University of California, Berkeley
Physics 110B Fall 2008 (*Strovink*)

SOLUTION TO FINAL EXAMINATION

Directions: Do all six problems, which have unequal weight. This is a closed-book closed-note exam except for Griffiths, Pedrotti, a copy of anything posted on the course web site, and anything in your own original handwriting (not Xeroxed). You may use the result of any assigned problem-set problem without proving it. Calculators, palmtops, laptops, and cellphones should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Show all your work. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (20 points)
Write down a master equation that is equivalent to the pair of Maxwell's equations in vacuum that involve *sources*. You need not prove this equivalence. Your equation must be *manifestly covariant*, with *no gauge condition* imposed. Any four-tensors that you use must further be defined in terms of four-vectors. Correspondingly, any four-vectors that you use (either in the master equation or in defining any four-tensors) must further be defined in terms of three-vectors and scalars.

Solution:

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \ \text{ with} \\
F^{\mu\nu} &\equiv \partial^\mu A^\nu - \partial^\nu A^\mu \\
\partial^\mu &\equiv \left(\frac{\partial}{c\,\partial t}, -\nabla\right) \\
\partial_\mu &\equiv \left(\frac{\partial}{c\,\partial t}, +\nabla\right) \\
A^\mu &\equiv \left(\frac{V}{c}, \vec{A}\right) \\
J^\mu &\equiv (c\rho, \vec{J})\ .
\end{aligned}$$

Problem 2. (45 points)
At $t = 0$ a relativistic point particle with rest mass m and charge q has velocity

$$c\vec{\beta}(t=0) = \hat{x}\, c\beta_0\ ,$$

where β_0 is a positive constant that is *not* $\ll 1$. The particle has been and continues to be immersed in a uniform static magnetic field

$$\vec{B} = \hat{z}\, B_0\ .$$

During the period $t < 0$, no electric field is applied.

(**a.**) (5 points)
Denote by $c\vec{\beta}_\perp$ the (vector) portion of the particle's velocity that is perpendicular to the $\hat{z}$ axis. For $t < 0$, is $\vec{\beta}_\perp$ conserved? Explain.

Solution: The Lorentz force is nonzero, since $\vec{v}$ and $\vec{B}$ are neither zero nor mutually parallel. Furthermore, since $\vec{B}$ lies along z, a nonzero Lorentz force acts in the perpendicular direction. Therefore $\vec{p}_\perp$ varies. On the other hand, according to the Lorentz force from a magnetic field, the rate of change of $\vec{p}_\perp$ is perpendicular to $\vec{p}_\perp$. Thus the magnitude of $\vec{p}_\perp$ cannot vary, so its direction must vary. Therefore $\vec{\beta}_\perp$ cannot be conserved.

(**b.**) (5 points)
For $t < 0$, is $|\vec{\beta}_\perp|$ conserved? Explain.

Solution: Because magnetic forces can do no work, $E(= \gamma mc^2)$ and therefore γ is conserved. Since $\beta_\parallel = 0$ for parts (**a.**)-(**c.**), γ is a function only of $|\vec{\beta}_\perp|$, which likewise must be conserved.

(**c.**) (10 points)
For $t < 0$, write an expression for $|\vec{p}_\perp|$, the absolute magnitude of the (vector) portion of the particle's momentum that is perpendicular to the $\hat{z}$ axis.

Solution: For $t < 0$, $|\vec{\beta}_\perp| \equiv \beta_0$ from part (**b.**), and

$$\begin{aligned}
|\vec{p}_\perp| &\equiv \gamma|\vec{\beta}_\perp|mc \\
&= \frac{\beta_0}{\sqrt{1-\beta_0^2}}\, mc\ .
\end{aligned}$$

(**d.**) (10 points)
For the later period $t > 0$, a uniform static electric field

$$\vec{E} = \hat{z}\,\tfrac{1}{2}\,cB_0$$

is applied (in addition to the magnetic field). It causes the point charge's motion to become more complicated.

As a strategy for determining this motion, is it possible to transform to a Lorentz frame $\mathcal{S}'$ in which one of the static fields vanishes? If so, which field will vanish? If not, why not?

Solution: Because $\vec{E}\cdot\vec{B}$ is a Lorentz invariant that does not vanish in the lab frame, neither field can vanish in any Lorentz frame.

(**e.**) (15 points)
For $t > 0$, after the electric field is applied, is $|\vec{\beta}_\perp|$ conserved? If so, why? If not, does $|\vec{\beta}_\perp|$ increase with time, or does it decrease? Why?

Solution: Now that the particle has acquired a nonzero velocity in the z ($\|$) direction, the magnitude of the transverse momentum is

$$|\vec{p}_\perp| = \frac{|\vec{\beta}_\perp|}{\sqrt{1-\beta_\perp^2-\beta_\|^2}}\,mc\,.$$

Because the perpendicular part of the Lorentz force remains perpendicular to $\vec{p}_\perp$, $|\vec{p}_\perp|$ is still conserved. Therefore, when $\beta_\|^2$ in the denominator increases due to the electric field, $|\vec{\beta}_\perp|$ must decrease.

Problem 3. (35 points)
At a symmetric positron-electron collider, a positron of charge $+e$, travelling with a (given) Lorentz factor $\gamma \equiv (1-v^2/c^2)^{-1/2}$ in the $+\hat{z}$ direction, collides at $t=0$ with an electron having the same γ, travelling in the $-\hat{z}$ direction. They decelerate instantaneously and stick together.

(**a.**) (20 points)
At $t_0 = 0$ an observer located at $(x,0,0)$, with $x>0$, measures the magnetic field $\vec{B}_0$ produced by *both* the positron and electron. Calculate the magnitude and direction of this field.

Solution: The observer, not yet aware that the electron and positron have collided, sees a "bottle-brush" electric field from the positron, together with its associated magnetic field. Since both its charge and its direction are reversed, the electron produces a cancelling *electric* field, but the same *magnetic* field as the positron. So we just double the positron's magnetic field.

From the positron, the "bottle-brush" electric field is

$$\vec{E} = \frac{e}{4\pi\epsilon_0}\frac{\gamma\vec{r}}{(\gamma^2 r_\|^2 + r_\perp^2)^{3/2}}\,.$$

With $r_\| = 0$ and $\vec{r}_\perp = \hat{x}x$, this becomes

$$\vec{E} = \frac{e}{4\pi\epsilon_0}\frac{\gamma\hat{x}}{x^2}\,.$$

Twice the positron's magnetic field is $2(\vec{\beta}/c)\times\vec{E}$, or

$$\vec{B}_{\rm tot} = \frac{e}{2\pi\epsilon_0 c}\frac{\gamma\beta\hat{y}}{x^2}\,.$$

(**b.**) (15 points)
At $t_1 = x/c$ the same observer measures the *dominant* component of the magnetic field $\vec{B}_1$ produced by *both* the positron and electron. Calculate the direction $\hat{B}_1$ of this field.

Solution: At time t_1 the observer sees mainly the (infinite) radiation fields associated with the particles' infinite deceleration. Again the *magnetic* fields from the positron and electron point in the same direction.

The positron's acceleration field $\vec{E}_a$ is directed along $\hat{x}\times[(\hat{x}-\vec{\beta})\times\dot{\vec{\beta}}]$, or $\hat{x}\times[\hat{x}\times(-\hat{z})] = \hat{z}$. Correspondingly its magnetic field is directed along $\hat{x}\times\hat{E}_a = -\hat{y}$, opposite to the earlier field.

Problem 4. (30 points)
A plane electromagnetic wave is described by

$$\mathbf{E}(z,t) = \mathrm{Re}\Big(\tilde{\mathbf{E}}\,\exp\big(i(kz-\omega t)\big)\Big)\,,$$

where

$$\tilde{\mathbf{E}} = E_0\big((2-i)\hat{\mathbf{x}} + (1-2i)\hat{\mathbf{y}}\big)\,,$$

and E_0, k, and ω are real constants. A linear polarizer is placed in the beam, and oriented so

that the largest possible fraction of the original beam's irradiance is transmitted. What is that fraction?

Solution:
The beam is described by the (unnormalized) Jones vector

$$J = \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} .$$

A linear polarizer with transmission axis oriented along the $\hat{\mathbf{x}}$ direction has the Jones matrix

$$M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

If the polarizer's transmission axis is oriented at angle ϕ with respect to the $\hat{\mathbf{x}}$ direction, it is represented by the Jones matrix

$$\begin{aligned} M(\phi) &= R^{-1} M(0) R \\ &= \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} , \end{aligned}$$

where the two-dimensional rotation matrix is

$$R \equiv \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} .$$

Upstream of the polarizer, the beam irradiance I is proportional to

$$\begin{aligned} I &\propto J^\dagger J \\ &= \begin{pmatrix} 2+i & 1+2i \end{pmatrix} \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} \\ &= (4+1) + (1+4) = 10 . \end{aligned}$$

Downstream of the polarizer, the irradiance I' is proportional to

$$\begin{aligned} I' &\propto (MJ)^\dagger MJ \\ &= J^\dagger (M^\dagger M) J . \end{aligned}$$

But $M^\dagger M = M$, as can easily be verified: $M^\dagger = M$, and adding a second ideal polarizer does nothing beyond the effect of the first, so $M^2 = M$. Thus

$$\begin{aligned} I &\propto J^\dagger M J \\ &= \begin{pmatrix} 2+i & 1+2i \end{pmatrix} \times \\ &\times \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} \\ &= 5 + 8\sin\phi\cos\phi \\ &= 9 \text{ (max)} \end{aligned}$$

when $\phi = \pi/4$. Therefore, at maximum, $I'/I = 9/10$.

Problem 5. (35 points)
A plane wave $U_0 \cos(kz - \omega t)$ is incident normally on a screen. Fraunhofer conditions apply. The diffracted wave is observed from $z \to \infty$ at various angles ψ with respect to the z axis.

(**a.**) (15 points)
Assume that the screen has three long parallel slits with equal spacing b and equal negligible width. Compute the irradiance ratio $I(\psi)/I(\psi = 0)$.

Solution:
In analogy to the standard double slit problem,

$$U(\psi) \propto 1 + e^{i\beta} + e^{-i\beta} ,$$

where $\beta = kb\sin\psi$. Therefore

$$\begin{aligned} U(\psi) &\propto 1 + 2\cos\beta \\ \frac{I(\psi)}{I(0)} &= \frac{(1+2\cos\beta)^2}{9} . \end{aligned}$$

This result is equivalent to the equally acceptable standard result $\frac{1}{9}\sin^2(3\gamma)/\sin^2\gamma$, where $\gamma = \beta/2$; for full credit this standard result may simply be written down.

(**b.**) (20 points)
Instead assume that the screen has five long parallel slits with equal spacing b. The slit widths are still negligible; however, they are a function of the slit location, so that the five slit areas vary according to the ratio 1:2:3:2:1. Compute the irradiance ratio $I(\psi)/I(\psi = 0)$.

Solution:
This configuration is equivalent to a triple-superposition of the triple-slit problem in (**a**), with the characteristic spacing of the superposition equal to the characteristic spacing of the

slit. Therefore it is a convolution of the arrangement in (**a**) with itself. Under Fraunhofer conditions, the image is a Fourier transform of the aperture function, and the Fourier transform of a convolution is the product of the individual Fourier transforms. Therefore

$$\frac{I(\psi)}{I(0)} = \frac{(1+2\cos\beta)^4}{81} \ .$$

This result may also be obtained by the brute-force methods of (**a**).

Problem 6. (35 points)
A monochromatic plane wave is normally incident on an aperture plane (with different aperture functions for parts (**a.**) and (**b.**)). Fraunhofer conditions do *not* apply. An observer is stationed at $(0, 0, R^2/\lambda)$, where λ is the wavelength, and R is a constant. In each part, calculate the ratio $I_{\text{aperture}}/I_{\text{open}}$ of irradiances seen by the observer with/without the aperture plane in place.

(**a.**) (15 points)
Here the aperture plane consists of a screen that is opaque except for a semicircular (half) hole of radius R, centered on the z axis.

Solution:
The half-hole radius corresponds to the first Fresnel zone. If it were a full zone (360° circumference), the optical disturbance U_1 from the first zone would be twice the unobstructed value U_0; however, since half (180°) of the zone is blocked, the optical disturbance is equal only to U_0. Therefore

$$\frac{I_{\text{aperture}}}{I_{\text{open}}} = 1 \ .$$

(**b.**) (20 points)
Here the aperture plane consists of a semicircular (half) penny of radius R, centered on the z axis; the rest is open.

Solution:
Half of the azimuth ("I") is open, while the other half ("II") is open except that its first Fresnel half-zone is blocked. As discussed in class, the magnitude of the optical disturbance from an infinite sum of adjacent zones is independent of the radius at which the first zone is defined to begin. (This argument was used to explain the bright point of white light on the axis downstream of a penny). Therefore

$$|U_I| = |U_{II}| \ .$$

What is the relative phase of U_I *vs.* U_{II}? Since each zone is 180° out of phase with the next, and "I" begins with zone 1 while "II" begins with zone 2, U_I and U_{II} are 180° out of phase with each other. Therefore

$$U_I = -U_{II}$$
$$\frac{I_{\text{aperture}}}{I_{\text{open}}} = \left|\frac{U_I + U_{II}}{U_0}\right|^2 = 0 \ .$$

An alternative approach takes advantage of Babinet's principle for complementary screens, which requires that $U_a + U_b = U_0$ (where "a" and "b" refer to the complementary screens of parts (**a.**) and (**b.**)). From part (**a.**), $U_a = U_0$. Therefore $U_b = 0$.