Physics 105: Classical Mechanics, Sec. 2 (Strovink)
Final Review Session Solutions
by Peter Battaglino

Problem 1: Yukawa Force Orbit: A particle of mass m moves in a circle of radius R under the influence of a central, attractive force

$$F = -\frac{K}{r^2}e^{-r/a},$$

where K and a are positive constants. (a) Determine the conditions on the constant a such that the circular orbit is stable. (b) Compute the frequency of small radial oscillations about this circular orbit.

<u>Solution:</u> (a) The condition for a stable circular orbit of radius R is that the effective potential (as a function of r) has a minimum at R. The effective potential for a central force problem has the form

$$U_{\text{eff}}(r) = \frac{\ell^2}{2mr^2} + U(r),$$

where ℓ is the conserved angular momentum conjugate to the polar orbit angle θ, and $U(r)$ is the potential energy. You can derive this effective potential by eliminating $\dot{\theta}$ from the Lagrangian in favor of ℓ and then using the transformation $\mathcal{H} = p_r\dot{r} + \ell\dot{\theta} - \mathcal{L}$ and reading off the terms in $\mathcal{H}$ that only depend on the coordinates. We can then express the conditions for stable circular orbits as

$$\left(\frac{\partial U_{\text{eff}}}{\partial r}\right)_{r=R} = 0 \qquad \text{and} \qquad \left(\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right)_{r=R} > 0.$$

The first derivative of the effective potential is

$$\frac{\partial U_{\text{eff}}}{\partial r} = -\frac{\ell^2}{mr^3} + \frac{\partial U}{\partial r} = -\frac{\ell^2}{mr^3} - F(r),$$

where we used the fact that $F(r) = -\partial U/\partial r$. Evaluating this at $r = R$ and moving things around gives us the condition that

$$\frac{\ell^2}{m} = KRe^{-R/a}.$$

This is just the condition that the orbit be a circle. For stability, we need the second derivative, which is

$$\frac{\partial^2 U_{\text{eff}}}{\partial r^2} = \frac{1}{r^4}\left[\frac{3\ell^2}{m} - \left(2Kr + \frac{Kr^2}{a}\right)e^{-r/a}\right].$$

Evaluating at $r = R$, using the relation found from the first condition, and enforcing that it be positive gives us the result

$$\frac{1}{R^4}\left[\frac{3\ell^2}{m} - \left(\frac{2\ell^2}{m} + \frac{R}{a}\frac{\ell^2}{m}\right)\right] > 0,$$

which reduces to the constraint that $R < a$ for stability.
(b) Letting $r(t) = R + \epsilon(t)$, where $\epsilon(t)$ has arbitrarily small amplitude, we can Taylor expand the equation of motion for r and keep only first order terms in ϵ. Such a calculation will always yield the result

$$m\ddot{\epsilon} + \left(\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right)_{\epsilon=0} \epsilon = 0.$$

Thus, our angular frequency is going to be equal to

$$\omega = \sqrt{\frac{1}{m}\left(\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right)_{r=R}} = \frac{\ell}{mR^2}\left(1 - \frac{R}{a}\right)^{1/2}.$$

Problem 2: Rotating Door: A uniform rectangular door of mass m with sides a and b ($b > a$) and negligble thickness rotates with constant angular velocity ω about a diagonal. Show that the torque

$$|\vec{\Gamma}| = \frac{m(b^2 - a^2)ab\omega^2}{12(a^2 + b^2)}$$

must be applied to keep the axis of rotation fixed.

<u>Solution:</u> Let's write down Euler's equations:

$$\begin{aligned}
\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 &= \Gamma_1 \\
\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 &= \Gamma_2 \\
\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 &= \Gamma_3.
\end{aligned}$$

Since we're holding the axis of the door's rotation fixed, the angular velocity will remain constant. Thus, $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. In addition, the angular velocity will always point along the axis of rotation, in the plane of the door. Letting $\hat{e}_1$ and $\hat{e}_2$ define the plane of the door, we have $\omega_3 = 0$ for all time. Thus, we get from Euler's equations that

$$\begin{aligned}
\Gamma_1 &= 0 \\
\Gamma_2 &= 0 \\
\Gamma_3 &= (\lambda_2 - \lambda_1)\omega_1\omega_2.
\end{aligned}$$

The components ω_1 and ω_2 of the angular velocity are just the projections onto the $\hat{e}_1$ and $\hat{e}_2$ basis vectors, respectively, which are

$$\omega_1 = \omega\frac{b}{\sqrt{a^2 + b^2}} \qquad \text{and} \qquad \omega_2 = \omega\frac{a}{\sqrt{a^2 + b^2}}.$$

In your tenth homework assignment, you calculated the principal moments λ_1 and λ_2. They are equal to $ma^2/12$ and $mb^2/12$, respectively. Putting it all together gives

$$\Gamma_3 = \frac{m(b^2 - a^2)\omega^2 ab}{12(a^2 + b^2)}.$$

Since the other two components of $\vec{\Gamma}$ are zero, we have $|\vec{\Gamma}| = \Gamma_3$, so we are finished.

Problem 3: Rolling Plate: A plate idealized as a uniform disk of radius a with negligible thickness and mass m rolls in a circle (imagine spinning a plate on a table and watching it wobble, but without any energy dissipation). The center of mass of the plate C moves in a circle of radius b and the axis of the plate is tilted at an angle θ with respect to the vertical. Find the angular velocity Ω of the center of mass of the plate.

<u>Solution:</u> We will solve this problem by computing the torque on the plate exerted by the table in two ways. One way is to calculate the force acting on the plate at its point of contact and using the formula

$$\vec{\Gamma} = \vec{r} \times \vec{F}.$$

Another way is to switch to a frame rotating with angular velocity Ω and use the fact that the plate's center of mass is fixed, yielding

$$\vec{\Gamma} = \left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{space}} = \left(\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right)_{\text{rot}} + \vec{\Omega} \times \vec{L} = \vec{\Omega} \times \vec{L}.$$

This is because in the rotating frame we've chosen, the angular momentum is constant. For the first method, we need the forces acting on the plate's point of contact with the table. One force is the normal force opposite the gravitational force, equal to $mg\hat{z}$. The other is the friction force opposing the centrifugal force caused by rotation of the center of mass with frequency Ω. This force is equal to $-m\Omega^2 b\hat{y}$. The vector pointing from the plate's center of mass to the point of contact is equal to $\vec{r} = a\sin\theta\hat{y} - a\cos\theta\hat{z}$. The torque will then be

$$\vec{\Gamma} = \vec{r} \times \vec{F} = (mga\sin\theta - \Omega^2 b\cos\theta)\hat{x}.$$

For the second method, we need the vectors $\vec{\Omega}$ and $\vec{L}$. The vector $\vec{\Omega}$ is given by

$$\vec{\Omega} = -\Omega\cos\theta\hat{e}_1 - \Omega\sin\theta\hat{e}_3.$$

To find $\vec{L}$, we need to know the inertia tensor of the plate and its total angular velocity. To find its inertia tensor, we remember that for a thin lamina, $I_3 = I_1 + I_2$, and that for a rigid body with an azimuthal symmetry we have $I_1 = I_2$ (taking the $\hat{e}_3$ axis to be the azimuthal axis). The value of I_3 is given by

$$I_3 = \frac{m}{\pi a^2}\int_0^a r^2 2\pi r \mathrm{d}r = \frac{1}{2}ma^2,$$

so we have the values

$$I_1 = I_2 = \frac{1}{4}ma^2.$$

The total angular velocity is the sum of the angular velocity in the rotating frame, $\omega\hat{e}_3$, and the angular velocity of that frame with respect to the space frame, $\vec{\Omega}$:

$$\vec{\omega}_{\text{tot}} = \vec{\Omega} + \omega\hat{e}_3 = -\Omega\cos\theta\hat{e}_1 + (\omega - \Omega\sin\theta)\hat{e}_3.$$

Now we can use the fact that $\vec{L} = I \cdot \vec{\omega}_{\text{tot}}$ to get

$$\vec{L} = -\frac{1}{4}ma^2\Omega\cos\theta\hat{e}_1 + \frac{1}{2}ma^2(\omega - \Omega\sin\theta)\hat{e}_3.$$

Then the cross product $\vec{\Omega} \times \vec{L}$ is equal to

$$\vec{\Gamma} = \vec{\Omega} \times \vec{L} = ma^2\Omega^2\left[\frac{1}{4}\sin\theta\cos\theta + \frac{1}{2}\cos\theta\left(\frac{\omega}{\Omega} - \sin\theta\right)\right]\hat{e}_2.$$

We can simplify this further by using the fact that the plate is rolling without slip, given by the condition

$$\omega a = \Omega(b + a\sin\theta),$$

or

$$\frac{\omega}{\Omega} = \frac{b}{a} + \sin\theta.$$

This reduces our expression for the torque to

$$\vec{\Gamma} = ma^2\Omega^2\left[\frac{1}{4}\sin\theta\cos\theta + \frac{1}{2}\frac{b}{a}\cos\theta\right]\hat{e}_2.$$

Comparing this to the first equation for the torque and realizing that instantaneously, the $\hat{e}_2$ and $\hat{y}$ axes are aligned, we can solve for Ω^2 as

$$\Omega^2 = \frac{4g\tan\theta}{a\sin\theta + 6b}.$$

Problem 4: Coupled Oscillators in a Circle: Three pointlike objects of mass m are distributed equidistant from one another on a circle of radius 1 (in appropriate units). Between each pair of masses is a spring of constant k that lies along the circle. The masses are constrained to move on the circle. Find the normal frequencies and modes of oscillation. One of these modes is actually not oscillatory. Is there a way you could have simplified the problem to ignore this mode from the start and only diagonalize a 2×2 matrix?

<u>Solution:</u> We need to write down the $\mathcal{K}$ and $\mathcal{M}$ matrices for the system. We can do this by looking at the coefficients of quadratic terms in the potential and kinetic energies. First we

need some generalized coordinates. Choose θ_1, θ_2, and θ_3 to be the angular displacements from equilibrium of each mass. Then

$$\begin{aligned} T &= \frac{m}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \\ U &= \frac{k}{2}\left((\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 + (\theta_1 - \theta_3)^2\right) \\ &= \frac{k}{2}(2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3 - 2\theta_3\theta_1). \end{aligned}$$

Now we can readily write down the $\mathcal{K}$ and $\mathcal{M}$ matrices as

$$\begin{aligned} \mathcal{M} &= m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathcal{K} &= k \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \end{aligned}$$

We must now solve the eigenvalue problem $\det(\mathcal{K} - \omega^2\mathcal{M}) = 0$. We can write the matrix $\mathcal{K} - \omega^2\mathcal{M}$ as

$$\mathcal{K} - \omega^2\mathcal{M} = k \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix},$$

where $\lambda = \omega^2 m/k$. Now we must solve for the roots of

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0.$$

We can expand the determinant and factor it as

$$\lambda(\lambda - 3)^2 = 0.$$

This implies eigenfrequencies

$$\omega_1 = 0 \qquad \text{and} \qquad \omega_2 = \omega_3 = \sqrt{\frac{3k}{m}}.$$

Thus, we have a zero mode and two degenerate modes. The zero mode should correspond to rotations of the circle. Plugging in the value $\lambda = 0$ gives the equation

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0,$$

which has the solution $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$. Thus we see that this is indeed a mode in which all the masses rotate in unison around the circle. Plugging in $\lambda = 3$ gives the equation

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = 0,$$

which has the solution $\beta_1 + \beta_2 + \beta_3 = 0$. We must now pick two normal modes which are orthogonal with respect to the inner product $\mathcal{M} \propto I$ and satisfy this condition. Although these choices are not unique, one mode is $(0, 1, -1)$. The second choice can then be $(2, -1, -1)$. You can check that these are orthogonal and both satisfy the condition $\beta_1 + \beta_2 + \beta_3 = 0$.

In many situations, we will be able to pick out zero modes before we actually start to solve the eigenvalue problem. In such a case, it may be advantageous to use our knowledge to introduce a constraint among the generalized coordinates. In the case of this problem, the zero mode corresponds to a constant, non-zero angular momentum. We can eliminate this zero mode from our problem by restricting the angular momentum to be zero. This gives us the condition $\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 = 0$, which we can integrate to $\theta_1 + \theta_2 + \theta_3 = 0$. This allows us to express, say, θ_3 in terms of the coordinates θ_1 and θ_2. We can then insert this constraint into our Lagrangian to reduce the original 3×3 eigenvalue problem to a much simpler 2×2 eigenvalue problem. The normal modes can then be computed as before, and we can just use the equation $\theta_3 = -(\theta_1 + \theta_2)$ to return to a description involving all three coordinates.