced thin slits.

Hint:

*See part (**d.**) of the previous problem.*

(**b.**)

Consider the full diffracted *amplitude* to be the superposition of the diffracted *amplitudes* from a pair of slits at $x = \pm\frac{d}{2}$ and a pair of slits at $x = \pm\frac{3d}{2}$. Write down $\mathcal{R}(\psi_x)$ as a quantity proportional to the modulus^2 of the sum of the diffracted amplitudes from the two pairs of slits.

Hint:

The standard result for a pair of thin slits of full separation D (Young's experiment) is

$$U(\psi_x) \propto \cos\left(\tfrac{kD}{2}\sin\psi_x\right) \ .$$

(**c.**)

Consider the aperture function for these four slits to be the convolution of a pair of δ-functions separated by d and another pair of δ-functions separated by $2d$ (both pairs are symmetric about $x = 0$). Write down $\mathcal{R}(\psi_x)$ as the product of two two-slit $\mathcal{R}$'s.

(**d.**)
Are your answers to parts (**a.**), (**b.**), and (**c.**) equivalent? Why or why not?
Hint:
Do all three methods represent valid approaches to the same physical problem?

71. Fuzzy thick slit.
Please use the notation and results of Problem 70. Consider a trapezoidal aperture function

$$\begin{aligned} g(x) &= 1 \qquad |x| < \tfrac{a}{2} \\ &= 0 \qquad |x| > a \\ &= \tfrac{2}{a}(x+a) \qquad -a < x < -\tfrac{a}{2} \\ &= \tfrac{2}{a}(a-x) \qquad \tfrac{a}{2} < x < a \ . \end{aligned}$$

Fraunhofer conditions apply. Under these conditions, calculate the slit's diffraction pattern

$$\mathcal{R}(\psi_x) \equiv \frac{I(\psi_x)}{I(\psi_x = 0)} \ .$$

Hint:
The convolution of two identical slits of thickness $\frac{D}{2}$ is an isosceles triangle of base D. This could be considered to be a trapezoid with a plateau of zero length. How would you adjust the two slit thicknesses to obtain the trapezoid base and plateau that the problem specifies?

72. Thick slits with wave plates.
A linearly ($\hat{x}$) polarized plane EM wave traveling along $\hat{z}$ is incident on an opaque baffle located in the plane $z = 0$. The baffle has two slits cut in it, which are of infinite extent in the $\hat{y}$ direction. In the $\hat{x}$ direction, the slit widths are each a and their center-to-center distance is d. (Obviously $d > a$, but you may *not* assume that $d \gg a$.) The top and bottom slits are each an equal distance from $x = 0$. The diffracted image is viewed on a screen located in the plane $z = L$, where $L \gg d$; also $\lambda L \gg d^2$, where λ is the EM wavelength.

Quarter-wave plates are placed in each slit. They are identical, except that the top plate's "slow" (high-index) axis is along $(\hat{x}+\hat{y})/\sqrt{2}$ ($+45°$ with respect to the $\hat{x}$ axis), while the bottom plate's slow axis is along $(\hat{x}-\hat{y})/\sqrt{2}$ ($-45°$ with respect to the $\hat{x}$ axis).

(**a.**)
What is the state of polarization of the diffracted light that hits the center of the screen, at $x = y = 0$? Explain.
Hint:
Divide each thick slit into N contiguous thin slits, where $N \to \infty$. Out of N, consider the nth-above-the-origin together with the nth-below-the-origin thin slits as sources of diffracted light. When observed at the symmetry point $x = 0$, the optical paths from these two thin slits are the same. Therefore the polarization of the light from these two thin slits will be given by the sum of the Jones vectors for the light emerging from each slit.
(**b.**)
At what diffracted angle ψ_x does the first minimum of the irradiance occur?
Hint:
Does the light from the top slit interfere with the light from the bottom slit? See the result of problem 52.