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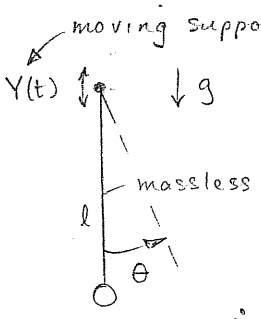
(105 Sec. 2 lecture notes)

Inverted pendulum

Notation: Hand & Finch p. 388 Fig 10.2



$$x = l\sin\theta$$
$$\dot{x} = l\cos\theta\,\dot{\theta}$$
$$y = Y(t) + l(1-\cos\theta)$$
(wrt bottom)
$$\dot{y} = \dot{Y}(t) + l\sin\theta\,\dot{\theta}$$

$$\therefore \frac{2T}{m} = l^2\dot{\theta}^2 + 2l\sin\theta\,\dot{\theta}\dot{Y} + \dot{Y}^2$$

$$U = mgy = mg\left[Y + l(1-\cos\theta)\right]$$

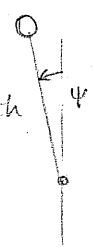
$$\frac{2U}{m} = 2\omega_0^2 l\left[Y + l(1-\cos\theta)\right]$$

where $\omega_0^2 = g/l$

So far this parallels Hand & Finch, p. 396.

We will be interested in the inverted case, with θ near π, so $(\theta-\pi)$ will be small. Define

$$\psi = \theta - \pi$$



Rotation by two quadrants, both cos and sin change sign:

$$\frac{2T}{m} = l^2\dot{\psi}^2 - 2l\sin\psi\,\dot{\psi}\dot{Y} + \dot{Y}^2$$

$$\frac{2U}{m} = 2\omega_0^2 l\left[Y + l(1+\cos\psi)\right]$$

E-L equation

$$\left\{\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{\partial\mathcal{L}}{\partial\psi}\right\} \times \frac{2}{m}$$

$$\frac{d}{dt}\left[2l^{\cancel{2}}\dot{\psi} - 2l\sin\psi\,\dot{Y}\right] = +2\omega_0^2 l^{\cancel{2}}\sin\psi - 2l\cos\psi\,\dot{\psi}\dot{Y}$$

(2(−) signs including $\mathcal{L} = T\ominus U$)

$$l\ddot{\psi} - \cancel{\dot{Y}\cos\psi\,\dot{\psi}} - \sin\psi\,\ddot{Y} - \omega_0^2 l\sin\psi + \cancel{\dot{Y}\cos\psi\,\dot{\psi}} = 0$$

If (following Hand & Finch) we plug in

$$Y(t) \equiv Y_0\cos\Omega t$$

$$a \equiv \left(\frac{2\omega_0}{\Omega}\right)^2 \qquad q \equiv \frac{2Y_0}{l}$$

we get

$$l\ddot{\psi} - \sin\psi\left(l\omega_0^2 - Y_0\Omega^2\cos\Omega t\right) = 0$$

$$\frac{\ddot{\psi}}{\Omega^2} - \sin\psi\left(\frac{\omega_0^2}{\Omega^2} - \frac{Y_0}{l}\cos\Omega t\right) = 0$$

$$\frac{4\ddot{\psi}}{\Omega^2} - \sin\psi\left(a - 2q\cos\Omega t\right) = 0$$

Still following Hand & Finch, we substitute $\tau = \frac{\Omega t}{2}$ so that

$$\frac{d^2}{d\tau^2} = \frac{4}{\Omega^2}\frac{d^2}{dt^2}$$

Then "Mathieu Eq$^{\underline{n}}$"

$$\boxed{\frac{d^2\psi}{d\tau^2} \ominus (a - 2q\cos 2\tau)\sin\psi = 0} \quad (1)$$

This is the same as Hand & Finch Eq. (10.36), except for $\ominus$ from $\theta \to \psi$. Note that NO approximations have been made so far!

We now depart from Hand & Finch, and follow the (web published) notes of Jon Juel Thomsen, a Danish engineer:

http://www.fam.web.mek.dtu.dk/html/FVP/Jon/Theory1.html

Define $s \equiv \omega_0 t - \pi/2$ (JJT's "τ")
$p \equiv q/2 = Y_0/l$ (JJT's "q")
$\gamma \equiv \Omega/\omega_0$ (JJT's "Ω")

(1) becomes

$$\boxed{\frac{d^2\psi}{ds^2} - (1 - p\gamma^2\sin\gamma s)\sin\psi = 0} \quad (2)$$

(corresponding to JJT's Eq. (1) without damping)

Split $\psi(s)$ into

$$\boxed{\psi(s) = z(s) + \gamma^{-1}\varphi(s, \gamma s)} \quad (3)$$

If we made the assumption that

$\gamma \gg 1$ ($\Omega \gg \omega_0$, "fast" vibration)

$p \ll 1$ ($Y_0 \ll l$, "small" amplitude of vibration)

then it would be true that

$z(s)$ is the "slow" component of ψ
$\varphi(s)$ is the "fast" component of ψ because it depends on γs as well as s.
Note that we multiply φ by $\gamma^{-1} \ll 1$, anticipating a smaller response to the "fast" vibration.

Substituting (3) into (2), considering s and γs to be two <u>independent variables</u>,

$$\frac{d^2\psi}{ds^2} = \frac{d^2}{ds^2}\left(z(s) + \frac{s}{\gamma s}\varphi(s,\gamma s)\right)$$
$$= \frac{d^2 z}{ds^2} + \frac{d^2}{ds^2}\left(\frac{s}{\gamma s}\varphi(s,\gamma s)\right)$$

Working on the last term, we note that

$$\frac{d}{ds} = \frac{\partial}{\partial s} + \overset{\gamma}{\cancel{\frac{d(\gamma s)}{ds}}}\frac{\partial}{\partial(\gamma s)}$$

$$\frac{\partial}{\partial s}\left(\frac{s}{\gamma s}\varphi\right) = \frac{\varphi}{\gamma s} + \frac{s}{\gamma s}\frac{\partial\varphi}{\partial s}$$

$$\gamma\frac{\partial}{\partial(\gamma s)}\left(\frac{s}{\gamma s}\varphi\right) = \frac{-\gamma s\varphi}{(\gamma s)^2} + \frac{\gamma s}{\gamma s}\frac{\partial\varphi}{\partial(\gamma s)}$$

So the first derivative of the last term is

$$\cancel{\frac{\varphi}{\gamma s}} + \frac{s}{\gamma s}\frac{\partial\varphi}{\partial s} - \cancel{\frac{\varphi}{\gamma s}} + \frac{\partial\varphi}{\partial(\gamma s)} = \frac{d}{ds}\left(\frac{s\varphi}{\gamma s}\right)$$

Now we need to take another $\frac{d}{ds}$ of the last term. Again

$$\frac{d}{ds} = \frac{\partial}{\partial s} + \gamma\frac{\partial}{\partial(\gamma s)}$$

$$\frac{\partial}{\partial s}\left(\frac{s}{\gamma s}\frac{\partial\varphi}{\partial s} + \frac{\partial\varphi}{\partial(\gamma s)}\right) =$$
$$= \cancel{\frac{1}{\gamma s}\frac{\partial\varphi}{\partial s}} + \frac{1}{\gamma}\frac{\partial^2\varphi}{\partial s^2} + \frac{\partial^2\varphi}{\partial s\,\partial(\gamma s)}$$

$$\gamma\frac{\partial}{\partial(\gamma s)}\left(\frac{s}{\gamma s}\frac{\partial\varphi}{\partial s} + \frac{\partial\varphi}{\partial(\gamma s)}\right) =$$
$$= \cancel{\frac{-\gamma s}{(\gamma s)^2}\frac{\partial\varphi}{\partial s}} + \frac{\partial^2\varphi}{\partial s\,\partial(\gamma s)} + \gamma\frac{\partial^2\varphi}{\partial(\gamma s)^2}$$

Putting it together and taking advantage of the second cancellation,

$$\frac{d^2}{ds^2}\left(\frac{s}{\gamma s}\varphi(s,\gamma s)\right) = \frac{1}{\gamma}\frac{\partial^2\varphi}{\partial s^2} + 2\frac{\partial^2\varphi}{\partial s\,\partial(\gamma s)} + \gamma\frac{\partial^2\varphi}{\partial(\gamma s)^2}$$

To simplify the notation, define

$$\dot{(\ )} \equiv \frac{\partial(\ )}{\partial s}$$
$$(\ )' \equiv \frac{\partial(\ )}{\partial(\gamma s)}$$

Then the first term in Eq. (2) is

$$\ddot{z} + \gamma^{-1}\ddot{\varphi} + 2\dot{\varphi}' + \gamma\varphi'' \quad \checkmark$$

The second term in Eq. (2) contains a $\sin\psi$, which we expand:

$$\sin\psi = \sin(z + \gamma^{-1}\varphi)$$
$$= \sin z\cos(\gamma^{-1}\varphi) + \cos z\sin(\gamma^{-1}\varphi)$$

Putting it all together,

$$\ddot{z} + \gamma^{-1}\ddot{\varphi} + 2\dot{\varphi}' + \gamma\varphi'' =$$
$$= (1 - p\gamma^2\sin\gamma s)\left(\sin z\cos(\gamma^{-1}\varphi) + \cos z\sin(\gamma^{-1}\varphi)\right)$$

Multiplying thru by γ^{-1} and isolating φ'' on the LHS,

$$\boxed{\begin{aligned}\varphi'' = &-p\gamma\sin z\sin\gamma s\cos(\gamma^{-1}\varphi)\\ &-\gamma^{-1}\Big[\ddot{z} + 2\dot{\varphi}' - \sin z\cos(\gamma^{-1}\varphi)\\ &\qquad + p\gamma^2\cos z\sin(\gamma^{-1}\varphi)\sin\gamma s\\ &\qquad + \gamma^{-1}\ddot{\varphi} - \cos z\sin(\gamma^{-1}\varphi)\Big]\end{aligned}} \quad (4)$$

Eq (4) is Eq (2) after the substitution (3).

We now make approximations. Assuming $\gamma \gg 1$ ("fast" vibration) and $\varphi = \mathcal{O}(1)$,

$$\sin(\gamma^{-1}\varphi) \approx \gamma^{-1}\varphi$$
$$\cos(\gamma^{-1}\varphi) \approx 1$$

Since $p \ll 1$ we assume that $p\gamma$ is of order unity.
The first term on the RHS then is of order unity. We decide to neglect any other term on the RHS that is of order γ^{-2} or smaller. This is the last term:

$$\gamma^{-1}\cos z\sin(\gamma^{-1}\varphi) \qquad (\mathcal{O}(\gamma^{-2}))$$

Making these approximations,

$$\boxed{\begin{aligned}\varphi'' = &-p\gamma\sin z\sin\gamma s - \gamma^{-1}\Big[\ddot{z} + 2\dot{\varphi}' - \sin z\\ &\qquad + p\varphi\gamma\cos z\sin\gamma s + \gamma^{-1}\ddot{\varphi}\Big]\end{aligned}} \quad (5)$$

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So far, the "slow" component $z(s)$ and "fast" component $\gamma^{-1}\varphi(s,\gamma s)$ of $\psi(s)$ are not uniquely defined. For example, at this point we could choose $\varphi \equiv 0$. Eq. (5) would become

$$0 = -p\gamma \sin z \sin\gamma s - \gamma^{-1}[\ddot{z} - \sin z]$$

$$\therefore \ddot{z} - (1 - p\gamma^2 \sin\gamma s)\sin z = 0$$

so $z(s)$ satisfies the same Eq. (2) as the original $\psi(s)$.

We can take advantage of the freedom in φ by choosing it so that we get an even simpler equation in z. To lowest order, Eq. (5) is

$$\frac{\partial^2\varphi}{\partial(\gamma s)^2} \cong -p\gamma \sin z(s) \sin\gamma s$$

Still considering s and γs to be independent variables, this equation is solved by

$$\varphi(s,\gamma s) = p\gamma \sin z(s) \sin\gamma s$$

We choose φ to be this solution.

The remainder of Eq. (5) becomes

$$\boxed{0 = \ddot{z} - \sin z + (p\varphi\gamma \sin\gamma s)\cos z + 2\dot{\varphi}' + \gamma^{-1}\ddot{\varphi}} \quad (6)$$

We would like to eliminate the last 2 terms and simplify the 3rd term on the RHS. We do this by taking the average value of Eq. (6) over one fast cycle $\left(\Delta s = \frac{2\pi}{\gamma}\right)$. Denote this average by $\langle\ \rangle$.

We don't bother to denote explicitly the average of z or its derivatives, since z varies only slightly over 1 fast cycle. We get

$$0 = \ddot{z} - \sin z + p\cos z\langle\varphi \sin\gamma s\rangle + 2\langle\dot{\varphi}'\rangle + \gamma^{-1}\langle\ddot{\varphi}\rangle.$$

Computing the last 3 terms,

$$\langle\varphi \sin\gamma s\rangle = p\gamma \sin z\langle\sin^2\gamma s\rangle = \tfrac{1}{2}p\gamma \sin z$$

$$\langle\dot{\varphi}'\rangle = p\gamma\dot{z}\cos z\langle\cos\gamma s\rangle = 0$$

$$\langle\ddot{\varphi}\rangle = -p\gamma\ddot{z}\sin z\langle\sin\gamma s\rangle = 0$$

Eq. (6) becomes

$$0 = \ddot{z} - \sin z + \tfrac{1}{2}p^2\gamma^2 \sin z\cos z$$

$$\boxed{\ddot{z} + \left(\tfrac{1}{2}p^2\gamma^2\cos z - 1\right)\sin z = 0} \quad (7)$$

So far we have made no approximation concerning the size of the "slow" angle z. Suppose now that the pendulum starts from a small angle z_0 from the vertical. Approximating

$$\sin z \approx z$$

$$\cos z \approx 1$$

we have

$$\ddot{z} + z\left(\tfrac{1}{2}p^2\gamma^2 - 1\right) = 0$$

which yields SHM if $p\gamma > \sqrt{2}$ and exponential growth of z if not. So the inverted pendulum is <u>stable</u> if

$$p\gamma > \sqrt{2} \Rightarrow \frac{Y_0}{\ell}\frac{\Omega}{\omega_0} > \sqrt{2}, \text{ or}$$

$$\frac{q\,\cancel{\ell}}{\cancel{\ell}\sqrt{a}} > \sqrt{2}, \quad \boxed{q^2 > 2a} \quad \left(\text{Hand \& Finch notation}\right)$$

So the amplitude Y_0 of the vibrating support must be at least a fraction $\sqrt{2}\,\frac{\omega_0}{\Omega}$ of the pendulum length ℓ.

All the while, the vibration frequency Ω must be $\gg \omega_0$, the natural frequency of the pendulum, in order for this method of analysis to be valid.

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