University of California, Berkeley
Physics 110B Fall 2008 (*Strovink*)

SOLUTION TO EXAMINATION 2

Directions: Do all 3 problems, which have unequal weight. This is a 50-minute closed-book closed-note exam except for Griffiths and Pedrotti, a copy of anything posted on the course web site, and anything in your own handwriting (not a Xerox of someone else's writing). You may use the result of any assigned problem-set problem without proving it. Calculators, palmtops, laptops, and cellphones should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Show all your work. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (25 points)
Consider a pair of coaxial circular rings of radius b, separated by a distance $\sqrt{2}\,b$. Put the origin on their axis, halfway between the rings. Each ring carries a uniformly distributed static charge q; a static point charge $-2q$ is placed at the origin.

In most directions, at distance $r \gg b$ from the origin, the static electric field is found to vary approximately as r^{-n}. What is n, and why?
Solution:
Take $\hat{z}$ to be the ring axis. Since the charge distribution is cylindrically symmetric about that axis, all $m \neq 0$ moments vanish. In the usual spherical polar coordinates, the $m = 0$ moments are proportional to the integral of $P_l(\cos\theta)$ over the charge distribution. $P_l(\cos\theta)$ is odd in $\cos\theta$ when l is odd, while the charge distribution is even. Therefore all odd-l $m = 0$ moments vanish. Of course the monopole moment also vanishes in the absence of net charge.

The $m = 0$ quadrupole moment q_{20} is proportional to

$$\begin{aligned} q_{20} &\propto \int d\tau\, \rho\, r^2 P_2(\cos\theta) \\ &\propto \int d\tau\, \rho\, r^2 (3\cos^2\theta - 1) \\ &= \int d\tau\, \rho\, (3z^2 - r^2)\ . \end{aligned}$$

However, the rings of charge lie at $z = \pm\sqrt{\frac{1}{2}}b$ and at $r = \sqrt{b^2 + z^2} = \sqrt{\frac{3}{2}}b$, so the integrand and q_{20} vanish. Thus the lowest nonvanishing multipole is the $m = 0$ hexadecapole moment q_{40}, for which $V \propto r^{-5}$ and

$$|\vec{E}| \propto r^{-6}\ .$$

Problem 2. (35 points)
The irradiance I_0 of a mystery light beam is attenuated by each of four devices, applied one at a time: (A) a grey filter passing half the incident irradiance; (B) an $\hat{x}$ polarizer; (C) a $\frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ polarizer; and (D) a device consisting of a quarter-wave plate (QWP) with slow axis at $+45°$ to $\hat{x}$, followed by an $\hat{x}$ polarizer, followed by a QWP with slow axis at $-45°$ to $\hat{x}$. The attenuated irradiances observed are, respectively,

$$\begin{aligned} I_A &= \tfrac{1}{2} I_0 \\ I_B &= \tfrac{1}{2}\big(1 + \tfrac{1}{\sqrt{2}}\big) I_0 \\ I_C &= \tfrac{1}{2} I_0 \\ I_D &= \tfrac{1}{2}\big(1 + \tfrac{1}{\sqrt{2}}\big) I_0\ . \end{aligned}$$

With devices (A) through (D) no longer in the picture, device (E) is inserted into the beam. It is the same as device (D) except that the $\hat{x}$ polarizer is rotated to become a $\hat{y}$ polarizer. What irradiance I_E is observed?
Solution:
For devices (D) and (E), the wave plate upstream of the linear polarizer cannot reduce I_0, the irradiance incident upon the polarizer (wave plates have $\det(M_W) \equiv 1$). In general, irrespective of the degree of polarization,

$$I_0 \propto \langle E_x^2 \rangle + \langle E_y^2 \rangle\ ,$$

where $E_{x,y}$ are the plane wave's physical field components and $\langle\rangle$ denotes the time average. The ideal $\hat{x}$ polarizer that is part of device (D) admits all of $\langle E_x^2\rangle$ that is incident upon it, while absorbing all of $\langle E_y^2\rangle$. Conversely, the ideal $\hat{y}$ polarizer that is part of device (E) admits all of $\langle E_y^2\rangle$ while absorbing all of $\langle E_x^2\rangle$. Therefore the sum of the irradiances admitted by devices (D) and (E) is simply the total irradiance I_0:

$$\begin{aligned} I_D + I_E &= I_0 \\ I_E &= I_0 - \tfrac{1}{2}\big(1 + \tfrac{1}{\sqrt{2}}\big) I_0 \\ &= \tfrac{1}{2}\big(1 - \tfrac{1}{\sqrt{2}}\big) I_0 \ . \end{aligned}$$

With more effort, this problem alternatively can be solved by using the given $I_{A,B,C,D}$ to obtain the beam's Stokes vector $\mathcal{S}$. In one approach, the Mueller matrix (appendix to Problem 59) for a left-hand circular analyzer (device (E)) can be applied to $\mathcal{S}$; the first element of the resulting $\mathcal{S}'$ is I_E. In an even more tedious approach, the result of Problem 58 can be used to demonstrate that the beam is fully polarized, and the result of Problem 57, with some algebra, can be used to deduce the elements α and β of the beam's Jones vector J, within an overall phase. Finally, the Jones matrix for a left-hand circular analyzer (Problem 56) can be applied to J to yield J'; the ratio I_E/I_0 is equal to $(J')^\dagger J'/J^\dagger J$.

Problem 3. (40 points)
A monochromatic beam traveling in medium "0" is normally incident upon a substrate *"T"*. Two films ("1"and "2") are interposed between the two media, such that film 1 adjoins medium 0 and film 2 adjoins medium T. The refractive indices are frequency-independent and equal, respectively, to n_0, n_1, n_2, and n_T, with $n_0 \neq n_T$. You may assume that all materials are insulating and nonabsorbing, and that they all have the same magnetic permeability. Film 2 has thickness $\lambda_2/4$ (where λ_2 is the wavelength of the beam in the particular material of which film 2 is made). The thickness l_1 of film 1 is *nonzero*.

(a.) (20 points)
In this part, the refractive index n_1 of film 1 is not supplied by the problem, but its thickness is specified to be equal to $\lambda_1/4$ (where λ_1 is the wavelength of the beam in the particular material of which film 1 is made). Identify a combination $\{n_1, n_2\}$ for which the full beam irradiance is transmitted without any reflection. (Any correct answer, properly justified, receives full credit.)
Solution:
If film 2 is given the same index

$$n_2 = n_T$$

as the substrate, it may be considered to be part of the substrate. The remaining single film 1 is $\frac{1}{4}$ wavelength thick; to yield zero reflectance it should have

$$n_1 = \sqrt{n_0 n_T}$$

as in Problem 63(a).

(b.) (20 points)
In this part, the thickness l_1 of film 1 is not supplied by the problem, but its refractive index n_1 is specified to be equal to n_T. Identify a combination $\{l_1, n_2\}$ for which the full beam irradiance is transmitted without any reflection. (Any correct answer, properly justified, receives full credit.)
Solution:
The right-to-left transfer matrix (Pedrotti[2] Eq. (19-24)) corresponding to a single film is proportional to the unit matrix if the film thickness is chosen to be $\frac{1}{2}$ wavelength: adding such a film has no effect on the reflectance of any film stack. Therefore, if

$$\begin{aligned} l_1 &= \tfrac{\lambda_1}{2} \\ n_2 &= \sqrt{n_0 n_T} \ , \end{aligned}$$

the reflectance vanishes as in part (**a.**).