University of California, Berkeley
Physics 105 Fall 2006 Section 2 (*Strovink*)

ASSIGNMENT 4

Reading:
SCCM 3.1-3.4
Taylor 5.1-5.8

1. Consider a rigid body whose orientation, relative to a fixed pivot point, is parametrized by the Euler angles $\{\varphi, \theta, \psi\}$. If the body is cylindrically symmetric around the same axis about which ψ is measured, its kinetic energy is

$$T = \tfrac{1}{2} I(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2} I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2 \ ,$$

where I_3 is the body's moment of inertia about its symmetry axis, and I is its moment about any axis $\perp$ to the symmetry axis and passing though the pivot point. [This follows from PS1 Problem 7(b).] Under the influence of gravity, the body's potential energy is $mgh\cos\theta$, where h is the distance from the pivot point to the body's CM.

(**a.**)
Using the Euler-Lagrange equations, find the constants of the motion that correspond to the two cyclic coordinates.

(**b.**)
Construct the Hamiltonian $\mathcal{H} \equiv \dot{q}_i\, \partial\mathcal{L}/\partial\dot{q}_i - \mathcal{L}$ and use it to find a third constant of the motion.

2. Taylor Problem 5.13.

3. Consider a mass m attached to a spring of relaxed length x_s, which in turn is attached to a wall at coordinate x_1 that undergoes cosinusoidal motion $x_1(t) = x_0 \cos\omega t$. The mass slides on a frictionless surface.

(**a.**)
Taking the generalized coordinate to be the position $x(t)$ of the mass, use the Euler-Lagrange equation to obtain the differential equation of motion.

(**b.**)
Construct the Hamiltonian $\mathcal{H}(x, \dot{x}, t)$ for this system. Is $\mathcal{H}$ equal to the total energy of the mass + spring system? Is $\mathcal{H}$ constant? Use the third Hamilton equation

$$\frac{d\mathcal{H}}{dt} = -\frac{\partial\mathcal{L}}{\partial t}$$

to obtain the same differential equation of motion.

4. Taylor Problem 5.45.

5. A damped driven oscillator is characterized by the quality factor $Q = \omega_0/\gamma$, where $-\gamma m\dot{x}$ is the damping force and $m\omega_0^2$ is the spring constant. It is driven at resonance ($\omega = \omega_0$).

For any value of Q, prove that the ratio of the total energy E stored in the oscillator to the work W done in one period by the driving force is equal to $Q/2\pi$.

6. An underdamped ($Q = \frac{1}{\sqrt{2}}$) oscillator, initially motionless at the origin, is driven at resonance by a sinusoidal force that begins at $t = 0$. Within a constant factor, solve for $x(t > 0)$.

7. An overdamped ($Q = \frac{6}{13}$) oscillator, initially motionless at the origin, is driven at resonance by a cosinusoidal force that begins at $t = 0$. Within a constant factor, solve for $x(t > 0)$.