University of California, Berkeley
Physics 110B Fall 2008 (*Strovink*)

SOLUTION TO EXAMINATION 1

Directions: Do all 3 problems, which have unequal weight. This is a 50-minute closed-book closed-note exam except for Griffiths, a copy of anything posted on the course web site, and anything in your own handwriting (not a Xerox of someone else's writing). Calculators, palmtops, laptops, and cellphones should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Show all your work. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer. *Without supplying additional proof, you may use the result of any assigned problem-set problem.*

Problem 1. (30 points)
A laser pointer has initial mass m_0. With a *very* advanced design, this laser pointer is able to emit a blast of collinear photons whose energy is provided by a reduction of $0.01m_0$ in its rest mass (no other energy is emitted). (Note that up to 99 blasts are possible before its rest mass is exhausted).

Approximately how many blasts will accelerate the laser pointer to 96% of the speed of light?
Solution:
Applying the rocket equation

$$\eta_f - \eta_0 = \beta_1 \ln \frac{m_0}{m_f}$$

with $\beta_1 = 1$ and $\eta_0 = 0$, and taking $m_0/m_f \equiv R$,

$$\begin{aligned}
\beta_f &= \tanh \eta_f \\
0.96 &= \tanh (\ln R) \\
&= \frac{\exp (\ln R) - \exp (-\ln R)}{\exp (\ln R) + \exp (-\ln R)} \\
&= \frac{R - 1/R}{R + 1/R} \quad . \\
0.96(R + 1/R) &= R - 1/R \\
1.96/R &= 0.04R \\
49 &= R^2 \\
7 &= R
\end{aligned}$$

The laser pointer rest mass needs to be reduced by a factor of 7, to $\approx 14\%$ of its original mass. This requires 86 blasts.

Problem 2. (35 points)
A dielectric with permeability μ_0 and dielectric constant ϵ fills all space. It is uniform, except that the region $z > 0$ is infused with a uniform low density of free electrons whose sole effect is to create a nonzero ohmic conductivity σ. This conductivity is small in the sense that, for frequencies ω of interest here, $\sigma/(\epsilon\omega) \ll 1$.

A plane-wave packet of monochromatic EM radiation, directed along $\hat{z}$, impinges on the interface between the two media. For $z < 0$, as usual the incident $\vec{E}$ and $\vec{B}$ are in phase – but for $z > 0$, the transmitted $\vec{B}$ is observed to be out of phase with the transmitted $\vec{E}$ by a small phase shift δ (its sign is unimportant here).

In terms of δ, what fraction of the wave packet's energy is reflected by the interface? (*Hint: use Faraday's law to solve for* $\sigma/(\epsilon\omega)$ *in terms of* δ.)
Solution:
Expressed in terms of complex fields $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$, Faraday's law,

$$i\tilde{\mathbf{k}} \times \tilde{\mathbf{E}}_0 = i\omega\tilde{\mathbf{B}}_0 \ ,$$

reveals that the phase shift between $\vec{B}$ and $\vec{E}$ is simply the argument of $\tilde{\mathbf{k}}$. According *e.g.* to Griffiths Eq. (9.125-9.126),

$$\arg(\tilde{\mathbf{k}}) = \Big(\frac{\sqrt{1+\gamma^2} - 1}{\sqrt{1+\gamma^2} + 1}\Big)^{1/2} \ ,$$

where $\gamma \equiv \sigma/(\epsilon\omega) \ll 1$. To lowest nonvanishing order in γ, a Taylor expansion of this quotient yields

$$\arg(\tilde{\mathbf{k}}) \approx \gamma/2 \ .$$

Therefore $\gamma \approx 2\delta$.

The *amplitude* reflection coefficient is

$$\tilde{r} = \frac{\tilde{Z}_1^{-1} - \tilde{Z}_2^{-1}}{\tilde{Z}_1^{-1} + \tilde{Z}_2^{-1}} \; ,$$

where the complex admittance $\tilde{Z}^{-1}$ is proportional to $\tilde{k}$ with the same constant of proportionality in both regions. Again, by Taylor expansion,

$$\tilde{r} \approx -\frac{i\gamma}{4} \; .$$

(This is the result of Problem (**-3.**), which you may use without additional proof.)

Therefore the energy or irradiance reflection coefficient is

$$R = |\tilde{r}|^2 \approx \frac{\gamma^2}{16} \approx \frac{\delta^2}{4} \; .$$

Problem 3. (35 points)
In quark-antiquark ($q\bar{q}$) annihilation, two photons are produced. In the $q\bar{q}$ center of mass $\mathcal{S}'$, the q ($\bar{q}$) arrives along $\hat{z}'$ (-$\hat{z}'$) and the photons γ_1 and γ_2 leave at polar angles $\theta_1' = 60°$ and $\theta_2' = 120°$, measured with respect to the z' axis.

Unfortunately, the q or $\bar{q}$ are not beam particles, but rather constituents of protons (p) or antiprotons ($\bar{p}$). Because the q and $\bar{q}$ carry varying fractions of the p and $\bar{p}$ momentum, the $q\bar{q}$ CM system $\mathcal{S}'$ is boosted along $\hat{z}' = \hat{z}$ with respect to the $p\bar{p}$ (lab) system $\mathcal{S}$.

In the lab, the second photon γ_2 is observed to emerge perpendicular to $\hat{z}' = \hat{z}$: $\theta_2 = \pi/2$. At what lab polar angle θ_1 does the first photon emerge? (It is immaterial whether you supply θ_1 or $180° - \theta_1$; you may leave trigonometric functions unevaluated.)
Solution:
See Problem (**20.**). Because the longitudinal rapidity y (the boost along z) is the additive parameter of a Lorentz transformation along z, the rapidity difference Δy between the two photons is the same in the lab as it is in the CM:

$$\Delta y \equiv y_1 - y_2 = y_1' - y_2' \; .$$

Furthermore, for a particle like the photon that travels at the speed of light, the rapidity and pseudorapidity are the same:

$$y = -\ln\left(\tan\tfrac{\theta}{2}\right) \; .$$

In the CM,

$$\begin{aligned} y_1' = -y_2' &= -\ln\left(\tan\tfrac{60°}{2}\right) \\ &= -\ln\left(1/\sqrt{3}\right) \\ \Delta y &= -2\ln\left(1/\sqrt{3}\right) = \ln 3 \; . \end{aligned}$$

In the lab, $y_2 = 0$ because the second photon emerges perpendicular to the beam direction. Therefore

$$\begin{aligned} \ln 3 &= y_1 \\ &= -\ln\left(\tan\tfrac{\theta_1}{2}\right) \\ \tfrac{1}{3} &= \tan\tfrac{\theta_1}{2} \\ 2\arctan\tfrac{1}{3} &= \theta_1 = 36.9° \; . \end{aligned}$$