University of California, Berkeley
Physics 105 Fall 2006 Section 2 (*Strovink*)

ASSIGNMENT 11

Reading:
SCCM 14.4-14.6, 15.1
Taylor 16.4, 12.1-12.8

1. Taylor Problem 16.12.

2. Taylor Problem 16.13.

3. Consider a uniform cube of side L. Inside the cube is a scalar field ϕ that satisfies the wave equation with characteristic wavespeed c. At the surfaces of the cube, ϕ is required to vanish.
(**a**)
Show that for this system the total number of modes of vibration corresponding to frequencies between ν and $\nu + d\nu$ is $4\pi L^3\nu^2 d\nu/c^3$, if $\pi c/L \ll d\nu \ll \nu$.
(**b**)
What would the result be for a (two-dimensional) square?
(**c**)
A (one-dimensional) rod?

4. Taylor Problem 12.16.

5. & 6. If a pendulum is started in motion, it is possible for its amplitude of oscillation to be increased by a *bootstrap* method in which the pendulum's own length $l(t)$ is varied periodically.

(**a.**)
Taking θ and l as generalized coordinates, write the Lagrangian. Applying the Euler-Lagrange equation in θ, show that the equation of motion (EOM) is

$$\frac{\mathrm{d}}{\mathrm{d}t}(l^2\dot{\theta}) + gl\theta = 0 \ .$$

(**b.**)
Substitute

$$l(t) \equiv \bar{l}\big(1 + \delta(t)\big) \ ,$$

where $\bar{l}$ is an average value and $\delta(t)$ averages to zero. Scale to dimensionless time $\tau \equiv \omega_0 t$, where $\omega_0^2 \equiv g/\bar{l}$, and show that the EOM becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big((1+\delta)^2\dot{\theta}\big) + (1+\delta)\theta = 0 \ .$$

(**c.**)
Defining $\phi \equiv \theta l$, show that the EOM becomes

$$\ddot{\phi} + \Big(\frac{1-\ddot{\delta}}{1+\delta}\Big)\phi = 0 \ .$$

(**d.**)
Taylor expanding $1/(1+\delta)$ and retaining the lowest-order terms in δ and $\ddot{\delta}$, show that the EOM becomes

$$\ddot{\phi} + \phi = (\delta + \ddot{\delta})\phi \ .$$

(**e.**)
Now tune the frequency of δ, the perturbation on l, to the *parametric resonance frequency* $2\omega_0$:

$$\delta(\tau) = \delta_0 \cos 2\tau \ .$$

Write ϕ as the sum of a constant-l solution with amplitude A and a perturbation term that is of order δ_0:

$$\phi \equiv Al(0)\cos\tau + \mathcal{O}(\delta_0) \ .$$

Using the trig identity

$$\cos\tau\cos 2\tau = \tfrac{1}{2}(\cos\tau + \cos 3\tau) \ ,$$

show that, to first order in δ_0, the EOM becomes

$$\ddot{\phi} + \phi = -\tfrac{3}{2}\delta_0 Al(0)(\cos\tau + \cos 3\tau) \ .$$

(**f.**)
The homogeneous solution to this equation is

$$\phi_h = B\sin\tau + C\cos\tau \ ,$$

where the constants B and C are (eventually) fixed by the initial conditions. The particular solution to the same equation is the sum of ϕ_1 and ϕ_3, where

$$\ddot{\phi}_1 + \phi_1 = -\tfrac{3}{2}\delta_0 A l(0)\cos\tau$$
$$\ddot{\phi}_3 + \phi_3 = -\tfrac{3}{2}\delta_0 A l(0)\cos 3\tau\ .$$

First solve the equation for ϕ_3, by substituting $\phi_3 = D\cos 3\tau$, and evaluate D. Then solve the equation for ϕ_1. Note that a similar solution $\phi_1 = E\cos\tau$ will fail because the left-hand side will vanish. Instead, using clairvoyance, attempt a solution of the form $\phi_1 = F\tau\sin\tau$ and evaluate F.

(g.)
Show that the general solution is

$$\phi = B\sin\tau + C\cos\tau + \delta_0 A l(0)\left(\tfrac{3}{16}\cos 3\tau - \tfrac{3}{4}\tau\sin\tau\right)\ .$$

(h.)
Finally, adjust B and C to satisfy the initial conditions

$$\phi(0) = A l(0)$$
$$\dot{\phi}(0) = 0\ .$$

You should obtain the complete solution

$$\phi(\tau) = A l(0)\left(1 - \tfrac{3}{16}\delta_0\right)\cos\tau + \tfrac{3}{16}A l(0)\delta_0\cos 3\tau - \tfrac{1}{4}A l(0)\delta_0\tau\sin\tau\ .$$

Indeed the magnitude of the coefficient of $\sin\tau$ in the last term increases linearly with time.

This is one model of how a child pumps a swing – without realizing, of course, that a parametric resonance is excited.

7. Consider a double pendulum with equal lengths and masses (Taylor Fig. 11.10). Using ϕ_1 and $\phi_3 \equiv \phi_2 - \phi_1$ as generalized coordinates, making *no* approximations,

(a.)
Write the Lagrangian.

(b.)
Write the Hamiltonian.

(c.)
Define the canonical momenta l_1 and l_3.

(d.)
Reëxpress the Hamiltonian in terms of ϕ_1, ϕ_3, l_1, and l_3.

(e.)
Write out all four of Hamilton's equations. (Because of their first-order character, these equations are the preferred starting point for numerical solution of the equations of motion for this simple but nevertheless nonlinear and chaotic system.)